Math 534
Problem Set 1

due Thursday, September 6, 2018

We denote the group operation of a group by multiplication, the identity of G by 1, and the inverse of an element x by x^{-1}, unless stated otherwise.

1. Suppose that G is a set with an associate operation, written as multiplication. Show that G is a group if and only if, for every $a, b \in G$, the equations $xa = b$ and $ay = b$ have solutions $x, y \in G$.

2. Suppose that G is a group in which $x^2 = 1$ for every $x \in G$. Show that G is abelian.

3. Let G be a group and let o be any element of G. Define a new operation $*$ on G by the formula $x * y = xoy$.

   (a) Show that G is a group under the new operation $*$. What is the identity of G for the new operation? If $x \in G$, what is the inverse of x for the new operation?

   (b) If G is a group and $o \in G$, let G_o denote the group defined in (a). Show that $G \cong G_o$.

4. (a) Let G be the set of linear polynomials $f(x) = ax + b$, where $a, b \in \mathbb{R}$ and $a \neq 0$. Show that G is a group under composition of functions.

   (b) Let
   $$H = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \;\middle|\; a, b \in \mathbb{R},\, a \neq 0 \right\}.$$
   Show that H is a group under matrix multiplication.

   (c) Show that G and H are isomorphic.

5. Let $\mathbb{C}^\times$ be the multiplicative group of the complex numbers, i.e. the non-zero elements of $\mathbb{C}$, under multiplication. Show that $\mathbb{C}^\times$ is isomorphic to the direct product of the two additive groups $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$.

6. Let $n \geq 2$ be an integer. Determine the center of the dihedral group D_{2n}. (Hint: The answer depends on whether n is even or odd.)

7. Let H and K be subgroups of G, with K normal in G.

(a) Show that $HK = \{\, hk \mid h \in H,\, k \in K \,\}$ is a subgroup of G.

(b) Suppose that H is also normal. Show that HK is a normal subgroup of G.

8. The *commutator subgroup* of a group G is the group G' generated by the elements $[x, y] = x^{-1}y^{-1}xy$, where x and y vary through the elements of G.

   (a) Show that G' is normal in G.

   (b) Show that G/G' is an abelian group.

   (c) Let H be a subgroup of G containing G'. Show that H is normal in G, and that G/H is abelian.

   (d) Suppose that N is a normal subgroup of G such that G/N is abelian. Show that $G' \subseteq N$.