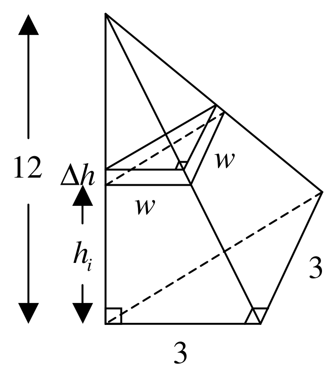
Solutions to Practice Exam 2

2.2 Each cross section parallel to the base is an isosceles right triangle.
The strategy is to sum up the volumes of each cross section with thickness Δh.

$V \approx \sum_{i=1}^{n}$ Volume of cross section at height h_i

$$V \approx \sum_{i=1}^{n} \tfrac{1}{2} w_i^{\,2} \Delta h$$



The height has been measured upward from the base of the pyramid.

Using similar triangles, we get $\dfrac{w}{3} = \dfrac{12-h}{12} \Rightarrow w = \tfrac{1}{4}(12-h)$

$$V \approx \sum_{i=1}^{n} \tfrac{1}{2} w_i^{\,2} \Delta h = \sum_{i=1}^{n} \tfrac{1}{2}\left(\tfrac{1}{4}(12-h_i)\right)^2 \Delta h = \sum_{i=1}^{n} \tfrac{1}{32}(12-h_i)^2 \Delta h$$

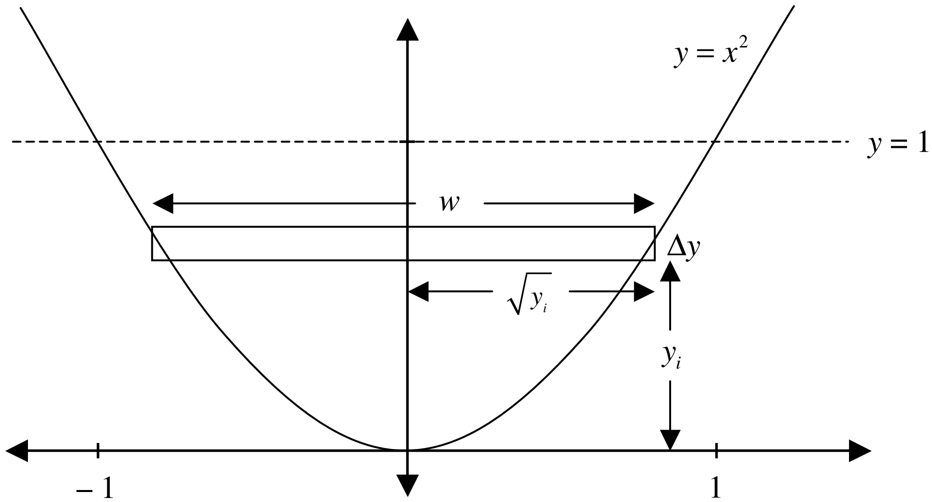
So, $V = \lim_{\substack{n\to\infty \\ \Delta h \to 0}} \sum_{i=1}^{n} \tfrac{1}{32}(12-h_i)^2 \Delta h = \tfrac{1}{32}\int_0^{12} (12-h)^2\, dh = 18$.

If the height is measured downward referring to the top of the pyramid as zero, using similar triangles, one would get $\dfrac{w}{3} = \dfrac{h}{12} \Rightarrow w = \tfrac{1}{4}h$.

Then, $V \approx \sum_{i=1}^{n} \tfrac{1}{2} w^2 \Delta h = \sum_{i=1}^{n} \tfrac{1}{2}\left(\tfrac{1}{4}h\right)^2 \Delta h = \sum_{i=1}^{n} \tfrac{1}{32} h^2 \Delta h$.

So, $V = \lim_{\substack{n\to\infty \\ \Delta h \to 0}} \sum_{i=1}^{n} \tfrac{1}{32} h^2 \Delta h = \tfrac{1}{32}\int_0^{12} h^2\, dh = 18$.

