On the algebraicity of compact Kähler varieties

I <u>Algebraicity</u>

Algebraic dimension

X : (connected) compact complex (analytic) variety

$\mathcal{M}(X) :=$ field of meromorphic functions on X

Algebraic dimension $a(X) := \operatorname{trdeg}_{\mathbb{C}} \mathcal{M}(X) \in [0, \dim X]$

Siegel, Thimm, Remmert

- $X \overset{f}{\dashrightarrow} Y \Rightarrow \mathcal{M}(Y) \xhookrightarrow{f^*} \mathcal{M}(X) \Rightarrow a(Y) \leq a(X)$
- $X \underset{\text{finite}}{\overset{\text{gen.}}{\dashrightarrow}} Y \Rightarrow \mathcal{M}(Y) \xhookrightarrow{\text{finite}} \mathcal{M}(X) \Rightarrow a(Y) = a(X)$

Algebraic varieties X : compact complex variety

We call X algebraic or Moishezon if

$$a(X) = \dim X$$

- X Moishezon $\Leftrightarrow$ $X \dashleftarrow\dashrightarrow$ projective variety
- $X \dashrightarrow\!\!\!\!\rightarrow Y$ Then X Moishezon $\Rightarrow$ Y Moishezon
- (Moishezon) Moishezon + Kähler $\Leftrightarrow$ projective
- (Artin) $\{$proper algebraic spaces $/\mathbb{C}\}$ $\xrightarrow[\text{on}]{\sim}$ $\{$Moishezon spaces$\}$

Campana's algebraicity criterion

Thm (Campana) X : compact complex variety. Equivalent:

- X Moishezon
- X algebraically connected and $\leftarrow - - \rightarrow$ compact Kähler

X is called algebraically connected if $\forall\, x, y \in X$ general, $\exists$ compact connected curve $C \subset X$ s.t. $x, y \in C$

"X Moishezon $\Longleftrightarrow$ X has many curves/subvarieties"

Sketch of proof $x \in X$ general point

X alg. conn. $\rightsquigarrow$
$$\begin{tikzcd} Z \arrow[r, two heads, "q"] \arrow[d, "p"', "\text{univ.}"] & X \\ S & \end{tikzcd}$$
S component of $\{\text{curves} \subset X \text{ through } x\}$

For $m \gg 0$,
$$\begin{tikzcd} Z \arrow[r, hook, dashed, "\text{Campana}"] \arrow[drr, two heads, "q"'] & \mathbb{P}(m\text{-jet bundle of } X) \arrow[dr] & \\ & & X \ni x \end{tikzcd}$$

$\Rightarrow$ $q^{-1}(x)$ algebraic

$$\begin{tikzcd} D := q^{-1}(x) \subset Z \arrow[r, hook, dashed, "N \gg 0"] \arrow[d, "p"] & \mathbb{P}(p_* \mathcal{O}_Z(ND)) \arrow[dl] \\ S & \end{tikzcd}$$
(divisor)

S alg $\Rightarrow$ $\mathbb{P}(\dots)$ alg

$\Rightarrow$ Z alg.

II Algebraicity & positivity

Kodaira embedding theorem

X : compact complex manifold

Thm (Kodaira) If $\exists$ Kähler form ω on X s.t.

$$[\omega] \in H^2(X, \mathbb{Q}),$$

then $X \hookrightarrow \mathbb{P}^N$

$[\omega] \in H^2(X,\mathbb{Q})$ Kähler $\underset{(1,1)}{\overset{\text{Lefschetz}}{\rightsquigarrow}}$ positive line bundle L

$\underset{\text{Vanishing}}{\overset{\text{Kodaira}}{\rightsquigarrow}}$ $L^{\otimes m}$ very ample, i.e. $\exists$ sufficiently many sections yielding

$$X \hookrightarrow \mathbb{P}^N$$

$\exists$ positive 1-dim rational class $\overset{?}{\Longrightarrow}$ projective

X compact Kähler manifold

$$\mathcal{K}(X) := \{\text{Kähler classes}\} \subset H^{1,1}(X,\mathbb{R}) := H^{1,1}(X) \cap H^2(X,\mathbb{R})$$

$$\mathcal{K}(X)^\vee \subset H^{d-1,d-1}(X,\mathbb{R}) \qquad (d := \dim X)$$

w.r.t. $H^{1,1}(X,\mathbb{R}) \otimes H^{d-1,d-1}(X,\mathbb{R}) \xrightarrow{\cup} H^{d,d}(X,\mathbb{R}) \simeq \mathbb{R}$

Problem (Oguiso-Peternell) Suppose

$$\mathrm{Int}(\mathcal{K}(X)^\vee) \cap H^{2d-2}(X,\mathbb{Q}) \neq \emptyset \qquad (\mathcal{K}^\vee)$$

How algebraic is X ?

