MATH 127 Solutions to Midterm 1

$\boxed{\text{12 pts}}$ 1. (a) Find the general solution to the differential equation $y'' + 2y' + 3y = 0$.

Solution: This is a second-order linear differential equation. Looking for solutions of the form $y = e^{\lambda x}$ leads to the characteristic polynomial

$$\lambda^2 + 2\lambda + 3 = 0.$$

The roots of the characteristic polynomial are the two complex numbers

$$\lambda = \frac{-2 \pm \sqrt{-8}}{2} = -1 \pm i\sqrt{2}.$$

The general solution to the differential equation therefore takes the form

$$y = C_1 e^{-x} \cos(\sqrt{2}x) + C_2 e^{-x} \sin(\sqrt{2}x),$$

where C_1 and C_2 are arbitrary constants.

$\boxed{\text{8 pts}}$ (b) Find the specific solution that satisfies the initial conditions $y(0) = 0$ and $y'(0) = 2$.

Solution: We determine C_1 and C_2 from the given initial conditions. First,

$$y(0) = C_1$$

and therefore $C_1 = 0$. Our formula for y simplifies to $y = C_2 e^{-x} \sin(\sqrt{2}x)$. Now

$$y' = C_2(-e^{-x}) \sin(\sqrt{2}x) + C_2 e^{-x}\sqrt{2}\cos(\sqrt{2}x),$$

and plugging in $x = 0$ gives

$$y'(0) = C_2\sqrt{2}.$$

We conclude that $C_2\sqrt{2} = 2$ or $C_2 = \sqrt{2}$. The specific solution is therefore

$$y = \sqrt{2}e^{-x}\sin(\sqrt{2}x).$$

$\boxed{\text{15 pts}}$ 2. Find *all* possible solutions y to the differential equation $\dfrac{dy}{dx} = -y^2 \ln x$.
(*Hint:* Are there any constant solutions?)

Solution: This is a separable differential equation. First, note that there is a constant solution $y = 0$, because the right-hand side of the equation equals zero if $y = 0$. We can find all the remaining solutions by the method of separating the variables. Indeed, if $y \neq 0$, we can rewrite the differential equation as

$$-\frac{dy}{y^2} = \ln x \, dx.$$

Integrating gives

$$\int -\frac{dy}{y^2} = \int \ln x \, dx,$$

and therefore

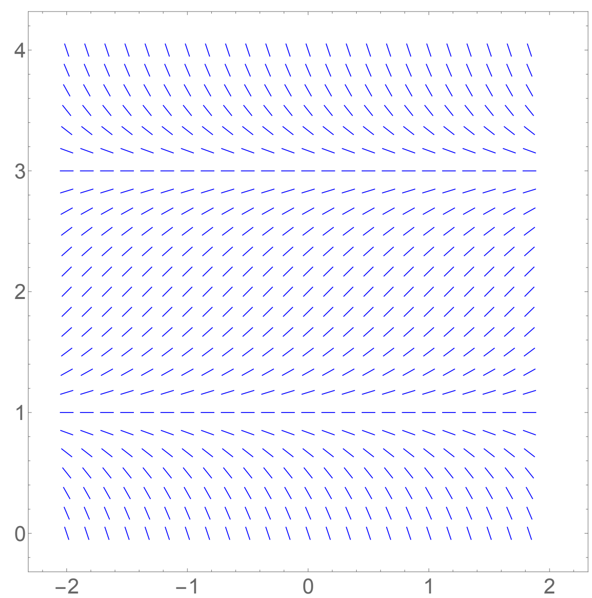
$$\frac{1}{y} = x \ln x - x + C,$$

where C is an arbitrary constant. This can be solved to give

$$y = \frac{1}{x \ln x - x + C}.$$

So the solutions to the differential equation are $y = 0$ and $y = 1/(x \ln x - x + C)$ for C an arbitrary constant.

3. The direction field for a certain first-order differential equation is shown below.



5 pts (a) Find all constant solutions to the differential equation. Justify your answer.

Solution: There are two constant solutions: $y = 1$ and $y = 3$. The graph of a constant solution is a horizontal line, with slope zero everywhere, and it is clear from the picture that there are two such horizontal lines.

10 pts (b) Suppose that $f(x)$ is a solution to the differential equation, and that $f(0) = y_0$. Determine all values of y_0 for which the function $f(x)$ is increasing. Justify your answer.

Solution: The function $f(x)$ is increasing when $1 \leq y_0 \leq 3$. To see why, imagine drawing the graph of $f(x)$ starting from the point $(0, y_0)$. Based on the direction field in the picture, we clearly get an increasing graph when $1 \leq y_0 \leq 3$, and a decreasing graph when $y_0 < 1$ or when $y_0 > 3$. (When $y_0 = 1$ or $y_0 = 3$, we get a horizontal line, which is considered to be "increasing".)

