Kähler-Ricci shrinkers and Fano fibrations

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Fort Collins, July 2025

This talk is based on joint work with Junsheng Zhang (NYU).

A general theme in Kähler geometry is to explore the connections between differential geometry and algebraic geometry.

A basic fact:

X: compact complex manifold
L: holomorphic line bundle admitting a hermitian metric with positive curvature

$\rightsquigarrow$ X is a projective algebraic subvariety of some $\mathbb{P}^N$.

(What about non-compact X?)

Deeper results:

$$\text{solvability of geometric PDE} \iff \text{algebraic stability.}$$

Philosophically, this is motivated by the Kempf-Ness theorem:

(M, L) projective, Kähler metric $\omega \in c_1(L)$, G compact, acting with a moment map

$$\mu : M \to Lie(G)^*$$

$\rightsquigarrow$

$$\mu^{-1}(0)/G = M^{ss}/G^{\mathbb{C}}.$$

Two examples:

Theorem (Donaldson-Uhlenbeck-Yau)

A holomorphic vector bundle over a compact Kähler manifold admits a unique Hermitian-Yang-Mill connection if and only if it is slope-polystable.

Here "G" is the gauge transformation group of a unitary connection.

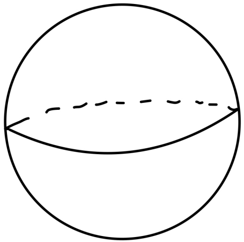
Theorem (Chen-Donaldson-S.)

A Fano manifold admits a unique Kähler-Einstein metric if and only if it is K-polystable.

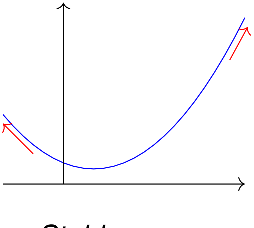
Here "G" is the Hamiltonian diffeomorphism group.

These results have many new proofs, variants, generalizations ...

Kähler-Einstein metric: *Ric*$(\omega) = \omega$.

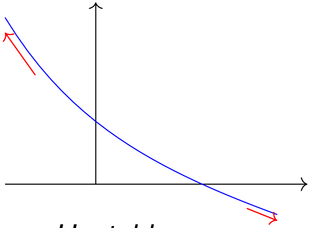


The notion of K-stability involves test configurations and Futaki invariant.



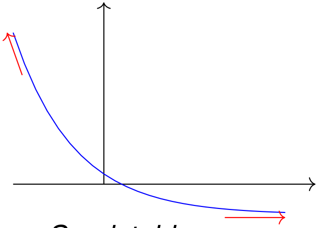
Stable

Critical point exists



Unstable

No critical points



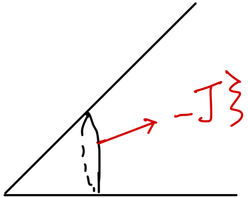
Semistable

No critical points

Local version of the CDS theorem:

Ricci-flat Kähler cones $\longleftrightarrow$ K-polystable Fano cones (Collins-Szekelyhidi, Li)

$Ric(\omega) = 0$

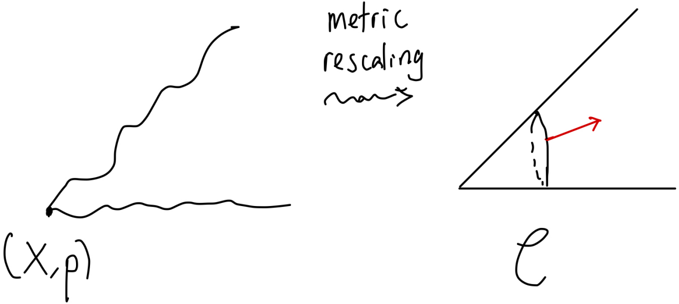


Martelli-Sparks-Yau: volume minimization principle $\rightsquigarrow$ volume is an algebraic number.

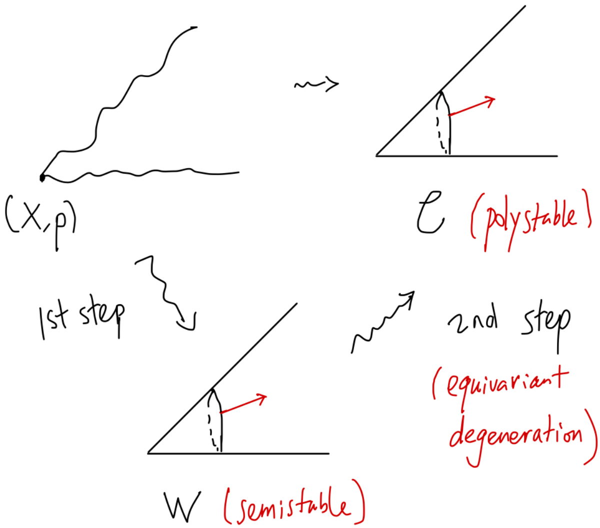
From singularities to cones:

2-step degeneration theory (Donaldson-S. 2015).