Remark $\omega \in \mathcal{K}(X) \Rightarrow \omega^{d-1} \in \mathrm{Int}(\mathcal{K}(X)^\vee)$ So projectivity $\Rightarrow (\mathcal{K}^\vee)$

Thm (Huybrechts, Oguiso-Peternell) X smooth compact Kähler surface

Then $(K^\vee)$ for X $\Leftrightarrow$ X projective

Pf • If $\exists$ line bundle L on X s.t. $c_1(L)^2 > 0$,

then X is projective

(R-R $\Rightarrow$ $|N \cdot L|$ or $|K_X - NL|$ has dim ≥ 1 for $N \gg 1$

$(NL)^2, (K_X - NL)^2 > 0$ $\Rightarrow$ X algebraically connected)

Thm (Huybrechts, Oguiso-Peternell) X smooth compact Kähler surface

Then $(K^\vee)$ for X $\Leftrightarrow$ X projective

Pf

- If $\exists$ line bundle L on X s.t. $c_1(L)^2 > 0$, then X is projective
- Suppose X non-proj. $\Longrightarrow$

$V := \mathbb{R} \cdot H^{1,1}(X, \mathbb{Q}) \subset H^{1,1}(X, \mathbb{R})$. $\quad \alpha^2 \leq 0 \quad \forall \alpha \in V$

$\operatorname{sgn}\left(H^{1,1}(X, \mathbb{R})\right) = (1, h^{1,1} - 1) \Rightarrow \exists\, w \in H^{1,1}(X, \mathbb{R}) \setminus 0$ s.t.

$w \in V^\perp$ and $w^2 \geq 0$

$\xRightarrow{\text{Demailly-Paun}}$ w or $-w \in \overline{K(X)} \setminus 0$ $\xRightarrow{w \in V^\perp}$ $\operatorname{Int}(K(X)^\vee) \cap H^{1,1}(X, \mathbb{Q}) = \emptyset$

X : compact Kähler manifold

$\mathrm{Psef}(X) := \{ [\text{closed positive } (1,1)\text{-current}] \} \subset H^{1,1}(X, \mathbb{R})$

Problem Suppose

$\mathrm{Inf}(\mathrm{Psef}(X)^\vee) \cap H^{2d-2}(X, \mathbb{Q}) \neq \emptyset \qquad (P^\vee)$

Is X algebraic ?

$\mathcal{K}(X) \subset \mathrm{Psef}(X) \Rightarrow \mathrm{Psef}(X)^\vee \subset \mathcal{K}(X)^\vee$

Thus X projective $\Rightarrow (P^\vee) \Rightarrow (K^\vee)$.

Both "$\Leftarrow$" still unknown

Hodge classes in $\mathrm{Psef}(X)^\vee$ and movable curves

X : compact Kähler manifold

Question $\forall \gamma \in \mathrm{Int}(\mathrm{Psef}(X)^\vee) \cap H^{2d-2}(X, \mathbb{Q})$

$\exists$? very movable curve $C \subset X$ s.t. for some $N \gg 0$

$[C] = N \cdot \gamma$

$\forall x, y \in X$ connected by deformations of C

- YES if X projective (Boucksom - Demailly - Păun - Peternell)
- By Campana's criterion, YES $\Longleftrightarrow$ $\left((\mathbb{P}') \Rightarrow \text{projective} \right)$

2 No Hodge conjecture for curves when X non-projective

Example E, F elliptic curves, $X_0 = E \times F$

X : very general small deformation of X_0 s.t.

$$\alpha := [E \times pt] - [pt \times F] \in H^2(X_0, \mathbb{Q})$$

remains Hodge

Then $H^{1,1}(X, \mathbb{Q}) = \mathbb{Q} \cdot [\alpha]$, and $\alpha^2 = -2 < 0 \Rightarrow X$ has no curves

Question X compact Kähler manifold of dim. d

Let $\alpha \in H^{d-1,d-1}(X, \mathbb{Q})$ which is positive (e.g. $\in \mathcal{K}(X)^\vee$ or $\mathrm{Psef}(X)^\vee$).