2.3 The parabola and the line intersect at the points $(-1,1)$ and $(1,1)$.



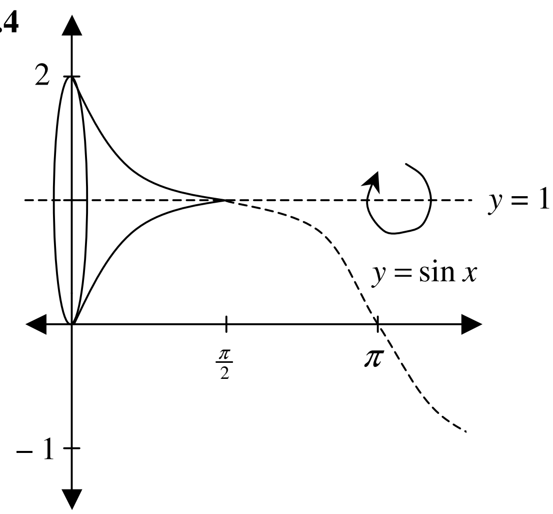
Each cross section perpendicular to the ***y*-axis** is a square.
The strategy is to sum up the volumes of each cross section with thickness Δy.
The length of each side of the square is $w = 2 \cdot \sqrt{y}$

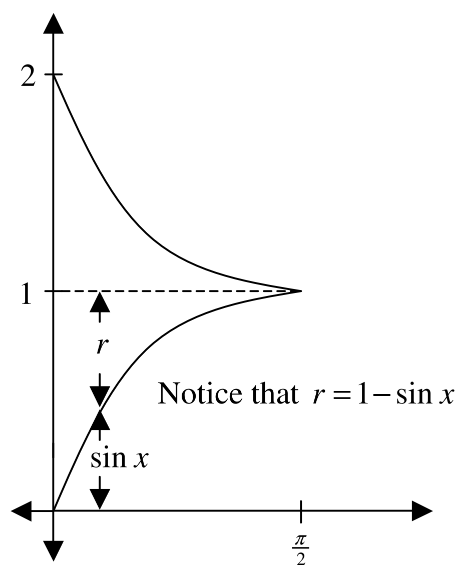
$V \approx \sum$ Volume of cross section at height y_i

$$V \approx \sum Area_{y_i} \Delta y = \sum w^2 \Delta y = \sum \left(2\sqrt{y}\right)^2 \Delta y = \sum 4y\Delta y$$

So, $V = 4\int_0^1 y \, dy = 2$.

2.4



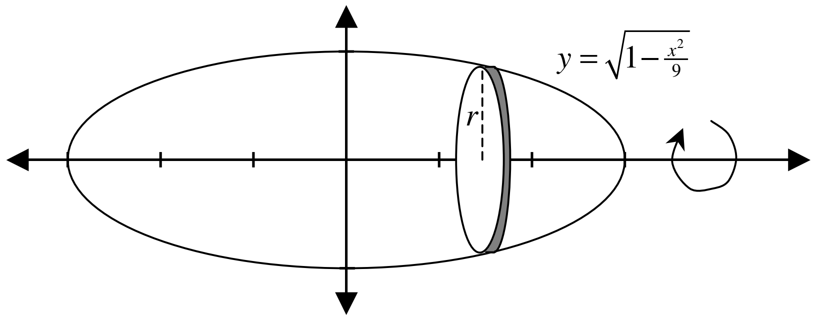


Notice that $r = 1 - \sin x$.

$$V \approx \sum \pi r^2 \Delta x = \sum \pi (1 - \sin x)^2 \Delta x.$$

$$V = \pi \int_0^{\frac{\pi}{2}} \left(1 - 2\sin x + \sin^2 x\right) dx = \pi \cdot \left[x + 2\cos x + \tfrac{1}{2}(-\sin x \cos x + x)\right]\Big|_0^{\frac{\pi}{2}} = \frac{\pi(3\pi - 8)}{4}$$

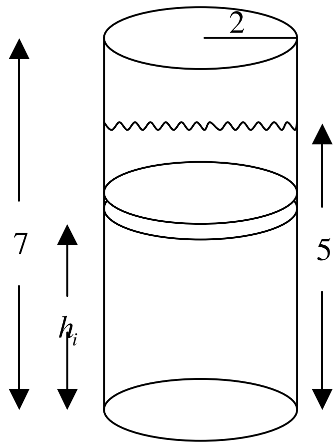
2.5



$$V \approx \sum \pi r^2 \Delta x = \sum \pi \left(\sqrt{1-\tfrac{x^2}{9}}\right)^2 \Delta x = \sum \pi \left(1-\tfrac{x^2}{9}\right) \Delta x$$

$$V = \pi \int_{-3}^{3} \left(1-\tfrac{x^2}{9}\right)\, dx = 2\pi \cdot \int_{0}^{3} \left(1-\tfrac{x^2}{9}\right)\, dx = 4\pi$$