Geometric picture: (X, p) a local (non-collapsed polarized) Kähler-Einstein singularity



Algebraic picture:



Donaldson-S. 2015 used this 2-step degeneration theory to prove the uniquenss of the metric tangent cone $\mathcal{C}$.

It was further conjectured W and $\mathcal{C}$ are algebraic invariants of singularities, which leads to a *stability notion* of local singularities.

The 1st step is a valuation canonically defined by the metric. Li 2015 gave an interpretation of this as a generalized volume minimization, and reformulated the conjecture in algebro-geometric terms.

The algebraic theory has been studied and further extended by Li, Liu, Blum, Xu, Zhuang and others.

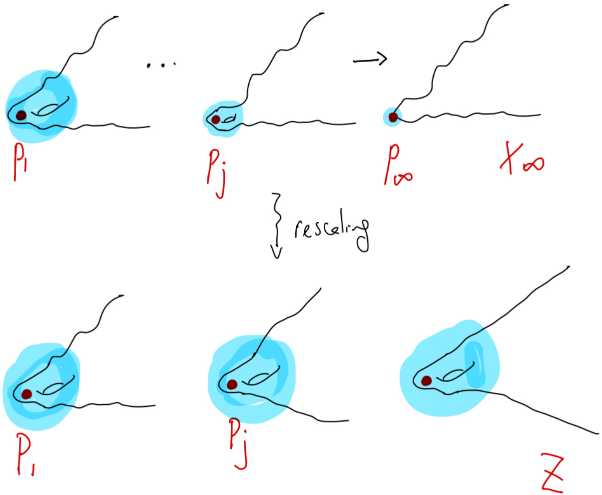
Bubble limits and degeneration of singularities:

Suppose a sequence of polarized Kähler-Einstein metrics (X_i, ω_i, p_i) converge to a non-collapsed limit (X_∞, p_∞).

One can rescale the whole sequence by $\lambda_i \to \infty$ and obtain non-compact Ricci-flat Kähler metrics as bubbles.

They are affine varieties and have asymptotic cones at infinity.

Bubbling:



One can iterate the procedure and obtain a bubble tree.

S. 2023 showed that the bubble tree terminates in finitely many steps.

Question: Give an algebro-geometric description of the bubble tree as an invariant of a degeneration of KLT singularities $\pi : X \to \Delta$ with a section $\sigma : (\Delta, 0) \to (X, p)$.

It is also an interesting question to understand the algebro-geometric meaning of (complete) non-compact Ricci-flat Kähler metrics.

Under a curvature decaying assumption, S.-Zhang discovered a new no semistability phenomenon, i.e., degeneration to the asymptotic cone only requires 1 step.

Ricci flow on Kähler manifolds (Tsuji, Song-Tian):

(X, L) projective, $\omega \in 2\pi c_1(L)$

$$\begin{cases} \frac{\partial}{\partial t}\omega_t = -Ric(\omega_t) \\ \omega_0 = \omega \end{cases}$$

Cohomologically:

$$[\omega_t] = 2\pi(c_1(L) + t \cdot c_1(K_X))$$

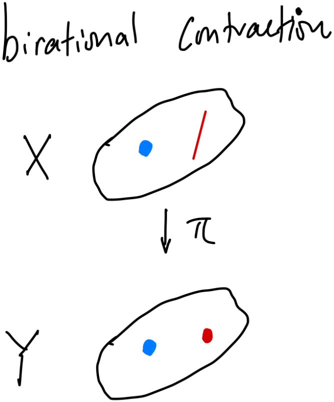
Maximal existence time

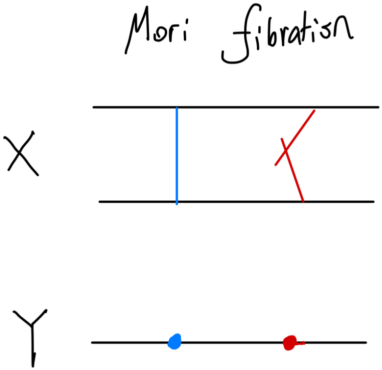
$$T_{max} = \sup\{t > 0 \,|L + t \cdot K_X > 0\}$$

