MATH 127 Solutions to the Final Exam

$\boxed{10\text{ pts}}$ 1. Write the infinite decimal $0.121212\ldots$ as a fraction.

Solution: The answer is $\frac{12}{99} = \frac{4}{33}$. The reason is that

$$0.121212\cdots = \frac{12}{100} + \frac{12}{100^2} + \frac{12}{100^3} + \cdots = \frac{12}{100}\left(1 + \frac{1}{100} + \frac{1}{100^2} + \cdots\right)$$
$$= \frac{12}{100} \cdot \frac{1}{1 - \frac{1}{100}} = \frac{12}{100 - 1} = \frac{12}{99},$$

by the formula for the sum of a geometric series.

$\boxed{10\text{ pts}}$ 2. Write a power series for the function e^{-x^2}.

Solution: Starting from the familiar Taylor series

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

and replacing x by $-x^2$, we get

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}.$$

This series still converges for all real numbers x.

$\boxed{10\text{ pts}}$ 3. The sequence $a_n = 1 + (-1)^n$ converges / diverges .
circle one
Justify:

Solution: The sequence looks like $2, 0, 2, 0, 2, 0, \ldots$, and so it diverges.

$\boxed{10\text{ pts}}$ 4. Find the solution to the initial-value problem $\frac{dy}{dt} = -2y$, $y(0) = 5$.

Solution: The solution is $y(t) = 5e^{-2t}$.

$\boxed{10\text{ pts}}$ 5. Solve the differential equation $\frac{dy}{dx} = xe^{-y}$.

Solution: The equation is separable. If we separate the variables, we get

$$\int e^y \, dy = \int x \, dx.$$

After integration, this becomes

$$e^y = \frac{1}{2}x^2 + C \qquad \text{or} \qquad y = \ln\left(\frac{1}{2}x^2 + C\right),$$

where C is an arbitrary constant.

10 pts 6. Find the value of the infinite sum $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$.

Solution: This is a geometric series with value $\dfrac{1}{1-\frac{1}{3}} = \dfrac{3}{2}$.

10 pts 7. The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges / diverges (circle one) by the ____________________ test.

Justify:

Solution: The series converges by the **Alternating Series Test**. We can apply this test because the series is alternating, and because the terms $\frac{1}{n}$ are obviously decreasing and going to zero.

10 pts 8. The series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ converges / diverges (circle one) by the ____________________ test.

Justify:

Solution: The series diverges by the **Integral Test**. We can apply this test because the function $\frac{1}{x\ln(x)}$ is positive and decreasing. So we need to look at the improper integral

$$\int_2^{\infty} \frac{1}{x\ln(x)} \, dx.$$

A simple substitution shows that the antiderivative of $\frac{1}{x\ln(x)}$ is $\ln(\ln x)$; therefore

$$\int_2^{\infty} \frac{1}{x\ln(x)} \, dx = \lim_{b\to\infty} \int_2^{b} \frac{1}{x\ln(x)} \, dx = \lim_{b\to\infty} \Big(\ln(\ln b) - \ln(\ln 2)\Big) = \infty.$$

Because the integral diverges, the series also diverges.

10 pts 9. Determine all values of r for which the function $y = e^{rx}$ solves the differential equation $y'' + 3y' + y = 0$.

Solution: We have $y' = re^{rx}$ and $y'' = r^2e^{rx}$, and after substituting this into the differential equation, we get

$$r^2e^{rx} + 3re^{rx} + e^{rx} = 0,$$

or equivalently, $r^2 + 3r + 1 = 0$. The two solutions of this quadratic equation are

$$r = \frac{-3 \pm \sqrt{5}}{2},$$

by the quadratic formula.

10 pts 10. Iodine-131 has a half life of approximately 8 days. If we initially have 1 gram of this substance, then how much iodine-131 will be left after 40 days?

Solution: Since $40 = 5 \times 8$, the amount left will be $\frac{1}{2^5} = \frac{1}{32}$ grams.

15 pts 11. Write the first four terms in the Taylor series expansion of $\ln x$ around the point $a = 5$.

