Lecture notes for April 10

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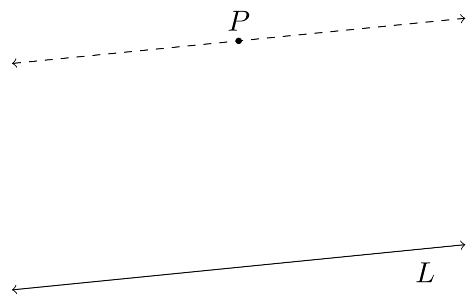
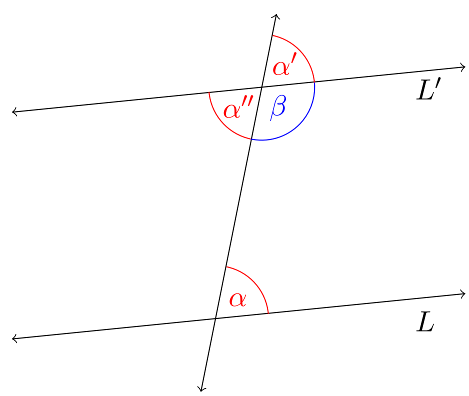
1 Parallel lines and alternate interior angles

Today's topic is the relationship between the parallel postulate and the fact that the angle measures in a triangle add up to 180°. The parallel postulate says the following: Given a line L, and a point P not on the line L, there is a unique line that passes through P and is parallel to L.



Here "parallel" means that the two lines do not intersect each other. (This is a better definition than saying that the distance between the two lines is constant, because it does not need the concept of "distance".)

The following drawing[1] illustrates a useful fact about parallel lines:



Namely, when a third line crosses two parallel lines, the alternate interior angles enclosed between the lines have equal measure. Let me briefly recall how this is proved. In the picture, the two angles marked α and α' are

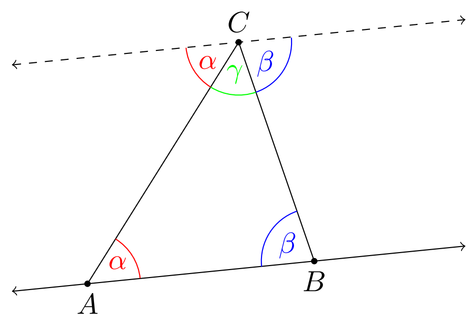
[1] All the drawings are done in LaTeX, with a package called `TikZ`.

congruent, because the lines L and L' are parallel. Congruent angles have equal measure, and so $\alpha = \alpha'$. We also know that $\alpha' + \beta = 180°$ and $\alpha'' + \beta = 180°$, because the two angles together make a straight line; consequently,

$$\alpha = \alpha' = \alpha''.$$

2 The angle sum in a triangle

Alternate interior angles can be used to prove that the sum of all the angle measures in a triangle is $180°$. Consider a triangle ΔABC, with angle measures α, β, and γ. Through the point C, draw the line parallel to AB.

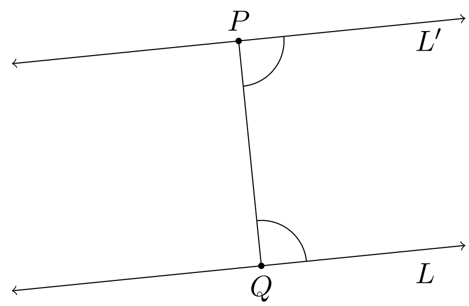


The two red angles are alternate interior angles, and therefore have the same measure α. For the same reason, the two blue angles have the same measure β. Since all three angles together form a straight line, we conclude that

$$\alpha + \beta + \gamma = 180°.$$

If we take a step back, we see that we have just proved the fact that the angle sum in a triangle is $180°$, starting from the parallel postulate. Interestingly, the result about the angle sum in a triangle can in turn be used to prove the parallel postulate. (In other words, the two statements are logically equivalent.)

Here is how this works. Consider a line L and a point P not on L. We can use the following simple construction to find the line parallel to L. First, draw a line through P that is perpendicular to L, in the sense that it makes a $90°$ angle with L. Then draw a line L' through P, perpendicular to the first line, as shown in the drawing below.



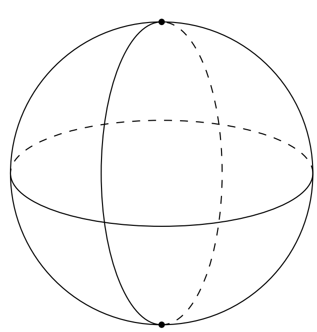
If we grant ourselves that the angle sum in every triangle is $180°$, we can prove that L' must be parallel to L. Indeed, suppose to the contrary that L and L' intersected in a point R. Consider the triangle ΔPQR. By construction, $m\angle P = m\angle Q = 90°$, and because

$$m\angle P + m\angle Q + m\angle R = 180°,$$

it follows that $m\angle R = 0°$. In other words, the two lines enclose an angle of measure $0°$. This implies that $L = L'$, and hence that P lies on L, contrary to our initial assumption.