Is $N \cdot \alpha$ algebraic for some $N > 0$?

Known results of $(K^\vee)$ or $(P^\vee)$ $\Rightarrow$?? (L.)

Thm (L.)

(1) $(K^\vee) \Rightarrow \mathrm{Alb}(X)$ is projective

(2) Suppose $K_X \equiv 0$. $(K^\vee) \Rightarrow X$ projective

(3) Suppose $\dim X = 3$. $(K^\vee) \Rightarrow a(X) \geq 2$

(Oguiso-Peternell : $[\text{curve}] \in \mathrm{Int}(\mathcal{K}(X)^\vee) \Rightarrow a(X) \geq 2$)

(4) Suppose $\dim X = 3$. $(P^\vee) \Rightarrow X$ projective

We will focus on (3) and (4)

Non-Kähler interlude

X : compact complex manifold

$$H^{p,p}_{BC}(X) := \frac{\{d\text{-closed } (p,p)\text{-forms}\}}{\operatorname{Im} \partial\bar\partial} \xrightarrow{\gamma} H^{2p}(X, \mathbb{C})$$

$$\operatorname{Psef}(X) := \{[\text{closed real positive } (1,1)\text{-currents}]\} \subseteq H^{1,1}_{BC}(X, \mathbb{R})$$

Thm (Ji - Shiffman)

$$\gamma\big(\operatorname{Int}(\operatorname{Psef}(X))\big) \cap H^2(X, \mathbb{Q}) \neq \emptyset \Rightarrow X \text{ Moishezon}$$

Let $d = \dim X$

$$H_A^{p,p}(X) := \frac{\{\partial\bar\partial\text{-closed } (p,p)\text{-forms}\}}{\operatorname{Im}\partial + \operatorname{Im}\bar\partial} \simeq H_{BC}^{d-p,d-p}(X)^\vee$$

(Lamari)
$$\begin{array}{ccc} H_{BC}^{1,1}(X,\mathbb{R}) & \xleftrightarrow{\text{dual}} & H_A^{d-1,d-1}(X,\mathbb{R}) \\ \cup & & \cup \\ \mathrm{Psef}(X) & & \overline{G(X)} \end{array}$$

where $G(X) := \{[\omega^{d-1}] \mid \omega : \text{Gauduchon metric on } X\}$

i.e. C^∞ real definite positive $(1,1)$-form s.t. $\partial\bar\partial(\omega^{d-1}) = 0$

$$H^{d-1,d-1}_{BC}(X,\mathbb{R}) \xrightarrow{\gamma} H^{2d-2}(X,\mathbb{R}) \xrightarrow{\delta} H^{d-1,d-1}_{A}(X,\mathbb{R})$$

Question (Dual Ji-Shiffman)

$$G(X) \cap \delta\left(H^{2d-2}(X,\mathbb{Q})\right) \neq \phi \overset{?}{\Rightarrow} \text{algebraicity of } X$$

Ex (Iwasawa 3-fold) R : ring

$$\mathrm{Heis}(R) := \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \right\} \leq GL_3(R) \qquad X := \mathrm{Heis}(\mathbb{C}) / \underbrace{\mathrm{Heis}(\mathbb{Z}[i])}_{=:\Gamma}$$

- $\alpha := dz_1$, $\beta := dz_2$, $\gamma := dz_3 - z_1 dz_2$ Γ- invariant

  $\omega := \frac{\sqrt{-1}}{2}(\alpha \wedge \bar{\alpha} + \beta \wedge \bar{\beta} + \gamma \wedge \bar{\gamma})$ satisfies $\omega^2 \in G(X) \cap \delta(H^2(X,\mathbb{Q}))$

- $a(X) = 2$ $\left(X \twoheadrightarrow \mathbb{C}_{z_1}/\mathbb{Z}[i] \times \mathbb{C}_{z_2}/\mathbb{Z}[i] \right)$

III Non-algebraic threefolds

X : compact Kähler 3-fold

$(K^\vee) \Rightarrow a(X) \geq 2$

$(P^\vee) \Rightarrow X$ projective

X compact Kähler 3-fold

Thm (Fujiki + Abundance for Kähler 3-folds CHP-DO-GP)

If $a(X) \leq 1$, then $X \dashleftarrow\dashrightarrow X'$ with X':

