Convergence of formal equivalence of submanifolds

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(i) *If there exists a $\mathcal{C}^\infty$ vector bundle isomorphism of normal bundles*

$$N_{C/X} \overset{\varphi}{\simeq} N_{\widetilde{C}/\widetilde{X}},$$

then there exists a $\mathcal{C}^\infty$-diffeomorphism of suitable neighborhoods

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(ii) *In fact, we can choose Φ such that φ is* **induced by** $\mathrm{d}\Phi : TO \simeq T\widetilde{O}$.

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What if we replace the condition to $TX|_C \overset{\varphi}{\simeq} T\widetilde{X}|_{\widetilde{C}}$? **Still false.**

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- The *ℓ-th infinitesimal neighborhood* $(C/X)_\ell$ is the space C equipped with the structure sheaf $\mathcal{O}_X/\mathcal{I}^{\ell+1}$.
  Example: $(C/X)_1 \ \sim \ TX|_C$.
- The *formal neighborhood* of C in X is the inverse limit $(C/X)_\infty := \lim_{\leftarrow}(C/X)_\ell$.
- For two submanifolds $C \subset X$ and $\widetilde{C} \subset \widetilde{X}$, a *formal isomorphism*

$$\varphi : (C/X)_\infty \to (\widetilde{C}/\widetilde{X})_\infty$$

  means a compatible collection of isomorphisms

$$\{\varphi_\ell : (C/X)_\ell \to (\widetilde{C}/\widetilde{X})_\ell, \ \ell \geq 1\}.$$

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does there exist a biholomorphic diffeomorphism of suitable neighborhoods

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$C \subset X$ satisfies the formal principle, if the answer to the above problem (i) is yes: for any $\widetilde{C} \subset \widetilde{X}$, if there is a **formal isomorphism**

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then there exists a **biholomorphic diffeomorphism** of suitable neighborhoods

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such that $\varphi = \Phi|_{(C/X)_\infty}$.

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- Formal Principle with Convergence $\not\Leftarrow$ Formal Principle

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Example For $0 \in \mathbb{C}$ with coordinate z and $\widetilde{0} \in \widetilde{\mathbb{C}}$ with coordinate $\widetilde{z}$, any formal power series

$$\widetilde{z} = a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$

with $a_1 \neq 0$ defines a formal isomorphism

$$(0/\mathbb{C})_\infty \overset{\varphi}{\simeq} (\widetilde{0}/\widetilde{\mathbb{C}})_\infty.$$

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But φ does not necessarily converge.

$\Rightarrow$ $0 \in \mathbb{C}$ **satisfies the Formal Principle**, but violates the Formal Principle with Convergence.

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  What are examples satisfying the Formal Principle with Convergence?

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- C is unbendable if $k_1, \ldots, k_{n-1} = 0$ or 1 ($\Rightarrow$ no deformations fixing two points of C).

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Question: Can we strengthen [H.2019] to **Formal Principle with Convergence**?

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$N_{C/X} \geq 0$, but any deformation of C in X violates the Formal Principle with Convergence because so does $0 \in \mathbb{C}$.

$\Rightarrow$ A fibered neighborhood is likely to be an obstruction to the Formal Principle with Convergence.

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Let X be a complex manifold and let $\mathcal{K}$ be an irreducible component of the space of smooth rational curves $C \subset X$ with $N_{C/X} \geq 0$.

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Then a general member of $\mathcal{K}$ satisfies the **Formal Principle with Convergence**.

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Remark A general line on a hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $= n$ is unbendable, but has fibered neighborhood. It satisfies the Formal principle (by [H.2019]), but violates the Formal Principle with Convergence.

How to prove the convergence of a formal isomorphism?

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Lemma (Kobayashi-Nomizu (1963) vol 1)

Let $y \in Y$ and $\widetilde{y} \in \widetilde{Y}$ be points on complex manifolds and let

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Question: Where are affine connections in the setting of Theorem 1??

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such that

- $\alpha : \mathrm{Univ}_{\mathcal{K}} \to \mathcal{K}$ is a $\mathbb{P}^1$-bundle; and
- each member $C \subset X$ of $\mathcal{K}$ and the corresponding point $[C] \in \mathcal{K}$ satisfies
$$C = \beta(\alpha^{-1}([C])).$$

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Theorem 2 is a generalization of this example!

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(2) A formal isomorphism $(C/X)_\infty \overset{\varphi}{\simeq} (\widetilde{C}/\widetilde{X})_\infty$ can be lifted to a formal isomorphism $(w/\mathcal{W})_\infty \overset{\varphi^\flat}{\simeq} (\widetilde{w}/\widetilde{\mathcal{W}})_\infty$ for any point $w \in C^\flat \cap \mathcal{W}$, by the **functoriality of Douady space (= Hilbert scheme).**

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(4) By Kobayashi-Nomizu Lemma, $\varphi^\sharp$ converges. Hence, so does $\varphi^\flat$. We conclude φ converges at a general point of C. Then it converges at all points of C by maximum principle.

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Example If $D \subset TM$ is a contact distribution, $\mathrm{symb}_x D$ is isomorphic to the Heisenberg algebra for all $x \in M$.

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Thank you very much !!