MATH 127 Solutions to Early Exam

1. $\dfrac{\frac{1}{2}+\frac{1}{3}}{\frac{2}{3}-\frac{1}{4}}$ equals (A) 2.

   Solution: Simplify the fraction.

2. $\tan\dfrac{\pi}{3}$ equals (D) $\sqrt{3}$.

   Solution: Compute $\tan\dfrac{\pi}{3} = \dfrac{\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}} = \dfrac{\frac{1}{2}\sqrt{3}}{\frac{1}{2}} = \sqrt{3}$.

3. $e^{-\ln 5}$ equals (D) $\dfrac{1}{5}$.

   Solution: Compute $e^{-\ln 5} = \dfrac{1}{e^{\ln 5}} = \dfrac{1}{5}$, remembering that $e^{\ln 5} = 5$.

4. The solution set for the inequality $|2x-1| < 9$ is (C) $\{\, x \mid -4 < x < 5 \,\}$.

   Solution: $|2x-1| < 9$ is saying that $2x-1$ lies between -9 and 9, so $-9 < 2x-1 < 9$. This simplifies to $-4 < x < 5$.

5. If $x, y > 0$, then $\sqrt{\dfrac{2x^{1/2}y^{1/3}}{x^{5/2}y^{-5/3}}}$ equals (B) $\sqrt{2}x^{-1}y$.

   Solution: We have
   $$\frac{2x^{1/2}y^{1/3}}{x^{5/2}y^{-5/3}} = 2x^{1/2-5/2}y^{1/3+5/3} = 2x^{-2}y^2,$$
   and after taking square roots, we end up with $\sqrt{2}x^{-1}y$.

6. $\dfrac{n}{n-1} - \dfrac{1}{n+1}$ equals (D) $\dfrac{n^2+1}{n^2-1}$.

   Solution: Compute
   $$\frac{n}{n-1} - \frac{1}{n+1} = \frac{n(n+1)-(n-1)}{(n-1)(n+1)} = \frac{n^2+n-n+1}{n^2-1} = \frac{n^2+1}{n^2-1}.$$

7. The graph of $\ln(x+2)-5$ is obtained by shifting the graph of $\ln x$ (B) 2 units to the left, 5 units down.

Solution: Think about the graph.

8. The equation of the line through the two points $(1, -2)$ and $(4, 3)$ is (E) $3y = 5x - 11$.

Solution: The slope of the line is $\frac{3-(-2)}{4-1} = \frac{5}{3}$. It goes through the point $(4, 3)$, so the equation of the line is

$$y - 3 = \frac{5}{3}(x - 4).$$

This simplifies to $3(y - 3) = 5(x - 4)$ or $3y = 5x - 11$.

9. The solution of the system of equations $\begin{cases} 3x + y = 1 \\ x - 2y = 5 \end{cases}$ is (B) $(1, -2)$.

Solution: If we solve the second equation for x, we get $x = 2y + 5$. Substituting this into the first equation gives $3(2y + 5) + y = 1$, which simplifies to $7y = -14$. Therefore $y = -2$ and $x = 2(-2) + 5 = 1$.

10. The sum $5 + 7 + 9 + 11 + \cdots + 47$ can be written as (B) $\displaystyle\sum_{n=1}^{22}(2n + 3)$

Solution: All the other formulas give something else.

11. $\displaystyle\lim_{x\to 1} \frac{3x - 3}{x^2 + x - 2}$ equals (E) 1.

Solution: As long as $x \neq 1$, we have

$$\frac{3x - 3}{x^2 + x - 2} = \frac{3(x - 1)}{(x - 1)(x + 2)} = \frac{3}{x + 2}.$$

We can then take the limit as $x \to 1$ by simply plugging in $x = 1$.

12. $\displaystyle\lim_{x\to +\infty} \frac{x^2}{e^x}$ equals (D) 0.

Solution: This is an indeterminate limit of the form $\frac{+\infty}{+\infty}$. We can use l'Hopital's rule:

$$\lim_{x\to +\infty} \frac{x^2}{e^x} = \lim_{x\to +\infty} \frac{2x}{e^x} = \lim_{x\to +\infty} \frac{2}{e^x} = 0.$$

(In each step, we replace numerator and denominator by their derivative, until the limit is no longer indeterminate.)

13. If $f(x) = \dfrac{\sin x}{x^2}$ then $f'(x)$ equals (B) $\dfrac{x \cos x - 2 \sin x}{x^3}$

Solution: Use the quotient rule

$$f'(x) = \frac{x^2 \cdot (\sin x)' - (x^2)' \cdot (\sin x)}{x^4} = \frac{x^2 \cos x - 2x \sin x}{x^4} = \frac{x(x \cos x - 2 \sin x)}{x^4},$$

and then cancel x.