1. $\mathbb{P}^1$-fibration
2. (surface × curve) / finite group
3. 3-torus / finite group
4. $X' \xrightarrow{f} B$ fibration, general fiber = 2-torus

   without multi-section

If $a(X) = 2$, then

5. $X \dashleftarrow\dashrightarrow$ elliptic fibration

Recall: $\mathrm{Inf}(\mathcal{K}(X)^\vee) \cap H^{2d-2}(X,\mathbb{Q}) \neq \emptyset$ $(K^\vee)$

$\mathrm{Inf}(\mathrm{Psef}(X)^\vee) \cap H^{2d-2}(X,\mathbb{Q}) \neq \emptyset$ $(P^\vee)$

Both $(K^\vee)$ and $(P^\vee)$ descend:

Lem (L.) X, Y : compact Kähler manifolds s.t. $X \dashrightarrow\!\!\!\!\!\!\rightarrow Y$

$(K^\vee)$ for $X \Rightarrow (K^\vee)$ for Y (dim = 3 : Oguiso-Peternell)

$(P^\vee)$ for $X \Rightarrow (P^\vee)$ for Y

Cor X : compact Kähler manifold.

Suppose $X \dashrightarrow\!\!\!\!\!\!\rightarrow S$ surface. Then

$$\underset{\substack{\Uparrow \\ (P^\vee)}}{(K^\vee)} \text{ for } X \Rightarrow a(X) \geq 2$$

X compact Kähler 3-fold

Thm (Fujiki + Abundance for Kähler 3-folds CHP-DO-GP)

If $a(X) \leq 1$ Then $X \dashrightarrow X'$ with X':

① $\mathbb{P}^1$-fibration

② (surface × curve) / finite group

impossible

Cor $\Leftarrow (K^\vee), (P^\vee)$

③ 3-torus / finite group

④ $X' \xrightarrow{f} B$ fibration, general fiber = 2-torus without multi-section

If $a(X) = 2$, then

⑤ $X \dashrightarrow$ elliptic fibration

IV Torus fibrations

X compact Kähler 3-fold

Thm (Fujiki + Abundance for Kähler 3-folds CHP-DO-GP)

If $a(X) \leq 1$ Then $X \dashleftarrow\dashrightarrow X'$ with X':

① $\mathbb{P}^1$-fibration

② (surface $\times$ curve) / finite group

③ 3-torus / finite group

④ $X' \xrightarrow{f} B$ fibration, general fiber = 2-torus without multi-section

impossible, if we can show that any 2-torus fibration $X \to B$ s.t. X satisfies $(K^\vee)$ has multi-sections

If $a(X) = 2$, then

⑤ $X \dashleftarrow\dashrightarrow$ elliptic fibration

Producing multisections for smooth torus fibrations

$X \xrightarrow{f} B$ smooth torus fibration

with X, B compact Kähler manifolds

Prop $g := \dim X - \dim B$. Equivalent:

① $f_* : H^{g,g}(X,\mathbb{Q}) \twoheadrightarrow H^0(B,\mathbb{Q})$

② f has (étale) multi-sections

Cor Suppose $\dim B = 1$. Then

$(K^\vee)$ for $X \Rightarrow f$ has a multi-section

Deligne cohomology

X complex manifold. Let $g \in \mathbb{Z}_{>0}$

Deligne complex $D_X(g)$:

$$\cdots \to 0 \to \underset{\text{deg } 0}{\mathbb{Z}_X} \xrightarrow{\times(2\pi i)^g} \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{g-1}_X \to 0 \to \cdots$$

$$\rightsquigarrow \qquad \Omega_X^{\bullet \le g-1}[-1] \to D_X(g) \to \mathbb{Z}_X \xrightarrow{+1}$$

Assume X compact Kähler, applying $R\Gamma$ to and truncating yields a short exact sequence

$$0 \to \underbrace{J^{2g-1}(X)}_{\substack{\text{intermediate} \\ \text{Jacobian}}} \to H^{2g}(X, D_X(g)) \to H^{g,g}(X, \mathbb{Z}) \to 0$$

e.g. if X = torus of dim g, $J^{2g-1}(X) \simeq X$

Relative Deligne complex

$X \xrightarrow{f} B$ smooth torus fibration

fiber dim $= g$

$$D_{X/B}(g): \quad \cdots \to 0 \to \underset{\deg 0}{\mathbb{Z}_X} \xrightarrow{\times(2\pi i)^g} \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/B} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{g-1}_{X/B} \to 0 \to \cdots$$

$$\rightsquigarrow \quad \Omega^{\bullet \le g-1}_{X/B}[-1] \to D_{X/B}(g) \to \mathbb{Z}_X \xrightarrow{+1}$$

