Harmonic Maps and rigidity

2025 Summer Research Institute in Algebraic Geometry

Colorado State University

July 2025

Chikako Mese
Johns Hopkins University

July 17, 2025

Assume:

- X is a Euclidean building.
- $\widetilde{M} = G/K$ is a symmetric space of non-compact type, excluding Euclidean space, real or complex hyperbolic space.
- $\Gamma \subset G = \mathrm{Isom}_0(\widetilde{M})$ is a discrete subgroup acting on the left.
- $M = \Gamma \backslash \widetilde{M}$ is compact. (If $\mathrm{rank}(\widetilde{M}) = 1$, we can relax this asumption to finite volume.)

Theorem (Gromov–Schoen 1992; Breiner–Dees–Mese 2025)

An isometric action $\Gamma \circlearrowleft X$ fixes a point in the visual compactification $\overline{X} = X \cup \partial X$.

Regularity Result

PROOF OF RIGIDITY. Harmonic maps into Euclidean buildings are regular enough to apply geometric differential methods.

Theorem (Gromov–Schoen 1992; Breiner–Dees–Mese 2025)

If $u : \Omega \to X$ is a harmonic map from a Riemannian domain into a Euclidean building, then the singular set of u is a closed set of Hausdorff codimension 2.

Harmonic maps between Riemannian manifolds

A smooth harmonic map $u : (M, g) \to (N, h)$ between Riemannian manifolds satisfies the harmonic map equation:

$$\triangle_M u^i + g^{\alpha\beta}\Gamma^i_{jk}\frac{\partial u^j}{\partial x^\alpha}\frac{\partial u^k}{\partial x^\beta} = 0.$$

This is the Euler-Lagrange equation of the energy functional:

$$E^u = \int_M |du|^2 d\mathrm{vol}_g.$$

$$du : T_pM \to T_{u(x)}N \quad \text{or} \quad du \in \Gamma(T^*M \otimes u^*(TN))$$

Example. A **harmonic function** $u : (M, g) \to \mathbb{R}$

$$\triangle_M u = \mathrm{div}(\nabla u) = 0.$$

This is the Euler-Lagrange equation of

$$E^u = \int_M |\nabla u|_g^2 \; d\mathrm{vol}_g.$$

Example. A **parameterized geodesic** $u : \mathbb{S}^1 \to (N, h)$ satisfies

$$\frac{d^2 u^i}{dt^2} + \Gamma^i_{jk} \frac{du^j}{dt} \frac{du^k}{dt} = 0.$$

This is the Euler-Lagrange equation of

$$E^u = \int_{\mathbb{S}^1} \left| \frac{du}{dt} \right|_h^2 dt.$$

Homotopy Problem

Given a continuous map $f : M \to N$, find a harmonic map $u : M \to N$ homotopic to f.

Theorem (Eells-Sampson 1964)

Assume:

- *M and N are compact Riemannian manifolds.*
- *N has non-positive sectional curvature.*

If $f : M \to N$ is a continuous map, there exists a harmonic map $u : M \to N$ homotopic to f.

Homotopy Problem

Given a continuous map $f : M \to N$, find a harmonic map $u : M \to N$ homotopic to f.

Theorem (Eells-Sampson 1964)

Assume:

- *M and N are compact Riemannian manifolds.*
- *N has non-positive sectional curvature.*

If $f : M \to N$ is a continuous map, there exists a harmonic map $u : M \to N$ homotopic to f.

Equivariant problem

- M, Riemannian manifold with universal cover $\tilde{M}$
- $\tilde{N}$, Hadamard manifold
- $\rho : \pi_1(M) \to \mathrm{Isom}(\tilde{N})$, homomorphism

Definition

A map $f : \tilde{M} \to \tilde{N}$ is said to be <u>ρ-equivariant</u> if

$$f(\gamma x) = \rho(\gamma) f(x) \quad \forall \gamma \in \pi_1(M),\ x \in \tilde{M}.$$

f is ρ-equivariant $\Rightarrow$ $|df|^2$ is ρ-invariant $\Rightarrow$ $|df|^2$ descends to a function defined on M.

$$\text{The energy of } f \text{ is } \; E^f := \int_M |df|^2 d\mathrm{vol}_g$$

Definition

A ρ-equivariant map $u : \tilde{M} \to \tilde{N}$ is harmonic if $E^u \leqslant E^f$ for any ρ-equivarinat map $f : \tilde{M} \to \tilde{N}$.

