CONVERGENCE AND DIVERGENCE OF INFINITE SERIES

1. Suppose we are given an infinite series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots$$

Of course, we cannot literally add infinitely many numbers; instead, we have to add the numbers one by one and see what happens. In other words, we look at the *partial sums*

$$s_1 = a_1, \quad s_2 = a_1 + a_2, \quad s_3 = a_1 + a_2 + a_3, \quad s_4 = a_1 + a_2 + a_3 + a_4, \quad \text{etc.}$$

and see whether they approach a limit or not. Because the formula for the value of the n-th partial sum is $s_n = a_1 + a_2 + \cdots + a_n$, the precise meaning of the infinite series is

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \bigl(a_1 + a_2 + \cdots + a_n\bigr).$$

If the limit exists, we say that the series *converges*; if it does not exist, we say that the series *diverges*.

2. In fact, we saw in class that there are three possibilities for what could happen:

(1) The series *diverges*.
(2) The series *converges conditionally*; this means that $\sum a_n$ converges, but the series $\sum |a_n|$ with positive terms diverges.
(3) The series *converges absolutely*; this means that $\sum |a_n|$ converges. If that is the case, the original series $\sum a_n$ itself also converges.

Deciding whether the series converges or diverges is usually the best we can do, since it is rarely possible to find the exact value of the series.

3. Earlier in the semester, we learned several convergence tests; here is a strategy for applying them to a given series.

(1) The first step is to have a careful look at the series. Is it a familiar one, such as a p-series or a geometric series? Do the terms a_n go to zero when $n \to \infty$, and if yes, quickly or slowly? If they go to zero very quickly (such as 2^{-n}), the series is most likely convergent; if they go to zero rather slowly (such as $1/n$), the series might be divergent. Are there pieces of a_n that are small compared to others? For instance, 1 is small compared to n^2, or n^3 is small compared to 2^n. Is the series alternating, as indicated by $(-1)^n$ or $(-1)^{n-1}$? It is useful to have a rough idea about all this before proceeding.
(2) Based on our analysis of the terms, we decide what to do. If the terms a_n do not seem to go to zero as $n \to \infty$, we can apply the **Divergence Test**. The conclusion is simple: If $\lim a_n \neq 0$, the series diverges. For example,

$$\sum_{n=1}^{\infty} \frac{n}{3n-1}.$$

(3) If the series is alternating, we can use the **Alternating Series Test**. We have to verify that $\lim a_n = 0$, and that $|a_1| \geq |a_2| \geq |a_3| \geq \cdots$ is decreasing; if both conditions hold, the series converges. For example,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}.$$

(4) If the terms $a_n = f(n)$ are given by a function $f(x)$, and if $\int f(x)\,dx$ can easily be computed, we can use the **Integral Test**. To apply the test, the function should be positive and decreasing; then we simply look at the improper integral $\int_1^\infty f(x)\,dx$, and if it converges/diverges, the series converges/diverges. For example,

$$\sum_{n=1}^{\infty} \frac{2n}{(n^2+1)\ln(n^2+1)}.$$

(5) If some parts of a_n are small compared to others, we can simplify the series by leaving them out; to justify this process, we apply the **Limit Comparison Test**. Leaving out the negligible parts, we get a new series $\sum b_n$. If we manage to show that $\lim a_n/b_n = 1$, the two series have the same behavior. For example,

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n+n}.$$

(6) Finally, we have another test for deciding absolute convergence. The **Ratio Test** can be used if the ratios a_{n+1}/a_n can be simplified, for instance when a_n contains $n!$ or small powers of n. To apply the test, we compute the quantity $\rho = \lim|a_{n+1}/a_n|$. If $\rho < 1$, the series converges absolutely; if $\rho > 1$, it diverges. When $\rho = 1$, we get no conclusion and have to try something else. For example,

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}.$$