$T_{max} = \infty \iff K_X$ is nef

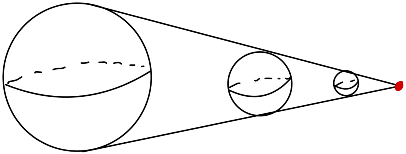
Suppose $T_{max} < \infty$, then

- Algebraically: we have a Fano fibration $\pi : X \to Y$, which is either a birational contraction or a Mori fibration.





- Geometrically, we encounter a singularity of the Ricci flow, and the mechanism is *expected* to be given by a metric contraction.



Spacetime $X \times [0, T_{max})$

Suitable parabolic rescaling leads to self-similar Ricci-flow limits, which are generated by shrinking gradient Ricci solitons, possibly noncompact and with singularities (Perelman, Bamler)

A gradient Ricci soliton is a complete Riemannian metric g satisfying the equation

$$Ric + \nabla^2 f = \lambda g$$

Shrinking: $\lambda > 0$;

Steady: $\lambda = 0$;

Expanding: $\lambda < 0$.

If g is Kähler, then the equation decouples into two equations

- (Complex geometry)

$$\nabla_g^{0,2} f = 0.$$

  Equivalently, $\xi = J\nabla f$ is a holomorphic Killing vector field

- (Elliptic PDE)

$$Ric + \sqrt{-1}\partial\bar{\partial} f = \lambda\omega$$

  $\Rightarrow c_1(X)$ has a definite sign

If X is compact, then

- $\xi = 0 \rightsquigarrow$ Kähler-Einstein metrics ($\lambda = 0 \rightsquigarrow$ Calabi-Yau metrics)
- $\xi \neq 0 \Rightarrow \lambda > 0$, X is Fano (in particular, projective)

In both cases these are *canonical* geometric structures, for which we have extensively studied connections to algebraic geometry.

If X is non-compact, then the situation is much more subtle:

- The sign of λ is not well-defined. For example, the standard metric on $\mathbb{C}^n$ can be viewed as a Ricci soliton for any $\lambda \in \mathbb{R}$.
- $\lambda = 0$, $\xi = 0 \rightsquigarrow$ complete Calabi-Yau metrics. The topology can be infinite even when $\dim_{\mathbb{C}} X = 2$. There is only a conjectural connection to algebraic geometry assuming Euclidean volume growth (S.-Zhang 2022).
- Uniqueness fails severely when $\lambda \leqslant 0$.
- Analysis on non-compact manifolds is in general not well-posed, unless some geometric information is prescribed at infinity.

In this talk we focus on the case $\lambda > 0$. Can normalize so that $\lambda = 1$.

$$Ric + \sqrt{-1}\partial\bar{\partial}f = \omega$$

Such a ω is called a Kähler-Ricci shrinker.

- when $\xi = 0$, $Ric = \omega \rightsquigarrow$ Kähler-Einstein metric on a Fano manifold
- when $Ric = 0$, $g = \nabla^2 f \rightsquigarrow$ Ricci-flat Kähler cone metric

A Kähler cone is naturally an affine variety (Van Coevering 2011).

We prove a foundational result connecting Kähler-Ricci shrinkers and algebraic geometry.

Theorem (S.-Zhang 2024)

A Kähler-Ricci shrinker is naturally a quasi-projective variety.

In particular, X is of finite topological type.

Compare: Yau's compactification conjecture for complete Calabi-Yau metrics, which is only known to be true under extra assumptions.

In dimension 2, Kähler-Ricci shrinkers are classified only very recently (combination of works of Conlon-Deruelle-S. 2019, Cifarelli-Conlon-Deruelle 2022, Bamler-Cifarelli-Conlon-Deruelle 2022, Y.Li-B.Wang 2025):

Compact: del Pezzo surfaces

Non-compact:

- $\mathbb{C}^2$ with the flat metric, $\xi = (1, 1)$
- $\mathbb{C} \times \mathbb{CP}^1$ with product metric, $\xi = (1, 0)$
- $Bl_p\mathbb{C}^2$: Feldman-Ilmanen-Knopf 2003, $\xi = (\sqrt{2}, \sqrt{2})$
- $Bl_p(\mathbb{C} \times \mathbb{CP}^1)$: Bamler-Cifarelli-Conlon-Deruelle 2022, $\xi = (2, 1)$

To explain the content of the main result we need to recall and introduce several terminologies.