14. If $f(x) = e^{3x} \ln x$, then $f'(x)$ equals (A) $3e^{3x} \ln x + \dfrac{e^{3x}}{x}$.

Solution: Use the product rule and the chain rule (for e^{3x}):

$$f'(x) = (e^{3x})' \cdot \ln x + e^{3x} \cdot (\ln x)' = 3e^{3x} \ln x + e^{3x} \cdot \frac{1}{x}.$$

15. $\displaystyle\int e^{4x}\, dx$ equals (A) $\dfrac{1}{4}e^{4x} + C$.

Solution: The derivative of e^{4x} is $4e^{4x}$, so the derivative of $\frac{1}{4}e^{4x}$ is e^{4x}.

16. $\displaystyle\int x \cos x\, dx$ equals (D) $x \sin x + \cos x + C$.

Solution: Use integration by parts:

$$\int u\, dv = uv - \int v\, du$$

If we take $u = x$ and $dv = \cos x\, dx$, then $du = dx$ and $v = \sin x$. Therefore

$$\int x \cos x\, dx = x \sin x - \int \sin x\, dx = x \sin x - (-\cos x) + C = x \sin x + \cos x + C.$$

17. $\displaystyle\int \sin^3 x\, dx$ equals (C) $\dfrac{1}{3} \cos^3 x - \cos x + C$.

Solution: The trick is to write $\sin^3 x = \sin^2 x \cdot \sin x = (1 - \cos^2 x) \sin x$. We can then do the integral using the substitution $u = \cos x$ and $du = -\sin x\, dx$.

$$\int \sin^3 x\, dx = \int (1 - \cos^2 x) \sin x\, dx = \int (1 - u^2)(-du) = \int (u^2 - 1)\, du.$$

This evaluates to $\frac{1}{3}u^3 - u + C$; to get the answer, we put $u = \cos x$ back.

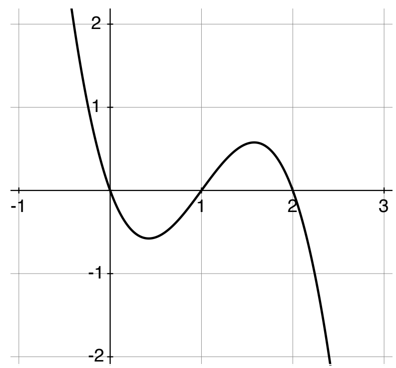
18. $\displaystyle\int_0^1 \frac{x}{\sqrt{4 - x^2}}\, dx$ equals (E) $2 - \sqrt{3}$.

Solution: Use the substitution $u = 4 - x^2$. Then $du = -2x\,dx$, and therefore

$$\int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx = \int_4^3 \frac{-\frac{1}{2}\,du}{\sqrt{u}} = \int_3^4 \frac{1}{2\sqrt{u}}\,du = \sqrt{u}\,\Big|_3^4 = \sqrt{4} - \sqrt{3}.$$

(The upper and lower limits in the u-integral are derived from the formula $u = 4 - x^2$: if $x = 0$, then $u = 4$, and if $x = 1$, then $u = 3$.)

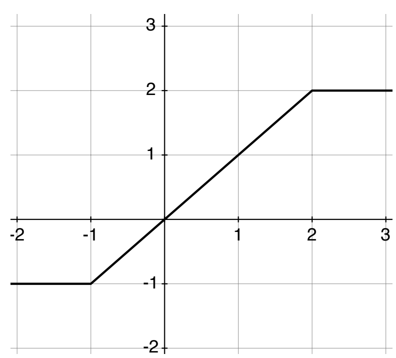
19. The figure below shows the derivative f' of a certain function.



Which of the following statements about the function f is correct? (B) f has a local maximum at the point $x = 0$.

Solution: The correct answer is (B). By looking at the graph, we see that near the point $x = 0$, the derivative f' is positive for $x < 0$, and negative for $x > 0$. This means that f is increasing for $x < 0$, and decreasing for $x > 0$, and so f has a local maximum at the point $x = 0$.

20. The graph of a function $g(x)$ on the interval $[-2, 3]$ is shown in the figure.



Then $\displaystyle\int_{-2}^{3} g(x)\,dx$ equals (C) $\dfrac{5}{2}$.

Solution: The integral is the signed area under the graph, with areas above the y-axis counted with a $(+)$-sign, and areas belowe the y-axis counted with a $(-)$-sign. Looking at the graph, we get $4 - \frac{3}{2} = \frac{5}{2}$ for the signed area.