Math 534
Problem Set 11

due Thursday, December 20, 2018

1. Let R be a unique factorization domain, and $d \in R$ a nonzero element. Show that there are only finitely many distinct principal ideals containing d.

2. Let $A \in \mathrm{GL}_n(F)$ be an invertible $n \times n$-matrix with entries in a field F. Prove that there is a polynomial $P(x) \in F[x]$, of degree $\leq n$, such that $A^{-1} = P(A)$.

3. Describe the field of fractions of the ring $\mathbb{C}[x, y]/(y^2 - x^3)$.

4. Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be a linear map such that the minimal polynomial of T is $x^4 + 1$. Determine the number of linear subspaces $W \subseteq \mathbb{R}^4$ such that $T(W) \subseteq W$.

5. Show that a group of order 72 cannot be simple.

6. Find, up to isomorphism, all finite groups that have precisely three conjugacy classes.

7. Let K be a field of characteristic p, and $f(x) = x^p - x - a$, where $a \in K$. Show that f is either irreducible in $K[x]$, or decomposes into a product of linear factors.

8. Consider the set $\mathbb{Q}$ of rational numbers as a group under addition. Prove that there is no proper subgroup $G \subset \mathbb{Q}$ of finite index.

9. Let p be a prime number. The symmetric group S_p acts in a natural way on the set $X = \{1, 2, \ldots, p\}$. Prove that the induced action of a subgroup $G \subseteq S_p$ is transitive if and only if G contains a p-cycle.

10. Classify, up to isomorphism, all groups of order 45.

11. Let $R = \mathbb{F}_2[x]$. List, up to isomorphism, all R-modules with exactly 8 elements.