A Fano fibration is a surjective projective morphism $\pi : X \to Y$ between normal varieties such that $\pi_* \mathcal{O}_X = \mathcal{O}_Y$, X has klt singularities, and $-K_X$ is a relatively ample $\mathbb{Q}$-Carier divisor.

Contractions of K_X-negative extremal faces are natural Fano fibrations.

A Fano fibration germ consists of the data $(\pi : X \to Y, p)$, where $\pi : X \to Y$ is a Fano fibration and p is a point in Y.

Special cases:

- $Y = \{p\}$: X is a Fano variety
- $\pi = \mathrm{Id}$: (X, p) is a klt singularity

Definition (Collins-Szekelyhidi 2012)

A polarized affine cone (Y, ξ) consists of

- a normal affine variety $Y = \text{Spec}(R)$
- a compact torus $\mathbb{T}$-action with a unique fixed point O
- a Reeb vector field $\xi \in Lie(\mathbb{T})$, i.e. ξ acts with positive weights on R.

This is the algebraic set-up for studying Kähler cone metrics in Sasaki geometry.

When Y has KLT singularities (Y, ξ) is called a Fano cone

Ex: $Y = \mathbb{C}^n$, $\xi = \text{Im}(\sum_\alpha a_\alpha z_\alpha \partial_{z_\alpha})$ for $a_\alpha > 0$, $\alpha = 1, \cdots, n$.

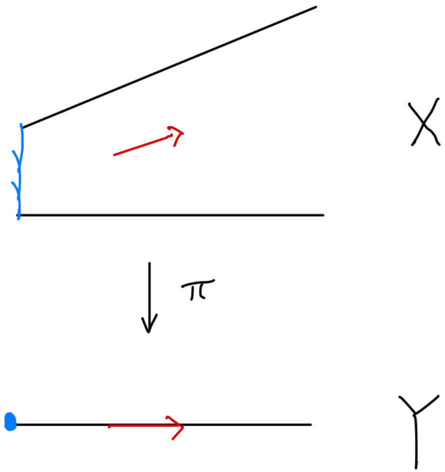
Definition (S.-Zhang 2024)

A polarized Fano fibration $(\pi : X \to Y, \xi)$ consists of

- a Fano fibration $\pi : X \to Y$
- an equivariant torus $\mathbb{T}$-action on X and Y
- an element $\xi \in \mathit{Lie}(\mathbb{T})$, such that (Y, ξ) is a polarized affine cone.

Special cases:

- $Y = \{O\}$: (X, ξ) is a polarized Fano variety
- $\pi = \mathrm{Id}$: (X, ξ) is a Fano cone


X
π
Y

Theorem (S-Zhang, 2024)

A Kähler-Ricci shrinker (X, J, ξ, ω) *defines a natural polarized Fano fibration* $(\pi : X \to Y, \xi)$*.*

Corollary (Esparza, S.-Zhang)

A Kähler-Ricci shrinker is simply-connected.

Traditional method goes by constructing holomorphic functions using analysis, which requires extra assumptions on the geometry at infinity.

Our proof instead uses symplectic geometry and birational algebraic geometry.

One can define the notion of K-stability for a polarized Fano fibration.

Conjecture (S.-Zhang 2024)

A polarized Fano fibration $(\pi : X \to Y, \xi)$ *admits a Kähler-Ricci shrinker, which is unique up to the action of* $\mathrm{Aut}(X, \xi)$, *if and only if it is K-polystable.*

This unifies the YTD type conjectures for Kähler-Einstein metrics, Ricci-flat Kähler cone metrics and compact Kähler-Ricci shrinkers. The latter have been extensively studied previously.

Even the uniqueness question is quite non-trivial, since we do not impose specific asymptotics at infinity.

Theorem (Cifarelli 2020)

A smooth toric polarized Fano fibration admits at most one toric Kähler-Ricci shrinker.

Theorem (Esparza 2025)

A smooth polarized Fano fibration admits at most one Kähler-Ricci shrinker with quadratic curvature decay.

On a polarized Fano fibration $(\pi : X \to Y, \xi)$, we have a weighted volume function $\mathbb{W}(\eta)$ on the space of Reeb vector fields $\eta \in Lie(\mathbb{T})$.