2.8



$$\text{work} \approx \sum_{i=1}^{n} \text{work on slice at height } h_i$$

$$\approx \sum_{i=1}^{n} \underbrace{(7-h_i)}_{\text{distance}} \cdot \underbrace{\delta \cdot \pi \cdot 2^2 \Delta h}_{\text{force}} \qquad \left(\text{ft} \cdot \tfrac{\text{lbs}}{\text{ft}^3} \cdot \text{ft}^3 = \text{ft} \cdot \text{lbs}\right)$$

$$\approx \sum_{i=1}^{n} 200\pi (7-h_i)\Delta h$$

$$\text{work} = 200\pi \int_0^5 (7-h)\, dh = 4500\pi \text{ ft} \cdot \text{lbs}$$

2.9 $\text{work} \approx \sum_{i=1}^{n} \text{work to raise the chain, with length } y_i + 3 \text{ ft, vertically } \Delta y \text{ ft}$

$$\approx \sum_{i=1}^{n} \underbrace{\delta(y_i+3)}_{\text{force (weight)}} \cdot \underbrace{\Delta y}_{\text{distance}} \qquad \left(\tfrac{\text{lbs}}{\text{ft}} \cdot \text{ft} \cdot \text{ft} = \text{ft} \cdot \text{lbs}\right)$$

$$\text{work} = 5\int_0^{17} (y+3)\, dy = 997.5 \text{ ft} \cdot \text{lbs}$$

2.10 $\dfrac{\Delta h}{\Delta t} = 1 \tfrac{\text{ft}}{\text{sec}}$ and $\dfrac{\Delta F}{\Delta t} = -20 \tfrac{\text{lbs}}{\text{sec}}$. So, $\dfrac{\Delta F}{\Delta h} = -20 \tfrac{\text{lbs}}{\text{ft}}$.

$$\text{work} \approx \sum_{i=1}^{n} \text{work to raise the bucket, at height } h_i \text{ ft, vertically } \Delta h \text{ ft}$$

$$\approx \sum_{i=1}^{n} \underbrace{(1000-20h_i)}_{\text{force (weight)}} \cdot \underbrace{\Delta h}_{\text{distance}} \qquad \left(\left(\text{lbs} - \tfrac{\text{lbs}}{\text{ft}} \cdot \text{ft}\right) \cdot \text{ft} = \text{ft} \cdot \text{lbs}\right)$$

$$\text{work} = \int_0^{30} (1000-20h)\, dh = 21{,}000 \text{ ft} \cdot \text{lbs}$$

2.12 $y = f(x) = \tfrac{2}{3} x^{\frac{3}{2}}$. $f'(x) = x^{\frac{1}{2}}$.

$$\text{Arclength} = \int_0^3 \sqrt{1 + (f'(x))^2}\, dx = \int_0^3 \sqrt{1+x}\, dx = \tfrac{14}{3}.$$

2.15 $A=\int_{\frac{3\pi}{4}}^{\pi}\frac{1}{2}r^2\ d\theta=\frac{1}{2}\int_{\frac{3\pi}{4}}^{\pi}\left(\sqrt{\theta}\right)^2\ d\theta=\frac{7\pi^2}{64}$.

2.16 **(a)** $A=\int_0^{\frac{\pi}{3}}\frac{1}{2}r^2\ d\theta=\frac{1}{2}\int_0^{\frac{\pi}{3}}\left(2\sin\left(3\theta\right)\right)^2\ d\theta=2\int_0^{\frac{\pi}{3}}\sin^2\left(3\theta\right)\ d\theta$

$A=2\cdot\left(\frac{1}{3}\cdot\frac{1}{2}\left(-\sin\left(3\theta\right)\cos\left(3\theta\right)+3\theta\right)\right)\Big|_0^{\frac{\pi}{3}}=\frac{\pi}{3}$