Harmonic maps into a complete metric space X

Korevaar-Schoen 1993:

For $u : \Omega \to X$ from a Riemannian domain, define

$$e_\varepsilon : \Omega \to \mathbb{R}, \quad e_\varepsilon(x) = \begin{cases} \int_{y \in \partial B_\varepsilon(x)} \frac{d^2(u(x),u(y))}{\varepsilon^2} \frac{d\sigma_{x,\varepsilon}}{\varepsilon^{n-1}} & x \in \Omega_\varepsilon \\ 0 & \text{otherwise} \end{cases}$$

$$E^u_\epsilon : C_c(\Omega) \to \mathbf{R}, \quad E^u_\epsilon(\varphi) = \int_\Omega \varphi e_\epsilon d\mathrm{vol}_g.$$

We say u has finite energy (or that $u \in W^{1,2}(\Omega, X)$) if

$$E^u := \sup_{\varphi \in C_c(\Omega), 0 \leqslant \varphi \leqslant 1} \limsup_{\epsilon \to 0} E^u_\epsilon(\varphi) < \infty.$$

Harmonic maps into a complete metric space X

Korevaar-Schoen 1993:

For $u : \Omega \to X$ from a Riemannian domain, define

$$e_\varepsilon : \Omega \to \mathbb{R}, \quad e_\varepsilon(x) = \begin{cases} \displaystyle\int_{y \in \partial B_\varepsilon(x)} \frac{d^2(u(x), u(y))}{\varepsilon^2} \frac{d\sigma_{x,\varepsilon}}{\varepsilon^{n-1}} & x \in \Omega_\varepsilon \\ 0 & \text{otherwise} \end{cases}$$

$$E^u_\epsilon : C_c(\Omega) \to \mathbf{R}, \quad E^u_\epsilon(\varphi) = \int_\Omega \varphi e_\epsilon d\mathrm{vol}_g.$$

We say u has finite energy (or that $u \in W^{1,2}(\Omega, X)$) if

$$E^u := \sup_{\varphi \in C_c(\Omega), 0 \leqslant \varphi \leqslant 1} \limsup_{\epsilon \to 0} E^u_\epsilon(\varphi) < \infty.$$

Harmonic maps into a complete metric space X

Weak limit of the measure $e_\varepsilon(x)d\mathrm{vol}_g$ as $\varepsilon \to 0$ is $|du|^2(x)d\mathrm{vol}_g$.

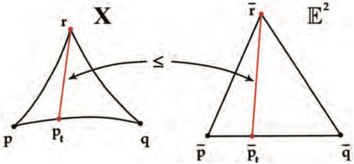
$$E^u[\Omega] = \int_\Omega |du|^2 \, d\mathrm{vol}_g.$$

Definition

We say a continuous map $u : \Omega \to X$ from a smooth Riemannian domain Ω is harmonic if it is locally energy minimizing.

NPC spaces

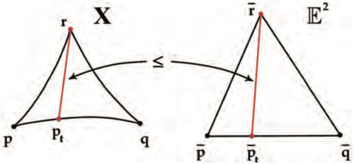
NPC space is a complete geodesic metric space such that geodesic triangles are “slimmer" than corresponding one in Euclidean space $\mathbb{E}^2$.