Solution: In general, the first four terms in the Taylor series expansion are

$$f(a) + f'(a) \cdot (x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3.$$

In our case, $f(x) = \ln x$ and $a = 5$. We first compute the necessary derivatives:

$$\begin{array}{cccc} f(x) = \ln x & f'(x) = \dfrac{1}{x} & f''(x) = -\dfrac{1}{x^2} & f'''(x) = \dfrac{2}{x^3} \\ f(5) = \ln(5), & f'(5) = \dfrac{1}{5} & f''(5) = -\dfrac{1}{25} & f'''(5) = \dfrac{2}{125} \end{array}$$

Putting it all together, the first four terms in the Taylor series are

$$\ln(5) + \frac{1}{5}(x-5) - \frac{1}{50}(x-5)^2 + \frac{1}{375}(x-5)^3.$$

20 pts 12. Find all values of x for which the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{n^2}$ converges. Do not forget to check the endpoints!

Solution: The first step is to find the radius of convergence. Set $a_n = \frac{(2x-1)^n}{n^2}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{(2x-1)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(2x-1)^n} = (2x-1) \cdot \frac{n^2}{(n+1)^2},$$

and therefore

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |2x-1| \cdot \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = |2x-1|.$$

By the ratio test, the series converges if $|2x-1| < 1$, and diverges if $|2x-1| > 1$. In other words, the series converges if $0 < x < 1$, and diverges if either $x < 0$ or if $x > 1$. (So the radius of convergence is $\frac{1}{2}$.)

The second step is to check the two endpoints $x = 0$ and $x = 1$. For $x = 0$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

which converges by the alternating series test. For $x = 1$, the series becomes

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

which converges by the p-series test (with $p = 2$).

The conclusion is that the power series converges for $0 \le x \le 1$, and diverges for all other values of x.

15 pts 13. Let $y = y(x)$ be the solution of the initial value problem $y' = x - y$, $y(0) = 1$.

(a) Use Euler's method with step size 0.2 to estimate $y(0.4)$.

Solution: We need to perform two steps of Euler's method; the step size is $h = 0.2$, and the initial value is $x_0 = 0$ and $y_0 = 1$.

First step: Since $x_0 = 0$ and $y_0 = 1$, the initial slope is $y_0' = x_0 - y_0 = -1$ (from the differential equation). Therefore

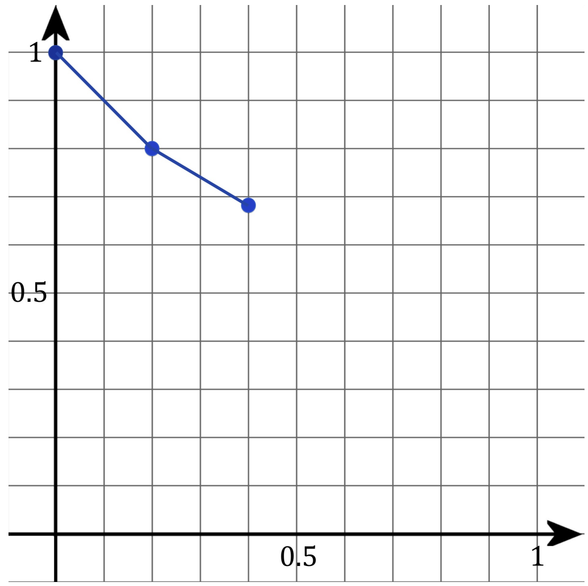
$$\begin{aligned} x_1 &= x_0 + h = 0.2 \\ y_1 &= y_0 + hy_0' = 1 - 0.2 \cdot 1 = 0.8. \end{aligned}$$

Second step: Now $x_1 = 0.2$ and $y_1 = 0.8$, and so the new slope is $y_1' = x_1 - y_1 = 0.2 - 0.8 = -0.6$. Therefore

$$\begin{aligned} x_2 &= x_1 + h = 0.4 \\ y_2 &= y_1 + hy_1' = 0.8 - 0.2 \cdot 0.6 = 0.68. \end{aligned}$$

The conclusion is that $y(0.4) \approx 0.68$.

(b) In the coordinate system below, plot the three points (x_0, y_0), (x_1, y_1), and (x_2, y_2) that you get from Euler's method.



25 pts 14. In this problem, we use the following two differential equations to model the populations of rabbits (R) and wolves (W).

$$\frac{dR}{dt} = 0.08R(1 - 0.0002R) - 0.001RW$$
$$\frac{dW}{dt} = -0.02W + 0.00002RW$$

(a) According to these equations, what happens to the rabbit population in the absence of wolves?

Solution: If there are no wolves, then $W = 0$, and the differential equation for the rabbit population becomes

$$\frac{dR}{dt} = 0.08R(1 - 0.0002R).$$

This is a logistic equation with growth constant $k = 0.08$ and carrying capacity $M = 1/0.0002 = 5000$. So in the absence of wolves, the rabbit population will converge to the carrying capacity of 5000 rabbits.