(b) $x=r\cos\theta=2\sin\left(3\theta\right)\cos\theta$ and $y=r\sin\theta=2\sin\left(3\theta\right)\sin\theta$

$\frac{dx}{d\theta}=6\cos\left(3\theta\right)\cos\theta-2\sin\left(3\theta\right)\sin\theta$ and $\frac{dy}{d\theta}=6\cos\left(3\theta\right)\sin\theta+2\sin\left(3\theta\right)\cos\theta$

$\frac{dy}{dx}=\frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}}=\left.\frac{6\cos\left(3\theta\right)\sin\theta+2\sin\left(3\theta\right)\cos\theta}{6\cos\left(3\theta\right)\cos\theta-2\sin\left(3\theta\right)\sin\theta}\right|_{\theta=\frac{\pi}{6}}=-\sqrt{3}$$

(c) $\frac{1}{2}\int_{\frac{\pi}{18}}^{\frac{5\pi}{18}}\left(4\sin^2\left(3\theta\right)-1\right)\ d\theta$

2.17 **(a)** $$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\frac{4n^2+3n}{2n^2+3n+1} = \lim_{n\to\infty}\frac{4n^2+3n}{2n^2+3n+1}\cdot\frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{4+\frac{3}{n}}{2+\frac{3}{n}+\frac{1}{n^2}} = 2$$

(b) $$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(3+\frac{(-1)^n}{\sqrt{n}}\right) = 3+\lim_{n\to\infty}\frac{(-1)^n}{\sqrt{n}} = 3$$

(c) $$\lim_{n\to\infty}\frac{n^2}{2^n} \overset{L.R.}{=} \lim_{n\to\infty}\frac{2n}{2^n\ln 2} \overset{L.R.}{=} \lim_{n\to\infty}\frac{2}{2^n(\ln 2)^2} = 0$$

2.18 **(a)** $$3-\tfrac{3}{2}+\tfrac{3}{2^2}-\tfrac{3}{2^3}+\tfrac{3}{2^4}-\ldots = 3+3\left(\tfrac{-1}{2}\right)+3\left(\tfrac{-1}{2}\right)^2+3\left(\tfrac{-1}{2}\right)^3+3\left(\tfrac{-1}{2}\right)^4+\ldots$$
$$=3\sum_{n=0}^{\infty}\left(\tfrac{-1}{2}\right)^n = \frac{3}{1-\left(\tfrac{-1}{2}\right)} = 2$$

(b) $$\sum_{n=1}^{21} 3a^{2n} = 3a^2+3a^4+3a^6+\ldots+3a^{42} = 3a^2\left(1+a^2+a^4+a^6+\ldots+a^{40}\right)$$
$$=3a^2\left(1+\left(a^2\right)^1+\left(a^2\right)^2+\left(a^2\right)^3+\ldots+\left(a^2\right)^{20}\right) = 3a^2\cdot\frac{1-\left(a^2\right)^{21}}{1-a^2}$$

Or, $\displaystyle\sum_{n=1}^{21} 3a^{2n} = 3a^2\sum_{n=0}^{20}\left(a^2\right)^n = 3a^2\cdot\frac{1-\left(a^2\right)^{21}}{1-a^2}$.

No calculators will be permitted at the exam.

3.1 A ping-pong ball is launched straight up, rises to a height of 15 feet, then falls back to the launch point and bounces straight up again. It continues to bounce, each time reaching a height 90% of the height reached on the previous bounce. Find the total distance that the ball travels.

The ball has gone 30 ft at the first return to the launch point, then 2(15)(.9) more feet at the second return, 2(15)(.9)(.9) at the third return, etc. The total distance is then

$$30 + 30(.9) + 30(.9)^2 + 30(.9)^3 + \cdots = \frac{30}{1 - .9} = 300\text{feet}.$$

3.2 Use the **integral test** to determine the convergence of the following series:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$

a) Series converges if and only the improper integral $\int_1^\infty x^{-3/2}\, dx$ converges. $\int_1^\infty x^{-3/2}\, dx = 2 < \infty$ so series *converges.*

b) Series converges if and only the improper integral $\int_1^\infty xe^{-x}\, dx$ converges. $\int_1^\infty xe^{-x}\, dx = 2/e < \infty$ (integrate by parts and use L'Hopital's rule) so series *converges.*