- A generalization of a Hadamard manifold.
- Important examples:
  - symmetric space of non-compact type
  - Euclidean buildings
  - hyperbolic buildings
  - Weil-Petersson completion of Teichmüller space

NPC spaces

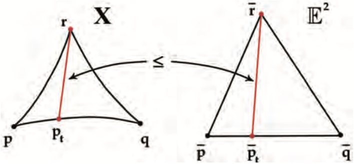
NPC space is a complete geodesic metric space such that geodesic triangles are “slimmer" than corresponding one in Euclidean space $\mathbb{E}^2$.



- A generalization of a Hadamard manifold.
- Important examples:
  - symmetric space of non-compact type
  - Euclidean buildings
  - hyperbolic buildings
  - Weil-Petersson completion of Teichmüller space

NPC spaces

NPC space is a complete geodesic metric space such that geodesic triangles are “slimmer" than corresponding one in Euclidean space $\mathbb{E}^2$.



- A generalization of a Hadamard manifold.
- Important examples:
  - symmetric space of non-compact type
  - Euclidean buildings
  - hyperbolic buildings
  - Weil-Petersson completion of Teichmüller space

Regularity Theorem

Let Ω be a Riemannian domain.

Theorem (Eells-Sampson 1964)

A harmonic map $u : \Omega \to \tilde{N}$ into a NPC Riemannian manifold is smooth.

Theorem (Korevaar-Schoen 1993)

A harmonic map $u : \Omega \to X$ into an NPC space is locally Lipschitz continuous.

Harmonic maps into Euclidean buildings enjoy a stronger regularity property than harmonic maps into an arbitrary NPC space.

Regularity Theorem

Let Ω be a Riemannian domain.

Theorem (Eells-Sampson 1964)

A harmonic map $u : \Omega \to \tilde{N}$ into a NPC Riemannian manifold is smooth.

Theorem (Korevaar-Schoen 1993)

A harmonic map $u : \Omega \to X$ into an NPC space is locally Lipschitz continuous.

Harmonic maps into Euclidean buildings enjoy a stronger regularity property than harmonic maps into an arbitrary NPC space.

Regularity Theorem

Let Ω be a Riemannian domain.

Theorem (Eells-Sampson 1964)

A harmonic map $u : \Omega \to \tilde{N}$ into a NPC Riemannian manifold is smooth.

Theorem (Korevaar-Schoen 1993)

A harmonic map $u : \Omega \to X$ into an NPC space is locally Lipschitz continuous.

Harmonic maps into Euclidean buildings enjoy a stronger regularity property than harmonic maps into an arbitrary NPC space.

A Euclidean building comes with:

- Euclidean space $\mathbb{E}^r$,
- Affine Weyl group $W_{aff} \subset \mathrm{Isom}(\mathbb{E}^r)$; i.e. the rotational part of W_{aff} is finite,
- Euclidean Coxeter complex, $(\mathbb{E}^r, W_{aff})$,
- Family of charts, i.e. a collection of isometric embeddings

$$\mathcal{A} = \{\iota : \mathbb{E}^r \to A := \iota(\mathbb{E}^r) \subset X\}$$

where A is called an apartment of X.

Euclidean buildings
$(\mathbb{E}^r, W_{aff})$ and $\mathcal{A}$ satisfy:

- X is a union of all the apartments.
- Two charts $\iota_1, \iota_2 \in \mathcal{A}$ are compatible, i.e.

$$\iota_1^{-1} \circ \iota_2 \in W_{aff}.$$

- Any two points is contained in an apartment and this defines a distance function d on X such that (X, d) is an NPC space.

Symmetric spaces vs. Euclidean buildings

- Lie group over Archimedean numbers ($\mathbb{R}$ and $\mathbb{C}$) isometrically act on symmetric spaces.

  Example: The group $\mathrm{GL}(n, \mathbb{C})$ acts on a symmetric space $\mathrm{GL}(n, \mathbb{C}) / U(n)$.

- Algebraic groups over non-Archimedean numbers ($\mathbb{Q}_p$ and $K := \mathbb{F}_{p^n}((t))$) act on Bruhat-Tits buildings.