For X smooth non-compact, under extra conditions, an analytic definition of weighted volume was defined and studied by Conlon-Deruelle-S. 2019, using the Duistermaat-Heckman localization formula in symplectic geometry.

$$\mathbb{W}(\eta) = \int_X e^{-\langle \mu, \eta \rangle} \omega^n.$$

- $\pi = \mathrm{Id}$, X is a Fano cone. $\mathbb{W}$ reduces to the volume of Sasaki manifolds studied by Martelli-Sparks-Yau.
- $Y = \{O\}$, (X, ξ) is a polarized Fano variety. $\mathbb{W}$ reduces to the weighted volume defined by Tian-Zhu.
- For a Kähler-Ricci shrinker, $\mathbb{W}(\xi)$ reduces to Perelman's μ-entropy.

Algebraically, for a Fano fibration germ $(\pi : X \to Y, p)$, one can define a weighted volume $\mathbb{W}$ for valuations centered on $\pi^{-1}(p)$, using the theory of Okounkov bodies.

The weighted volume $\mathbb{W}(\pi)$ is defined to be the minimum weighted volume among all valuations centered on $\pi^{-1}(p)$.

This unifies and extends the notion of normalized volume for a KLT singularity (Li) and the notion of $\tilde{\beta}$ invariant on a Fano variety (Han-Li).

Singularities of Kähler-Ricci flows and stability:

A finite time singularity $\rightsquigarrow$ Fano fibration $\pi : X \to Y$. Fix $p \in Y$ and $q \in \pi^{-1}(p)$

Conjecture (2-step degeneration for Kähler-Ricci flows)

- *The Kähler-Ricci flow $\omega(t)$ induces a canonical weighted volume minimizing valuation on the Fano fibration germ $(\pi : X \to Y, p)$, which defines a K-semistable polarized Fano fibration $(\bar{\pi} : Z \to W, \xi)$.*

- *The tangent flow of $\omega(t)$ at q is unique, and is given by the Kähler-Ricci shrinker associated to a K-polystable polarized Fano fibration $(\hat{\pi} : \mathcal{S} \to \mathcal{C})$, which is uniquely determined by the K-semistable polarized Fano fibration $(\bar{\pi} : Z \to W, \xi)$.*

There is also a purely algebro-geometric conjecture.

This unifies the 2-step degeneration theory for both the singularities of Kähler-Einstein metrics and the normalized limits of Ricci flow on Fano manifolds.

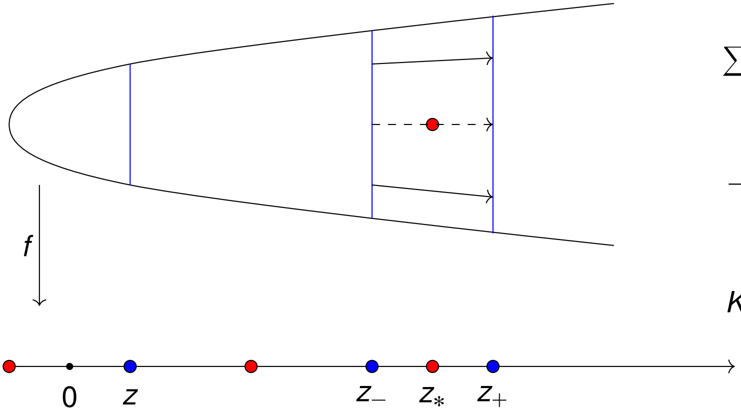
In the latter case, the result is known by Chen-Wang-S. 2015, Han-Li 2020. Algebro-geometric generalizations given by Blum-Liu-Xu-Zhuang 2022.

The weighted volume $\mathbb{W}(\pi)$ of a Fano fibration germ is an algebraic invariant that is worth studying in the future.

Theorem (S-Zhang, 2024)

A Kähler-Ricci shrinker (X, J, ξ, ω) *defines a natural polarized Fano fibration* $(\pi : X \to Y, \xi)$.

The key point of proof is to show the critical set of f is compact. This uses the result of Birkar on the boundedness of Fano type varieties and of Blum-Liu on the semicontinuity of local volume of KLT singularities.



$\sum_\alpha q_\alpha - \sum_\beta p_\beta = z_*.$

$-K_{X_z} + D_z + zL > 0$

$K_{X_z} + D_z + \Delta_z \sim_{\mathbb{Q}} 0$

(X_z, D_z)
weak log Fano

$X_{z_-} \dashrightarrow X_{z_+}$
birational transform