On the geometry of log Calabi-Yau manifolds

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Analytic methods in algebraic geometry

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Yau's theorem and applications

X compact Kähler manifold s.t. $c_1(X) = 0 \in H^2(X, \mathbb{R})$.

Theorem (Yau '78)

Given a Kähler class $\alpha \in H^2(X, \mathbb{R})$, $\exists!\, \omega \in \alpha$ such that $\mathrm{Ric}\, \omega = 0$.

Some applications

1. $\mathrm{Alb}_X : X \to \mathrm{Alb}(X)$ is a holomorphic submersion, étale trivial.
2. T_X is polystable wrt any Kähler class.
3. Any tensor $\sigma \in H^0(X, T_X^{\otimes p} \otimes \Omega_X^{\otimes q})$ is parallel wrt ω (i.e. $\nabla \sigma = 0$).
4. $\widetilde{X} \simeq \mathbb{C}^r \times Y$ with Y compact ($\Rightarrow \widetilde{X}$ can be compactified)

Question

Vague question

What are larger classes of Calabi-Yau varieties where all or some of these properties still hold ?

Log terminal CY varieties

If X is compact Kähler with log terminal singularities s.t. $c_1(X) = 0$, then :

1. X_{reg} admits an incomplete Ricci-flat Kähler metric (EGZ'06)
2. T_X is polystable with respect to any Kähler class (GKP'11, G'14)
3. $\mathrm{Alb}_X : X \to \mathrm{Alb}(X)$ is étale trivial* (K'81, CGGN'20)
4. Bochner principle holds for holomorphic tensors on X_{reg} (GGK'17, CGGN'20)

Log canonical varieties

If X is compact Kähler with log canonical singularities s.t. $c_1(X) = 0$ then

$$T_X \text{ is still semistable (G'14)}$$

but there are examples (Bernasconi, Filipazzi, Tsakanikas, Patakfalvi + Müller '25) s.t

1. T_X is not polystable with respect to any Kähler class.
2. $\mathrm{Alb}_X : X \to \mathrm{Alb}(X)$ is not isotrivial (not even birationally).

Log Calabi-Yau manifolds

Definition of a log CY manifold

A log smooth pair (X, D) with X compact Kähler of dimension n, $D = \sum_{i=1}^{k} D_i$ snc reduced such that $K_X + D \sim \mathcal{O}_X$.

Questions

1. Existence a complete CY metric on $X \setminus D$?
2. Validity of Bochner principle ? What is the holonomy ?
3. Is $\mathrm{Alb}_{(X,D)}$ locally trivial ?
4. Structure of $\widetilde{X \setminus D}$?

Today's goal

Focus on two examples of log CY pairs :

1. X Fano, D smooth
2. X Fano, $D = D_1 + D_2$ with D_i mutually proportional.

Upshot

In both cases, $X \backslash D$ admits a complete CY metric but the geometry of (X, D) is vastly different in each case.

D smooth, first observations

If X is Fano and $D \in |-K_X|$ is smooth, then

- $H^0(X, \Omega^1_X(\log D)) = 0$ ($\Leftarrow$ residue theorem)
- $\pi_1(X \backslash D)$ finite ($\Leftarrow$ Nori theorem)

$\rightsquigarrow$ $\mathrm{Alb}_{(X,D)}$ is trivial, $\widetilde{X \backslash D}$ is quasi-projective.

Next : Existence of CY metric, holonomy, Bochner principle, stability of $T_X(-\log D)$?

The Tian-Yau metric

Assume X Fano and $D \in |-K_X|$ smooth.

Theorem (Tian-Yau '90)

There exists a complete Ricci flat Kähler metric ω_{TY} on $X \backslash D$.

Asymptotics. Near $D = (z_1 = 0)$, we have

$$\omega_{\mathrm{TY}} \simeq \frac{i dz_1 \wedge d\bar{z}_1}{|z_1|^2(-\log|z_1|)^{1-\frac{1}{n}}} + (-\log|z_1|)^{\frac{1}{n}} \omega_D$$

Volume growth : $\mathrm{vol}(B(R)) \simeq R^{\frac{2n}{n+1}}$ $(\ll R^2)$ as $R \to +\infty$.
Asymptotic tangent cone : $(\mathbb{R}, dt^2)$

Holonomy

1. Asymptotics $\Rightarrow \nexists$ non-zero parallel p-form on $X \backslash D$ for $p \neq 0, n$.
[alternatively use vanishing $H^0(X, \Omega_X^p(\log D)) = 0$]
2. $\pi_1(X \backslash D)$ finite + Berger-Simons classification $\Rightarrow$

$$\mathrm{Hol}_x(X \backslash D, \omega_{\mathrm{TY}}) \simeq \mathrm{SU}(n)$$

for any fixed point $x \in X \backslash D$.

Bochner principle and stability

X Fano, $D \in |-K_X|$ smooth, ω_{TY} complete CY metric with holonomy $G := \mathrm{SU}(n)$. Pick $p, q \in \mathbb{N}$ and set $E := T_X(-\log D)^{\otimes p} \otimes \Omega_X(\log D)^{\otimes q}$, $V := (\mathbb{C}^n)^{\otimes p} \otimes ((\mathbb{C}^n)^*)^{\otimes q}$.

Theorem (Collins-G. '25)

1. [Bochner] There is a 1 :1 correspondence

$$H^0(X, E) \xrightarrow[\simeq]{\mathrm{ev}_x} V^G$$

$$H^0(X, E) \xrightarrow{\mathrm{rest}_{X \setminus D}} \{\sigma \in \mathcal{C}^\infty(X \setminus D, E); \nabla \sigma = 0\} \xrightarrow{\mathrm{ev}_x} V^G$$

2. [Stability] $T_X(-\log D)$ is stable wrt $-K_X$.