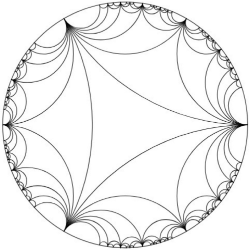
Example:

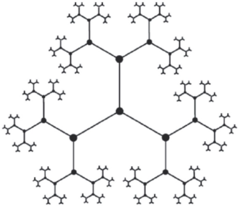
$\mathbb{Q}$

- $|\cdot|_\infty$ is the usual absolute value $\rightarrow$ metric completion is $\mathbb{R}$
- $|\cdot|_p$ is the p-adic absolute value $\rightarrow$ metric completion is $\mathbb{Q}_p$

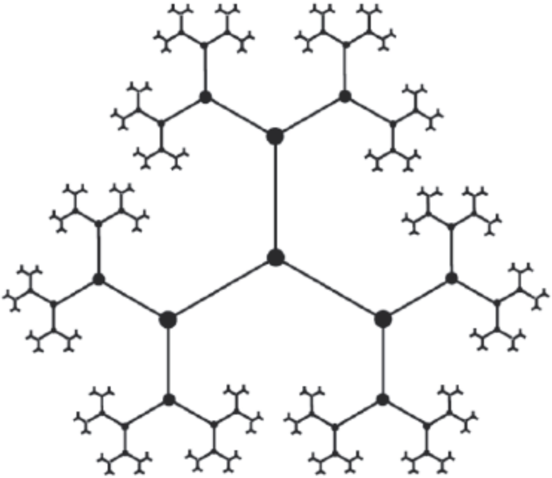
$\mathbf{G} = \mathrm{SL}_2$

- $G = \mathrm{SL}_2(\mathbb{R})$ acts on the upper half-plane
- $G = \mathrm{SL}_2(\mathbb{Q}_2)$ acts on a tree with valency 3.





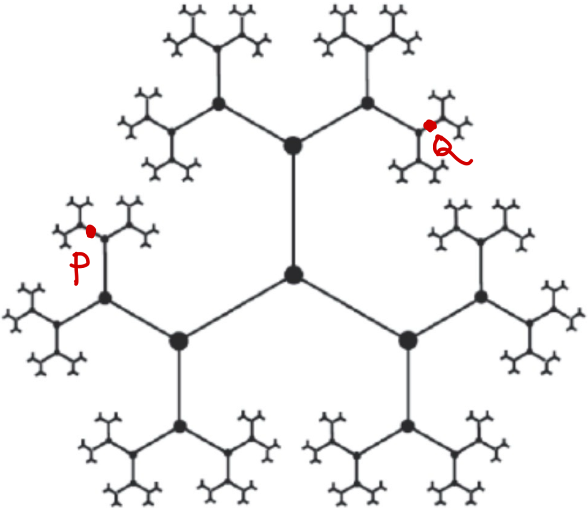
Euclidean buildings



Bruhat-Tits tree for the 2-adic Lie group $\mathrm{SL}(2, \mathbb{Q}_2)$.