$\overset{Rf_*}{\rightsquigarrow}$

$R^{2g} f_* \Omega^{\bullet \le g-1}_{X/B} = 0$

$$0 \to \mathcal{J} \to R^{2g} f_* D_{X/B}(g) \to \underset{\underset{\mathbb{Z}_B}{\wr\shortmid}}{R^{2g} f_* \mathbb{Z}_X} \to 0$$

where $\mathcal{J}$ is the sheaf of germs of holomorphic sections of the Jacobian fibration $J \to B$ of f

Deligne cohomology & J-torsors

$f : X \to B$ torus fibration
rel. dim $= g$

$$0 \to J \to R^{2g} f_* D_{X/B}(g) \to \mathbb{Z}_B \to 0$$

$D_X(g) \to D_{X/B}(g)$ yields

$$\begin{tikzcd} & H^{2g}(X, D_X(g)) \arrow[r, two heads] \arrow[d] & H^{g,g}(X, \mathbb{Z}) \arrow[d, "f_*"] & & \\ \rightsquigarrow & H^0(B, R^{2g} f_* D_{X/B}(g)) \arrow[r] & H^0(B, \mathbb{Z}) \arrow[r] & H^1(B, J) & \text{exact} \end{tikzcd}$$

$$1 \mapsto [f]$$ (Nakayama, Claudon)

Supp $f_* : H^{g,g}(X, \mathbb{Q}) \twoheadrightarrow H^0(B, \mathbb{Q})$

Then $m \cdot [f] = 0$ for some $m \in \mathbb{Z}_{>0}$ $\rightsquigarrow$

$$\begin{tikzcd} X \arrow[rr, two heads, "\times m"] \arrow[dr, "f"'] & & J \arrow[dl] \\ & B \arrow[ur, bend right, "\text{section}"'] & \end{tikzcd}$$

$\rightsquigarrow$ multisection of f $\square$

V Elliptic threefolds

Support of Hodge classes

Prop (Hodge theory) X : compact Kähler manifold

$Y \subset X$ algebraic subvariety, $j : \tilde{Y} \xrightarrow{\text{desing.}} Y \subset X$ with $\tilde{Y}$ projective

If $\beta \in \ker\left(H^{k,k}(X, \mathbb{Q}) \to H^{2k}(X \setminus Y, \mathbb{Q})\right)$, then

$$\beta = j_* \tilde{\beta}$$

for some $\tilde{\beta} \in H^{p,p}(\tilde{Y}, \mathbb{Q})$

- Proof based on (mixed) Hodge theory, and

$$H^\bullet(\text{sm. proj.}, \mathbb{Q}) = \bigoplus \text{polarized } \mathbb{Q}\text{-Hodge structures}$$

- Without the projectivity assumption, Proposition is unknown

Question (Property (H)) X : smooth compact Kähler 3-fold

$Y \subset X$ surface, $j : \tilde{Y} \xrightarrow{\text{desing}} Y \subset X$

Let $\beta \in \ker\left(H^{2,2}(X, \mathbb{Q}) \to H^4(X \setminus Y, \mathbb{Q})\right)$, $\exists ?$ $\tilde{\beta} \in H^{1,1}(\tilde{Y}, \mathbb{Q})$ s.t.

$$\beta = j_* \tilde{\beta}$$

Lem (L.) (Yes, if Y is irreducible

Thm (L.) X : compact Kähler 3-fold with an elliptic fibration.

- $(P^\vee)$ for $X \Rightarrow X$ projective (unconditional, based on Lem.)
- Assume (H) for X. Then $(K^\vee)$ for $X \Rightarrow X$ projective

Thm (L.) X : compact Kähler 3-fold with an elliptic fibration.

- $(P^\vee)$ for $X \Rightarrow X$ projective (unconditional, based on Lem.)
- Assume (H) for X. Then $(K^\vee)$ for $X \Rightarrow X$ projective

Sketch of proof $X \xrightarrow{f} S$ elliptic fibration with S smooth

Let $\alpha \in \mathrm{Int}(\mathcal{C}(X)^\vee) \cap H^4(X, \mathbb{Q})$ for $\mathcal{C}(X) = \mathcal{K}(X)$ or $\mathrm{Psef}(X)$

Then $f_*\alpha \in H^{1,1}(S, \mathbb{Q})$ is $\begin{cases} \text{ample} & \text{if } \mathcal{C}(X) = \mathrm{Psef}(X) \\ \text{big} & \text{if } \mathcal{C}(X) = \mathcal{K}(X) \end{cases}$

<u>Thm</u> (L.) X : compact Kähler 3-fold with an elliptic fibration.