Elements of proof

1. Take $\sigma \in H^0(X, E)$, then $\Delta|\sigma|^2 = |\nabla\sigma|^2$. Then $f := \log|\sigma|^2$ satisfies

$$\Delta f \geqslant 0, \quad \text{and} \quad f = O(\log R).$$

Choose χ_R cutoff function supported on $B(R)$; $\Delta\chi_R = O(R^{-2})$.

$$\begin{aligned} 0 \leqslant \int_X \chi_R \Delta f &= \int_X f \Delta\chi_R \\ &= O\Big(\mathrm{vol}(B(R)) \cdot \log R \cdot R^{-2}\Big) \to 0 \end{aligned}$$

$\rightsquigarrow \Delta f \equiv 0 \overset{Yau}{\Longrightarrow} f \equiv ct$ (as f grows sublinearly) $\Rightarrow |\sigma|^2 \equiv ct$
$\Longrightarrow 0 = \Delta|\sigma|^2 = |\nabla\sigma|^2$.

Elements of proof II

2. More involved (cf related work by Junsheng Zhang).

$T_X(-\log D)$ semistable $\Rightarrow$ only need to show $\nexists F \subset T_X(-\log D)$ proper s.t. $c_1(F) \cdot D^{n-1} = 0$. By contradiction, consider such an F (say $\mathrm{rk}(F) = 1$).

Step 1. $T_X(-\log D)|_D$ is polystable.

$\rightsquigarrow$ WLOG F is a subbundle near D.

Step 2. Use ω_{TY} and Griffiths formula.

ω_{TY} induces a singular metric h_F on $F|_{X\setminus D}$.

$$i\Theta_{h_F}(F) \wedge \omega_{\mathrm{TY}}^{n-1} = -i\beta \wedge \beta^* \wedge \omega_{\mathrm{TY}}^{n-1} \leqslant 0$$

with $\beta \in C^\infty(X\setminus D, \Omega_X^{1,0} \otimes \mathrm{Hom}(F^\perp, F))$.

Elements of proof III

Step 3. Show that $\int_{X\setminus D} i\Theta_{h_F}(F)\wedge\omega_{\mathrm{TY}}^{n-1}=0$.

The integral diverges a priori. One uses cutoff functions and

- Vanishing of D-slope of F
- Step 1
- Closeness of ω_{TY} to its model on N_D

Given Step 2 :

$$\int_{X\setminus D} i\beta\wedge\beta^*\wedge\omega_{\mathrm{TY}}^{n-1}=0$$

i.e. $F|_{X\setminus D}$ is parallel, which reduces the holonomy of ω_{TY}.

The two component case, first observations

X Fano of dimension $n \geqslant 3$, $D = D_1 + D_2$ snc s.t. $K_X + D \sim \mathcal{O}_X$.
Assume $d_1 D_1 \sim d_2 D_2$ for some $d_i \in \mathbb{N}^*$.

- The meromorphic function $a := s_{D_1}^{d_1} / s_{D_2}^{d_2}$ induces

$$a : X \backslash D \longrightarrow \mathbb{C}^* \simeq \mathbb{C}/\mathbb{Z}$$

which is the quasi-Albanese $\mathrm{Alb}_{(X,D)}$ of the pair (X, D).

- The logarithmic 1-form $d \log a$ generates $H^0(X, \Omega_X^1(\log D))$.
- $\pi_1(X \backslash D)$ is a finite extension of $\mathbb{Z}$.

Albanese has moduli + No polystability

$X = \mathbb{P}^n$ with $n \geqslant 9$ odd and D_1, D_2 general of degree $\frac{n+1}{2}$.

Proposition (Collins-G.)

1. Two general fibers of $\mathrm{Alb}_{(X,D)}$ are not birationally equivalent.
2. $T_X(-\log D)$ is not polystable.

Compare [BFTP+M].

The complete Calabi-Yau metric

X Fano of dimension $n \geqslant 3$, $D = D_1 + D_2$ snc s.t. $K_X + D \sim \mathcal{O}_X$.
(still assume $d_1 D_1 \sim d_2 D_2$ for some $d_i \in \mathbb{N}^*$)

Theorem (Collins-Li '24)

There exists a complete Ricci flat Kähler metric ω_{CL} on $X \backslash D$.

Volume growth : $R^{\frac{4n}{n+2}}$ ($\gg R^2$)
Asymptotic tangent cone : $((\mathbb{R}_{>0})^2$, explicit warped metric)

Bochner principle fails

The logarithmic 1-form $d \log a$ is not parallel wrt ω_{CL} !

Holonomy of ω_{CL} and universal cover

Theorem (Collins-G. '25)

1. $\mathrm{Hol}(X\backslash D, \omega_{\mathrm{CL}}) = \mathrm{SU}(n)$.
2. The universal cover $(\widetilde{X\backslash D}, \tilde{\omega}_{\mathrm{CL}})$ has volume growth $R^{\frac{6n}{n+2}}$ and asymptotic tangent cone $(\mathbb{R}_{>0})^2 \times \mathbb{R}$.
3. There exist examples (X, D) such that $\widetilde{X\backslash D}$ cannot be compactified analytically.

Summary

TABLE – Properties of log CY pairs (X, D)

	∃? CY metric	$\mathrm{Alb}_{(X,D)}$ loc. trivial	$T_X(-\log D)$ polystable	Bochner	$\widetilde{X \setminus D}$ q. proj
X smooth, $D = 0$	✓	✓	✓	✓	✓
X klt, $D = 0$	✓	✓	✓	✓	?
X lc, $D = 0$	?	×	×	NA	?
X Fano, D smooth	✓	✓	✓	✓	✓
X Fano, D singular	✓	×	× (s-stable)	×	×