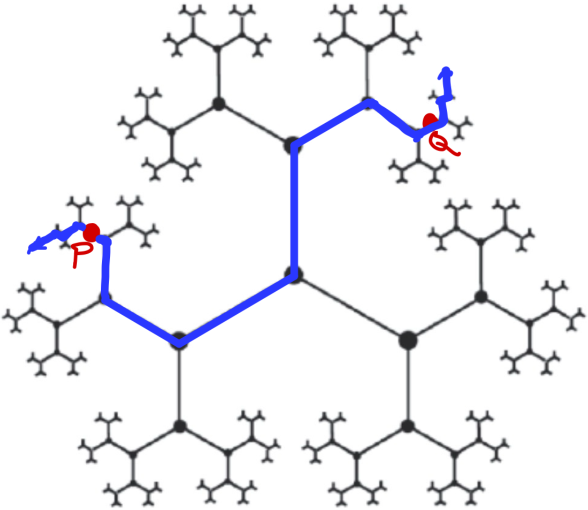
Euclidean buildings

Bruhat-Tits building for the 2-adic Lie group $SL(2, \mathbb{Q}_2)$

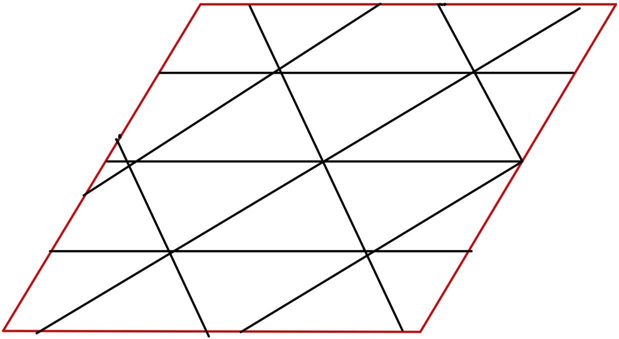


Euclidean buildings

Bruhat-Tits building for the 2-adic Lie group $SL(2, \mathbb{Q}_2)$

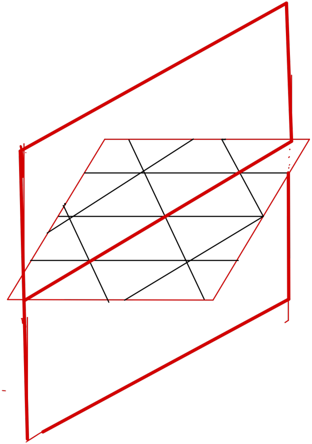


Euclidean buildings



- The fixed point set of $w \in W_{aff}$ are called **walls**.

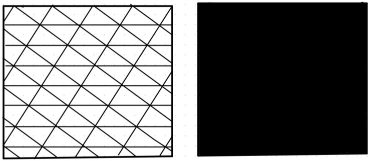
Euclidean buildings



- The intersection of two apartments is contained in a wall.

Euclidean buildings

Walls in an apartment: locally finite case vs. locally non-finite case.

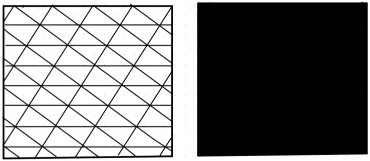


- The structure of a locally non-finite buildings is complicated:

  The convex hull of 3 points may not be contained in a finite number of apartments!

Euclidean buildings

Walls in an apartment: locally finite case vs. locally non-finite case.



- The structure of a locally non-finite buildings is complicated:

  The convex hull of 3 points may not be contained in a finite number of apartments!

Regular sets and Singular sets

- Let $\mathcal{A}$ be the set of charts of a Euclidean building X.
- Let $u : \tilde{M} \to X$ be a harmonic map.

Definition

- **Regular set**

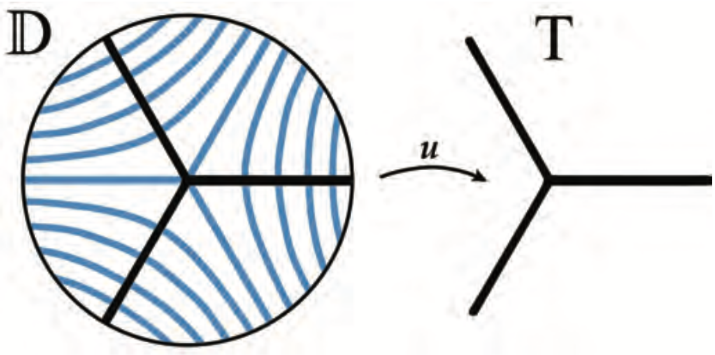
$$\mathcal{R}(u) = \{x \in \tilde{M} : \exists r > 0, \iota \in \mathcal{A} \text{ such that } u(B_r(x)) \subset A = \iota(\mathbb{R}^r)\}$$