Note: The derivative $f'(x)$ tells us the slope of the graph, and in particular, $f(x)$ is increasing exactly when $f'(x) \geq 0$. Because $f(x)$ is a solution to the differential equation, the value of $f'(x)$ at each point is determined by the slope of the little line segments in the direction field; as you can see from the picture, this slope is positive in the region between $y = 1$ and $y = 3$, and negative everywhere else.

$\boxed{\text{15 pts}}$ 4. A bacteria culture grows at a rate that is proportional to the number of bacteria in the culture. The bacteria count was 1000 after 2 hours, and 4000 after 4 hours. Determine the initial number of bacteria in the culture.

Solution: Let $P(t)$ be the number of bacteria in the culture after t hours. We are looking for the initial number of bacteria $P(0)$. The assumption about the growth rate means that

$$\frac{dP}{dt} = kP,$$

where k is a positive constant. Therefore $P(t)$ grows exponentially as

$$P(t) = P(0)e^{kt}.$$

We are given that

$$P(2) = P(0)e^{2k} = 1000 \qquad \text{and} \qquad P(4) = P(0)e^{4k} = 4000.$$

Dividing the second equation by the first gives

$$e^{2k} = \frac{P(0)e^{4k}}{P(0)e^{2k}} = \frac{4000}{1000} = 4.$$

This can be solved to give $k = \frac{1}{2}\ln 4 = \ln 2$. Now we can go back to the first equation

$$P(0)e^{2\ln(2)} = 4P(0) = 1000$$

and conclude that $P(0) = 250$.

Note: Here is a faster way to get the answer: The number of bacteria went up from 1000 to 4000, or by a factor of 4, in two hours. Because we know that $P(t)$ grows exponentially, we deduce that the population always goes up by a factor of 4 every two hours. In particular, $P(2) = 4P(0)$. Since $P(2) = 1000$, we get $P(0) = 250$.

5. Consider the differential equation $x^2y'' = 2xy' + 10y$.

$\boxed{\text{10 pts}}$ (a) For which real numbers a does the function x^a solve this differential equation?

Solution: We substitute $y = x^a$ into the differential equation to find out for which values of a we get a solution. Clearly $y' = ax^{a-1}$ and $y'' = a(a-1)x^{a-2}$, and therefore the condition for $y = x^a$ to be a solution to the differential equation is that

$$x^2 \cdot a(a-1)x^{a-2} = 2x \cdot ax^{a-1} + 10x^a.$$

Simplifying and factoring out the common term x^a, we arrive at the condition

$$x^a \cdot (a^2 - 3a - 10) = 0.$$

Therefore $y = x^a$ is a solution whenever $a^2 - 3a - 10 = 0$. This happens for two values: $a = 5$ and $a = -2$.

10 pts

(b) Find a solution y to the differential equation with $y(1) = 0$ and $y'(1) = 3$.

Solution: We have just shown that x^5 and x^{-2} are solutions to the differential equation. Since the differential equation is linear, every function of the form

$$y = C_1 x^5 + C_2 x^{-2}$$

is also a solution. We can determine the right values of the two constants C_1 and C_2 from the initial conditions. Plugging in $x = 1$ gives

$$y(1) = C_1 + C_2 = 0,$$

and therefore $C_2 = -C_1$. Also, $y' = 5C_1 x^4 - 2C_2 x^{-3}$, and so

$$y'(1) = 5C_1 - 2C_2 = 3.$$

After putting $C_2 = -C_1$ into the second equation, we get $7C_1 = 3$ or $C_1 = \frac{3}{7}$ and $C_2 = -\frac{3}{7}$. The specific solution is therefore

$$y = \frac{3}{7} x^5 - \frac{3}{7} x^{-2}.$$

15 pts

6. At elevated temperatures, nitrogen dioxide decomposes into nitric oxide and oxygen, according to the following chemical reaction:

$$2\,\mathrm{NO_2} \longrightarrow 2\,\mathrm{NO} + \mathrm{O_2}$$

It is known that the rate at which the reaction occurs is proportional to the square of the concentration of $\mathrm{NO_2}$.

Formulate a mathematical model, in the form of an initial value problem, that describes the concentration of $\mathrm{NO_2}$ as a function of time. For each variable/function/constant that appears in your model, state in a few words what quantity it represents.

Solution: In our model, the variable t will denote time, and the function $C = C(t)$ the concentration of $\mathrm{NO_2}$ at time t. The derivative dC/dt represents the rate at which the concentration is changing; this rate has to be negative, because $\mathrm{NO_2}$ is being used up in the reaction. We are given that the reaction occurs at a rate of kC^2, where k is a positive constant. Every occurence of the reaction uses up two molecules of $\mathrm{NO_2}$, and so the rate of change of the concentration equals

$$\frac{dC}{dt} = -2 \cdot (\text{the rate at which the reaction occurs}) = -2kC^2.$$

The initial condition is that $C(0)$ equals the initial concentration of NO_2, at time $t = 0$.

Note: It would also be okay to write the differential equation as

$$\frac{dC}{dt} = -kC^2$$

for a positive constant k; the important thing is the minus sign, because NO_2 is being used up by the reaction.