3 Spherical geometry

The parallel postulate is the last of the five axioms in Euclidean geometry. Historically, many people believed that it should be possible to deduce the parallel postulate from the other four axioms. There were many (incorrect) attempts to prove this, until the first half of the 19th century, when Gauss and Lobachevsky found examples of so-called "non-Euclidean" geometries: models for geometry with points and lines, in which the first four axioms of Euclidean geometry are true, but the parallel postulate is not true. This made people realize that the parallel postulate really is necessary, and cannot be deduced from the other axioms.



The simplest example of this kind is "spherical geometry", or geometry on the surface of a sphere. The lines in this geometry are great circles. There are two problems with this idea, however:

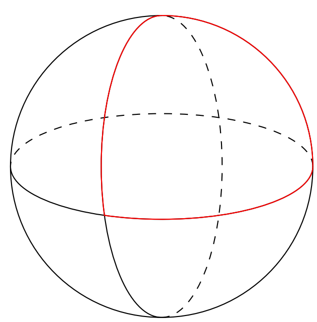
1. If you look at the intersection of two great circles, they intersect in two points, instead of in one point. The intersection points are "antipodal points", meaning that they are on opposite sides of the sphere. (The north pole and the south pole are antipodal points, for example.)

2. There is a great circle through any two points on the sphere, but if the two points are antipodal points, there is more than one great circle. (Again, think of the north pole and the south pole.)

The solution is to change our definition of points. If we declare that, from now on, any *pair of antipodal points* is to be considered as a single "point", then it is true that any two great circles intersect in a unique "point", and that there is a unique great circle through any two "points".

In this spherical geometry, the parallel postulate is completely false: there are no parallel lines whatsoever, because any two great circles intersect each other. (Note that if we simply "move up" from the equator by a certain distance, the resulting circle is no longer a great circle!)

Another surprising fact is that there are spherical triangles whose angle sum is greater than 180°. (The sides of a spherical triangle have to lie on great circles, of course, because lines are great circles.) In the drawing below,



for example, the three great circles all intersect each other at a 90° angle, and so the angle sum in the red triangle is

$$90^\circ + 90^\circ + 90^\circ = 270^\circ.$$

These two phenomena are related, because we said earlier that the parallel postulate is equivalent to the fact that the angle sum in a triangle is 180°.

4 The area of a spherical triangle

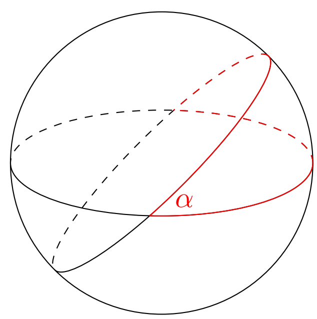
Interestingly, the angle sum in *any* spherical triangle is greater than 180°, and in fact, there is a formula that relates the angle sum to the area of the triangle. From now on, we measure angles in radians, so $180° = \pi$. We also restrict our attention to the unit sphere, meaning to a sphere of radius 1. Then the result is the following.

Theorem. *The area of a spherical triangle is equal to*

$$\alpha + \beta + \gamma - \pi,$$

where α, β, and γ are the measures of the interior angles.

The proof is not difficult at all, so let me explain how it works. First, let us consider the problem of computing the area of the crescent-shaped region, which is drawn in red in the figure below.

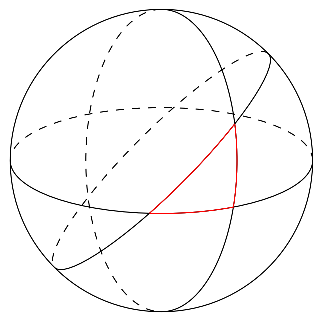


Since the angle measure is α, and since $360° = 2\pi$, the red figure clearly takes up a proportion of $\alpha/2\pi$ of the entire surface of the sphere. Now the surface area of the unit sphere is 4π, and so we get

$$\frac{\alpha}{2\pi} \cdot 4\pi = 2\alpha$$

for the area of the red figure. (Consistency check: If $\alpha = \pi$, the area is 2π, which is half of the area of the sphere, as it should be.)

Now let us consider a spherical triangle with interior angle measures α, β, and γ. Let us denote the area of this triangle by the letter A. If you look carefully at the drawing below, you will notice that the three great circles divide the sphere into six crescent-shaped regions – with opening angles measuring α, β, and γ, respectively – that overlap with each other in the red triangle, and in its antipodal image on the back of the sphere.



All six regions together cover the entire sphere, but the triangle and its antipodal image (on the back of the sphere) are both covered exactly three times. The total area of the six crescent-shaped regions is

$$2 \cdot 2\alpha + 2 \cdot 2\beta + 2 \cdot 2\gamma = 4(\alpha + \beta + \gamma).$$

Since the triangle (and its antipodal image) are each covered three times, this total area has to equal the area of the sphere, plus four times the area of the triangle. In other words,

$$4(\alpha + \beta + \gamma) = 4\pi + 4A,$$

from which we obtain the claimed formula for the area.

Note. What is interesting about this result is that it is totally unlike plane geometry: since we can rescale a triangle by any factor we like, there is no relation whatsoever between the area of a triangle and the measures of its interior angles. Ultimately, the reason is that the sphere is curved – whereas the plane is flat – which makes triangles at different scales behave differently.