Math 535
Problem Set 10

due Thursday, April 25, 2024

You may discuss problems with other students, but please write up your solutions on your own. Please try to write neatly. It is helpful if you staple all the pages together, and write your name on the first page.

1. Prove the so-called "Five Lemma": If

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{i} & M & \xrightarrow{p} & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & N' & \xrightarrow{j} & N & \xrightarrow{q} & N'' & \longrightarrow & 0 \end{array}$$

is a commutative diagram with exact rows, and if both f' and f'' are isomorphisms, then f is an isomorphism.

2. Let M be an A-module. The purpose of this problem is to show that morphisms between free resolutions are unique up to homotopy. Suppose that we have a commutative diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & F_2 & \xrightarrow{d} & F_1 & \xrightarrow{d} & F_0 & \xrightarrow{p} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle 0} & & \\ \cdots & \longrightarrow & G_2 & \xrightarrow{d} & G_1 & \xrightarrow{d} & G_0 & \xrightarrow{p} & M & \longrightarrow & 0 \end{array}$$

with exact rows in which F_n and G_n are free A-modules for every $n \geq 0$. Show that $f_\bullet$ is homotopic to zero: there are maps of A-modules $s_n \colon F_n \to G_{n+1}$ such that $f_n = ds_n + s_{n-1}d$ (setting $s_{-1} = 0$).

Problems from the textbook

All problems are from the 3rd edition of *Abstract Algebra* by Dummit and Foote.

1. From Section 17.1, 1, 3, 6, 14, 21, 25