3.3 Determine if the following series converge or diverge. Give your reasoning using complete sentences.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(n+2)!}$

a) Series converges if and only the improper integral $\int_1^\infty \frac{\ln x}{x^2}\,dx$ converges. Integrate by parts and use L'Hopital's rule to see that this integral converges to 1 so series *converges*. Alternately, $\ln x < x^{1/2}$ when x is large since by L'Hopital's rule $\lim_{x\to\infty} \frac{\ln x}{x^{1/2}} = 0$, so $\frac{\ln x}{x^2} \leq \frac{1}{x^{3/2}}$ so $\int_1^\infty \frac{\ln x}{x^2}\,dx$ is convergent by comparison with the integral in problem 2a) above

b) $\frac{n!}{(n+2)!} = \frac{1}{(n+2)(n+1)} < \frac{1}{n^2}$ so given series converges by comparison with p-series with $p = 2$ which is convergent.

3.4 For each of the following items a) and b) choose a correct conclusion and reason from among the choices (R), (C), (I) below and provide supporting computation. For example, if you choose (R) calculate and interpret a suitable ratio.

(R) Converges by the ratio test. (C) Diverges by a p-series comparison.

(I) Converges by the integral test.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 3^n}{n 4^n}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

a) (R). The ratio $\frac{a_{n+1}}{a_n} = \frac{n+1}{n}\frac{3}{4} \to \frac{3}{4} < 1$.

b) (C). For $n > 3$ $\ln n > 1$ so the series diverges by comparison with the p-series with $p = 1$.

3.5 a) Which of the following correctly classifies the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^n + 4^n}{e^n}$ and gives a valid reason? Circle your answer. No other reason required.

i) Diverges: $\lim_{n\to\infty} a_n \neq 0$.

ii) Converges: sum of two convergent geometric series.

iii) Converges: Comparison test with geometric series with ratio $3/e$.

iv) Converges: Comparison test with geometric series with ratio $e/4$.

v) Diverges: constant multiple of the harmonic series.

b) Which of the following correctly classifies the series $\sum_{n=1}^{\infty} \frac{n^3}{5+n^4}$ and gives a valid reason? Circle your answer. No other reason required.

i) Diverges: $\lim_{n\to\infty} a_n \neq 0$.

ii) Converges: Ratio test.

iii) Converges: Comparison test with a p-series, $p > 1$.

iv) Diverges: Ratio test.

v) Diverges: Comparison or limit comparison with the harmonic series.

a) Answer: i)   b) Answer: v)

3.6 Determine if the following series converge or diverge. Give your reasoning using complete sentences.

a) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$

b) $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{\ln n}$

a) An alternating series, but the terms do not approach 0 (they approach 1) so *divergent*.

b) An alternating series with terms decreasing to 0 so *convergent*.

3.7 a) Find the *interval* of convergence for the power series $\sum_{n=1}^{\infty} \frac{1}{n2^n}(x-3)^n$

b) Find the *radius* of convergence for $\sum_{n=1}^{\infty} \frac{n!}{(2n)!} x^n$

a)

$$\left| \frac{\frac{1}{(n+1)2^{n+1}}(x-3)^{n+1}}{\frac{1}{n2^n}(x-3)^n} \right| = \left| \frac{n2^n(x-3)^{n+1}}{(n+1)2^{n+1}(x-3)^n} \right| = \frac{1}{2} \cdot \frac{n}{n+1}|x-3| \to \frac{1}{2}|x-3|$$

as $n \to \infty$. $\frac{1}{2}|x-3| < 1$ for $|x-3| < 2$, that is, $1 < x < 5$ so series converges if $1 < x < 5$ and diverges if $x > 5$ or $x < 1$. At $x = 5$ the series becomes the harmonic series $\sum_n \frac{1}{n}$ which diverges, but at $x = 1$ it becomes the alternating series $\sum_n (-1)^n \frac{1}{n}$ which converges. The interval of convergence is therefore $[1, 5)$.

b)

$$\left| \frac{\frac{(n+1)!}{(2n+2)!}x^{n+1}}{\frac{n!}{(2n)!}x^n} \right| = \frac{n+1}{(2n+2)(2n+1)}|x| \to 0 < 1$$

for every x when $n \to \infty$. Thus the series converges for *all* x and the radius of convergence is ∞.