- $(P^\vee)$ for $X \Rightarrow X$ projective (unconditional, based on Lem.)
- Assume (H) for X. Then $(K^\vee)$ for $X \Rightarrow X$ projective

<u>Sketch of proof</u> $X \xrightarrow{f} S$ elliptic fibration with S smooth

Let $\alpha \in \mathrm{Inf}(\mathcal{C}(X)^\vee) \cap H^4(X, \mathbb{Q})$ for $\mathcal{C}(X) = \mathcal{K}(X)$ or $\mathrm{Psef}(X)$

Then $$N \cdot f_*\alpha = [C] \in H^{1,1}(S, \mathbb{Q}) \quad \text{for } N \gg 0$$

where $$C = A \cup E$$

A: very ample in S

E: $= 0$ if $\mathcal{C}(X) = \mathrm{Psef}(X)$

For a suitable model of $X \xrightarrow{f} S$, we show

$$\alpha \in \ker\left(H^4(X, \mathbb{Q}) \to H^4(X \setminus f^{-1}(C), \mathbb{Q})\right)$$

Thm (L.) X : compact Kähler 3-fold with an elliptic fibration.

- $(P^\vee)$ for $X \Rightarrow X$ projective (unconditional, based on Lem.)
- Assume (H) for X. Then $(K^\vee)$ for $X \Rightarrow X$ projective

Idea $X \xrightarrow{f} S$ elliptic fibration, $C = A \cup E \subset S$

A: very ample in S; $E = 0$ if $\mathcal{G}(X) = \mathrm{Psef}(X)$

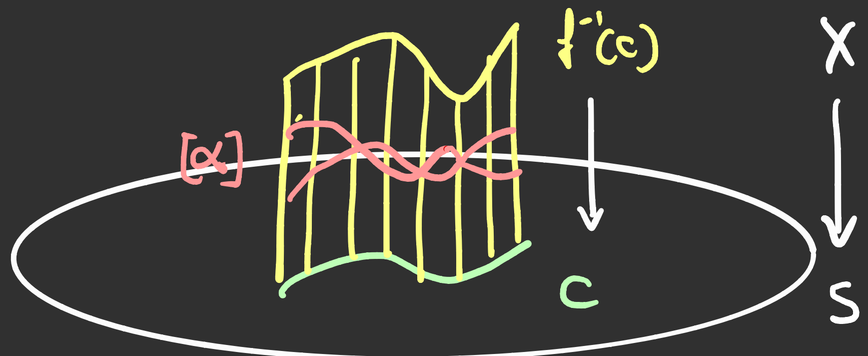
For a suitable model of $X \xrightarrow{f} S$, we show

$$\alpha \in \ker(H^4(X,\mathbb{Q}) \to H^4(X \setminus f^{-1}(C), \mathbb{Q}))$$

C irreducible or (H) $\Rightarrow$ α is supported on $f^{-1}(C)$ as Hodge class

As α dominates C, $f^{-1}(C)$ is algebraic

Finally, we vary the very ample curve A and apply Campana $\square$



Varying $C \rightsquigarrow X$ algebraically connected

$\Longrightarrow$ X algebraic

Campana

Conclusion

- Algebraicity $\Longleftrightarrow$ many algebraic subvarieties (curves)
- $\exists$ positive divisors $\Rightarrow$ algebraicity
- $\exists$ positive higher codimension classes

  $\overset{?}{\nRightarrow}$ positive higher codimension subvarieties $\overset{?}{\nRightarrow}$ algebraicity
- For 3-fold, $\mathrm{Inf}(\mathrm{Psef}(X)^\vee) \cap H^4(X, \mathbb{Q}) \neq \emptyset \Rightarrow X$ proj

  $\mathrm{Inf}(\mathcal{K}(X)^\vee) \cap H^4(X, \mathbb{Q}) \neq \emptyset \Rightarrow a(X) \geq 2$
- If $\exists$ non-alg. 3-fold X with $\mathrm{Inf}(\mathcal{K}(X)^\vee) \cap H^4(X, \mathbb{Q}) \neq 0$, it also yields a negative answer to the 1-cycle support problem

Thank you !