Math 535
Problem Set 8

due Thursday, April 4, 2024

You may discuss problems with other students, but please write up your solutions on your own. Please try to write neatly. It is helpful if you staple all the pages together, and write your name on the first page.

1. Let V be a k-vector space. Denote by $V^* = \mathrm{Hom}(V, k)$ the *dual vector space*, whose elements are the linear functionals $\varphi\colon V \to k$.
   (a) Show that there is a natural linear transformation $V \to V^{**}$.
   (b) If V is finite-dimensional, show that $V \cong V^{**}$.

2. Let V be a k-vector space. The notation $\langle \varphi, v \rangle = \varphi(v)$ is sometimes used to mean the value of a linear functional $\varphi \in V^*$ on a vector $v \in V$.
   (a) Show that a linear transformation $T\colon V \to V$ induces a linear transformation $T^*\colon V^* \to V^*$, called the *adjoint* of T, such that $\langle T^*\varphi, v \rangle = \langle \varphi, Tv \rangle$ for all $v \in V$ and all $\varphi \in V^*$.
   (b) Choose a basis $v_1, \ldots, v_n \in V$. Show that V^* has a unique basis $\varphi_1, \ldots, \varphi_n$ with the property that

$$\varphi_i(v_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

   (c) Let A be the matrix representing T with respect to $v_1, \ldots, v_n$. Show that the matrix representing T^* with respect to $\varphi_1, \ldots, \varphi_n$ is the transpose of the matrix A.
   (d) Show that $T^{**} = T$ under the isomorphism in Problem 1.

3. Let V be a finite-dimensional k-vector space. Suppose that the minimal polyonomial of a linear transformation $T\colon V \to V$ can be factored as $(x - \lambda_1)^{e_1} \cdots (x - \lambda_n)^{e_n}$ for distinct elements $\lambda_1, \ldots, \lambda_n \in k$. We proved in class that

$$V \cong V_1 \oplus \cdots \oplus V_n,$$

   where $V_i \subseteq V$ is the kernel of $(T - \lambda_i \,\mathrm{id})^{e_i}$. Show that the resulting coordinate projections $\pi_i\colon V \to V_i$ can be written in the form $\pi_i(v) = p_i(T)v$ for certain polynomials $p_i(x) \in k[x]$.

4. (a) Let $B \colon V \times V \to k$ be an alternating bilinear form on a finite-dimensional k-vector space V. Assume that $\operatorname{char}(k) \neq 2$. Show that there is a basis for V such that the matrix representing B has the block form

$$\begin{pmatrix} 0 & I_k & 0 \\ -I_k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where I_k is the identity matrix of size $k \times k$ and $2k \leq \dim V$.

   (b) Conclude that if B is nondegenerate, then $\dim V$ must be even.

5. Let B be a symmetrix $n \times n$-matrix over a field k, and suppose that B has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n \in k$. Show that there is a basis of eigenvectors for B in which the associated bilinear form is represented by a diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$.

Problems from the textbook

All problems are from the 3rd edition of *Abstract Algebra* by Dummit and Foote.

1. From Section 11.5, 13