- **Singular set**

$$\mathcal{S}(u) = \Omega \backslash \mathcal{R}(u)$$

Example

Vertical leaves of $q = zdz^2$, i.e. curves γ such that $q(\gamma', \gamma') < 0$



- The projection map $u : \mathbb{D} \to T$ is a harmonic map.
- $u^{-1}(\text{vertex})$ is not the singular set
- $\mathcal{S}(u) = \{0\}$

Regularity Theorem

Theorem (Gromov-Schoen 1993)

If X is a locally finite Euclidean building, then a harmonic map $u : \Omega \to X$ is smooth outside of a closed set $\mathcal{S}(u)$ of Hausdorff codimension 2.

Theorem (Breiner-Dees-Mese 2025)

Same conclusion without assuming local finiteness.

Regularity Theorem

Theorem (Gromov-Schoen 1993)

If X is a locally finite Euclidean building, then a harmonic map $u : \Omega \to X$ is smooth outside of a closed set $\mathcal{S}(u)$ of Hausdorff codimension 2.

Theorem (Breiner-Dees-Mese 2025)

Same conclusion without assuming local finiteness.

Tools in Gromov-Schoen's Proof

By local finiteness, we can assume the target space is $T_{x_0}X$.

- **blow up maps** $u_\sigma : B_1(0) \to T_{x_0}X$:

  For $x_0 = 0 \in \Omega$ and $\sigma > 0$,

  $$u_\sigma(x) = u(\sigma x)$$

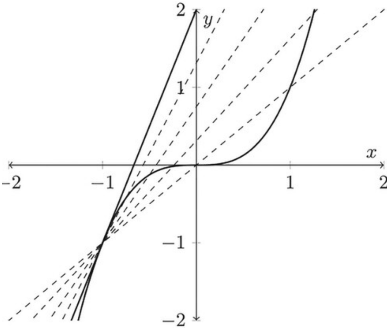
  and rescale the target $T_{x_0}X$ by a factor of μ_σ^{-1}.

  • Blow up maps generalize the difference quotients of functions.

- **tangent map** $u_* : B_1(0) \to T_{x_0}X$:

  Arzela-Ascoli says that a subsequence of blow up maps $u_\sigma : B_1(0) \to T_{x_0}X$ converges uniformly to a tangent map $u_* : B_1(0) \to T_{x_0}X$ in any compact subset of $B_1(0)$.

Tools in Gromov-Schoen's Proof



Review of the Gromov-Schoen's Proof

The two key components of Gromov and Schoen's proof are:

(1) A tangent map $u_* : B_1(0) \to T_{x_0}X$ is a **homogeneous degree 1 map**; i.e. the restriction of u_* to a radial ray is a constant speed geodesic.

The image of u_* is a flat = image of a isometric, totally geodesic map $\phi : \mathbb{R}^m \to T_{x_0}X$.

(2) The tangent map u_* at an order 1 point is **effectively contained** in a subbuilding $\mathbb{R}^m \times Y$ where Y is a lower dimensional Euclidean building.

KEY: Gromov-Schoen show that u must be contained in $\mathbb{R}^m \times Y$.

Using an inductive argument on the dimension of the Euclidean building, prove u is "well-approximated" by u_*; i.e. u is instrinsically differentiable.

Review of the Gromov-Schoen's Proof

The two key components of Gromov and Schoen's proof are:

(1) A tangent map $u_* : B_1(0) \to T_{x_0}X$ is a **homogeneous degree 1 map**; i.e. the restriction of u_* to a radial ray is a constant speed geodesic.

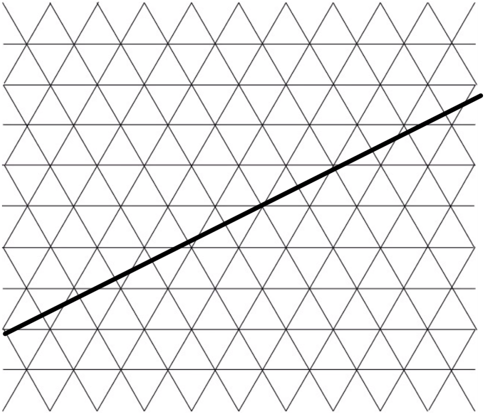
The image of u_* is a flat = image of a isometric, totally geodesic map $\phi : \mathbb{R}^m \to T_{x_0}X$.

