Math 534
Problem Set 6

due Thursday, October 25, 2018

1. Let R be a ring with 1. Show that if $0 = 1$, then R is the zero ring.

2. Let R be a ring. We use $-a$ to denote the additive inverse of $a \in R$.

   (a) Prove that $0 \cdot a = 0$ for every $a \in R$.

   (b) Prove that $(-a)(-b) = ab$ for every $a, b \in R$.

3. Prove that every ideal in the ring $\mathbb{Z}$ is of the form (n) for some $n \in \mathbb{Z}$.

4. Let F be a field, and let R be the set of all functions $a\colon \mathbb{N} \to F$. Here we think of $a \in R$ as representing the formal power series

$$\sum_{n=0}^{\infty} a(n)x^n = a(0) + a(1)x + a(2)x^2 + \cdots$$

   The ring of formal power series is usually denoted by $F[[x]]$.

   (a) Show that R is a commutative ring with 1, under the operations

$$(a+b)(n) = a(n) + b(n) \quad \text{and} \quad (a \cdot b)(n) = \sum_{i=0}^{n} a(i)b(n-i)$$

   (b) Determine the set of units $R^\times$.

   (c) Determine all ideals in the ring R.

5. Find a ring R_0 in ((commutative rings with 1)) such that for any commutative ring with 1, the set $\mathrm{Mor}(R_0, R)$ is in one-to-one correspondence with R.

6. Find a ring R_1 in ((commutative rings with 1)) such that for any commutative ring with 1, the set $\mathrm{Mor}(R_1, R)$ is in one-to-one correspondence with $R^\times$.

7. If R is a ring with 1, the set $C(R) = \{\, x \in R \mid xy = yx \text{ for all } y \in R \,\}$ is called the *center* of R.

   (a) Show that $C(R)$ is a subring of R (with 1).

(b) Let F be a field, and let $R = M_n(F)$ be the ring of $n \times n$-matrices with coefficients in F. What is the center of R?

(c) Let G be a finite group, and let R be the integral group ring $\mathbb{Z}[G]$ of G. What is the center of R in this case?

8. Suppose that $f(x, y)$ is a polynomial in $\mathbb{Z}[x, y]$, and $R = \mathbb{Z}[x, y]/(f(x, y))$. Show that $\mathrm{Mor}(R, \mathbb{R})$ is in one-to-one correspondence with the points on the graph of $f(x, y) = 0$ in $\mathbb{R}^2$.

9. Let R be a commutative ring with 1. An element $e \in R$ such that $e^2 = e$ is called an *idempotent*.

   (a) If $e \in R$ is an idempotent, show that R is isomorphic to the product ring $R/(e) \times R/(1 - e)$.

   (b) Let H be a subgroup of a finite group G. Show that the element

   $$\frac{1}{|H|} \sum_{g \in H} g$$

   is an idempotent in the group ring $\mathbb{Q}[G]$.

   (c) Find all idempotents in the group ring $\mathbb{Q}[S_3]$.

10. Let X be a compact Hausdorff space, and let $C(X)$ be the ring of all continuous functions $f \colon X \to \mathbb{R}$, with addition and multiplication defined pointwise.

   (a) Is $C(X)$ an integral domain?

   (b) Let $x \in X$ be any point. Show that the set of continuous functions $f \in C(X)$ such that $f(x) = 0$ is a maximal ideal.

   (c) Show that every maximal ideal of $C(X)$ is of this form.

11. Let R be a commutative ring with 1. Show that an element $r \in R$ is a unit if and only if r in not contained in any maximal ideal of R.