(b) Find all the equilibrium solutions and explain their significance.

Solution: An equilibrium solution is one where the populations of rabbits and wolves are constant. The condition for this is that $dR/dt = dW/dt = 0$. If we set the derivatives of R and W equal to zero, we get

$$\begin{aligned} 0.08R(1 - 0.0002R) - 0.001RW &= 0 \\ -0.02W + 0.00002RW &= 0. \end{aligned}$$

The second equation can be written in the form

$$0.00002W(R - 1000) = 0,$$

and so either $W = 0$ or $R = 1000$. Let us consider these two cases separately.

The first case is that $W = 0$. Here the first equation becomes

$$0.08R(1 - 0.0002R) = 0,$$

and so there are two possible values for R: either $R = 0$ or $R = 1/0.0002 = 5000$.
The second case is that $R = 1000$. Here the first equation becomes

$$0.08 \cdot 1000 \cdot (1 - 0.0002 \cdot 1000) - 0.001 \cdot 1000 \cdot W = 0,$$

which simplifies to $W = 64$.
In total, there are three equilibrium solutions:

1. $R = 0$ and $W = 0$. There are no rabbits and no wolves.
2. $R = 5000$ and $W = 0$. There are no wolves, and the rabbit population is exactly at the carrying capacity of 5000 rabbits.
3. $R = 1000$ and $W = 64$. What this means is that a population of 1000 rabbits is exactly big enough to support a population of 64 wolves.

15 pts 15. The value of the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is known to be $\dfrac{\pi^2}{6}$. Suppose we approximate $\dfrac{\pi^2}{6}$ by adding together the first n terms of the series. How big should n be, in order for the error in our approximation to be at most 0.001?

Solution: The function $f(x) = \frac{1}{x^2}$ is positive and decreasing, and so we can apply the error estimate from the integral test. This gives

$$\frac{\pi^2}{6} - \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}\right) \leq \int_n^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg|_n^\infty = \frac{1}{n}.$$

If we want to be sure that the error is at most $0.001 = \frac{1}{1000}$, we should take $n \geq 1000$.

15 pts 16. The initial-value problem

$$y'' = xy' + y, \qquad y(0) = 2, \quad y'(0) = -1$$

has a power series solution $y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$.
Determine the values of the first five coefficients c_0, c_1, c_2, c_3, c_4.

Solution: First, we compute the derivatives:

$$\begin{aligned} y &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots \\ y' &= c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots \\ y'' &= 2c_2 + 6c_3 x + 12c_4 x^2 + \cdots \end{aligned}$$

This also shows that $c_0 = y(0)$ and $c_1 = y'(0)$.

Then we substitute the three power series into the differential equation:

$$\begin{aligned}(2c_2 + 6c_3x + 12c_4x^2 + \cdots) &= x(c_1 + 2c_2x + 3c_3x^2 + \cdots) + (c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots)\\ &= c_0 + 2c_1x + 3c_2x^2 + 4c_3x^3 + \cdots\end{aligned}$$

By matching up the coefficients on both sides, we obtain

$$2c_2 = c_0 \qquad 6c_3 = 2c_1 \qquad 12c_4 = 3c_2.$$

The conclusion is that

$$\begin{aligned}c_0 &= y(0) = 2\\ c_1 &= y'(0) = -1\\ c_2 &= \frac{1}{2}c_0 = 1\\ c_3 &= \frac{1}{3}c_1 = -\frac{1}{3}\\ c_4 &= \frac{1}{4}c_2 = \frac{1}{4},\end{aligned}$$

and so the power series solution to the initial value problem looks like

$$y = 2 - x + x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots$$

$\boxed{\text{15 pts}}$ 17. Determine all values of x for which the infinite series $\displaystyle\sum_{n=1}^{\infty}(\ln x)^n$ is convergent.

Be sure to justify your answer.

Solution: The series is obtained from the geometric series

$$\sum_{n=1}^{\infty} x^n = x + x^2 + x^3 + \cdots + x^n + \cdots$$

by replacing x with $\ln x$. We know that the geometric series converges for $-1 < x < 1$, and diverges for all other values of x. Therefore the series in the problem converges exactly when $-1 < \ln x < 1$, which translates into $e^{-1} < x < e$.