(2) The tangent map u_* at an order 1 point is **effectively contained** in a subbuilding $\mathbb{R}^m \times Y$ where Y is a lower dimensional Euclidean building.

KEY: Gromov-Schoen show that u must be contained in $\mathbb{R}^m \times Y$.

Using an inductive argument on the dimension of the Euclidean building, prove u is "well-approximated" by u_*; i.e. u is instrinsically differentiable.

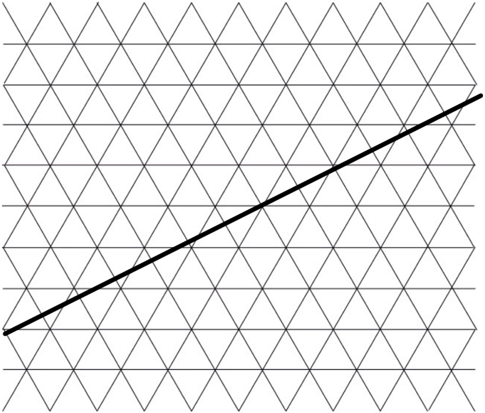
“Effectively contained"



In this diagram, the set of points on the thick line close to the complement of an apartment is small.

For the non-simplicial case, the situation is much more complicated.

"Effectively contained"



In this diagram, the set of points on the thick line close to the complement of an apartment is small.

For the non-simplicial case, the situation is much more complicated.

Why we need to modify Gromov-Schoen's proof

Gromov-Schoen assumes the buildings are:

- locally finite
- simplicial

For a general Euclidean building:

Issue 1 not locally finite $\Rightarrow$ the blow up maps u_σ and tangent map u_* do not have the same target as u.

Issue 2 not simplicial $\Rightarrow$ u_* is not effectively contained in a subbuilding.

Constructing a tangent map – Ultralimits

- ω finitely additive probability measure on $\mathbb{N}$
  - $\omega(S) = 0$ or 1 for all $S \subset \mathbb{N}$
  - $\omega(S) = 0$ for every finite set
- For a sequence $s = (s_1, s_2, \ldots)$,

$$\omega\text{-lim}s = s_*$$

means that $\omega(s^{-1}(U)) = 1$ for all neighborhoods $U \subset \mathbb{R}$ of s_*.

Constructing a tangent map – Ultralimits

- ω finitely additive probability measure on $\mathbb{N}$
  - $\omega(S) = 0$ or 1 for all $S \subset \mathbb{N}$
  - $\omega(S) = 0$ for every finite set
- For a sequence $s = (s_1, s_2, \dots)$,

$$\omega\text{-lim} s = s_*$$

means that $\omega(s^{-1}(U)) = 1$ for all neighborhoods $U \subset \mathbb{R}$ of s_*.

Constructing a tangent map – Ultralimits

- ω finitely additive probability measure on $\mathbb{N}$
  - $\omega(S) = 0$ or 1 for all $S \subset \mathbb{N}$
  - $\omega(S) = 0$ for every finite set
- For a sequence $s = (s_1, s_2, \dots)$,

$$\omega\text{-lim}s = s_*$$

means that $\omega(s^{-1}(U)) = 1$ for all neighborhoods $U \subset \mathbb{R}$ of s_*.

Constructing a tangent map – Ultralimits

- ω finitely additive probability measure on $\mathbb{N}$
  - $\omega(S) = 0$ or 1 for all $S \subset \mathbb{N}$
  - $\omega(S) = 0$ for every finite set
- For a sequence $s = (s_1, s_2, \dots)$,

$$\omega\text{-lim}s = s_*$$

means that $\omega(s^{-1}(U)) = 1$ for all neighborhoods $U \subset \mathbb{R}$ of s_*.

