Math 535
Problem Set 11

due Thursday, May 2, 2024

You may discuss problems with other students, but please write up your solutions on your own. Please try to write neatly. It is helpful if you staple all the pages together, and write your name on the first page.

1. Consider a short exact sequence of R-modules

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0.$$

   Prove that if there is a morphism $s\colon M \to M'$ with $s \circ f = \mathrm{id}$, then the short exact sequence is *split*, meaning that $M \cong M' \oplus M''$.

2. Given two morphisms

$$\begin{array}{ccc} M' & \xrightarrow{f} & M \\ \downarrow{\scriptstyle h} & & \\ N & & \end{array}$$

   we define their *pushout* P as the quotient of $M \oplus N$ by the image of the morphism $M' \to N \oplus M$, $x \mapsto (f(x), -h(x))$. The two natural morphisms $i\colon M \to P$ and $j\colon N \to P$ make a commutative diagram

$$\begin{array}{ccc} M' & \xrightarrow{f} & M \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle i} \\ N & \xrightarrow{j} & P. \end{array}$$

   Now suppose that we are given a short exact sequence

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0.$$

   and a morphism $h\colon M' \to N$. Show that the pushout P fits into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & N & \xrightarrow{j} & P & \xrightarrow{g} & M'' & \longrightarrow & 0 \end{array}$$

   whose bottom row is also a short exact sequence.

3. Let G be a group, let k be a field, and consider the vector space

$$\mathrm{Map}^f(G,k) = \left\{\, f\colon G \to k \;\middle|\; f(x) = 0 \text{ for all but finitely many } x \in G \,\right\}$$

of all k-valued functions on G with finite support. Show that the rule

$$(g \cdot f)(x) = f(xg)$$

defines a representation of G on $\mathrm{Map}^f(G,k)$. Is this representation isomorphic to the regular representation $k[G]$ or not?

4. Show that the decomposition of a representation into irreducible representations in unique, in the following sense: if $V_1, \ldots, V_n$ and $W_1, \ldots, W_m$ are irreducible representations of G, and if

$$V_1 \oplus \cdots \oplus V_n \cong W_1 \oplus \cdots \oplus W_m$$

are isomorphic as representations, then $m = n$ and there is a permutation $\sigma \in S_n$ such that $W_i = V_{\sigma(i)}$ for $i = 1, \ldots, n$.

Problems from the textbook

All problems are from the 3rd edition of *Abstract Algebra* by Dummit and Foote.

1. From Section 18.1, 2, 3, 5, 9, 15