Math 534
Problem Set 2

due Thursday, September 13, 2018

1. Let H and K be subgroups of a finite group G. Let $HK = \{ hk \mid h \in H, k \in K \}$, which need not be a subgroup of G. Show that

$$|HK| = \frac{|H| \cdot |K|}{|H \cap K|}$$

Here $|S|$ denotes the cardinality of a set S.

2. Let G be a group (with group operation written multiplicatively, as usual). Define a new operation $*$ on G by setting

$$x * y = yx.$$

   (a) Show that G is a group under the new operation. We write G^{op} for the set G with the new operation $*$.

   (b) Show that G^{op} is isomorphic to G.

   (c) Suppose that G acts on a set X by a left action, and let $\alpha\colon G \times X \to X$ be the map which defines the action; in other words, if gx denotes the action of $g \in G$ on $x \in X$, then $\alpha(g, x) = gx$. Show that α defines a *right* action of G^{op} on X.

3. Let G and H be groups, and let Φ be the set of all functions $\phi\colon G \to H$ with the property that $\phi(1) = 1$.

   (a) Let $\alpha\colon G \times \Phi \to \Phi$ be defined as follows: $\alpha(g, \phi)$ is the function that takes $x \in G$ to $\phi(g)^{-1}\phi(gx) \in H$. Prove that this defines an action of G on Φ. Is it a right action or a left action?

   (b) Describe the set Φ_0 of fixed points for this action.

4. Let p be a prime, and let G be a p-group, meaning a finite group whose order is a power of p. Suppose that G acts on a finite set X. Let $X_0 \subseteq X$ be the set of fixed points of G. Show that

$$|X_0| \equiv |X| \mod p.$$

5. Let G be a finite group of order n, let p be a prime dividing n, and let Z_p be the cyclic group of order p. Let Φ be the set of all functions $\phi\colon Z_p \to G$ with the property that $\phi(1) = 1$.

(a) Show that $|\Phi| = n^{p-1}$, where $n = |G|$.

(b) The group Z_p acts on the set Φ, as in Problem 3. Let Φ_0 be the set of fixed points in Φ for the action of Z_p. Show that the cardinality of Φ_0 is divisible by p.

(c) Show that Φ_0 is nonempty, and hence contains at least p elements. Use this to prove *Cauchy's theorem*: If p is a prime that divides the order of a finite group G, then G contains a subgroup isomorphic to the cyclic group Z_p.

6. Let G be a group, H a subgroup, and let X be the set of left cosets of H. G acts on X by the rule $(g, aH) \mapsto gaH$, a left action.

   (a) Show that G acts transitively on X.

   (b) What is the stabilizer of the coset $aH \in X$ in G?

   (c) Show that the homomorphism $G \to S_X$ corresponding to this action has kernel
   $$\bigcap_{g \in G} gHg^{-1}.$$

   (d) Suppose that G is a finite group, and that H is a proper subgroup. Suppose that $(G\colon H)!$ is not divisible by $|G|$. Show that G is not simple.

   (e) Suppose that G is finite, and that $(G\colon H)$ is the smallest prime dividing the order of G. Show that H is normal in G.

7. Let $G = \mathrm{GL}_2(\mathbb{C})$.

   (a) Give a complete set of representatives for the conjugacy classes of G, meaning a list of elements of G such that each element of G is conjugate to exactly one element of your list.

   (b) Given $A \in G$, how do you determine which element of your list A is conjugate to?