Constructing a tangent map – Ultralimits

- $x_0 \in X$
- scale factors μ_k, ω-lim $\mu_k = 0$
- $d_k(\cdot,\cdot) := \mu_k d(\cdot,\cdot)$, $k \in \mathbb{N}$
- $(X_\omega, d_\omega) = \omega$-lim (X, d_k) where
  - $X_\omega = \{x = (x_1, x_2, \dots) : d_k(x_k, x_0) < \infty\}$
  - $d_\omega(x,y) := \omega$-lim $d_k(x_k, y_k)$ where $x = (x_1, x_2 \dots), y = (y_1, y_2, \dots)$
  - $X_\omega := X_\omega / d_\omega$ is the quotient with respect to d_ω

Constructing a tangent map – Ultralimits

- $x_0 \in X$
- scale factors μ_k, ω-lim $\mu_k = 0$
- $d_k(\cdot,\cdot) := \mu_k d(\cdot,\cdot)$, $k \in \mathbb{N}$
- $(X_\omega, d_\omega) = \omega$-lim (X, d_k) where
  - $X_\omega = \{x = (x_1, x_2, \dots) : d_k(x_k, x_0) < \infty\}$
  - $d_\omega(x,y) := \omega$-lim $d_k(x_k, y_k)$ where $x = (x_1, x_2 \dots), y = (y_1, y_2, \dots)$
  - $X_\omega := X_\omega / d_\omega$ is the quotient with respect to d_ω

Constructing a tangent map – Ultralimits

- $x_0 \in X$
- scale factors μ_k, ω-lim $\mu_k = 0$
- $d_k(\cdot,\cdot) := \mu_k d(\cdot,\cdot)$, $k \in \mathbb{N}$
- $(X_\omega, d_\omega) = \omega$-lim (X, d_k) where
  - $X_\omega = \{x = (x_1, x_2, \dots) : d_k(x_k, x_0) < \infty\}$
  - $d_\omega(x, y) := \omega$-lim $d_k(x_k, y_k)$ where
    $x = (x_1, x_2 \dots), y = (y_1, y_2, \dots)$
  - $X_\omega := X_\omega / d_\omega$ is the quotient with respect to d_ω

Constructing a tangent map – Ultralimits

Fix $x_0 \in \Omega$ and choose normal coordinates centered at $x_0 = 0$.

Let $\sigma_k \to 0$.

For simplicity of notation, let

$$d_k(x_1, x_2) = \mu_k^{-1} d(\sigma_k x_1, \sigma_k x_2)$$

and

$$u_k : B_1(0) \to (X, d_k) \;\; u_k(x) := u(\sigma_k x).$$

- $(X_\omega, d_\omega) = \omega\text{-lim } (X, d_k)$

  Recall the X_ω is an equivalence class of bounded sequences $x = (x_1, x_2, \dots)$

- $u_\omega = \omega\text{-lim } u_k$ is defined as the map

$$u_\omega : B_1(0) \to (X_\omega, d_\omega), \quad u_\omega(p) = [(u_1(p), u_2(p), \dots)]$$

Constructing a tangent map – Ultralimits

Fix $x_0 \in \Omega$ and choose normal coordinates centered at $x_0 = 0$.

Let $\sigma_k \to 0$.

For simplicity of notation, let

$$d_k(x_1, x_2) = \mu_k^{-1} d(\sigma_k x_1, \sigma_k x_2)$$

and

$$u_k : B_1(0) \to (X, d_k) \;\; u_k(x) := u(\sigma_k x).$$

- $(X_\omega, d_\omega) = \omega\text{-lim } (X, d_k)$

  Recall the X_ω is an equivalence class of bounded sequences $x = (x_1, x_2, \dots)$

- $u_\omega = \omega\text{-lim } u_k$ is defined as the map

$$u_\omega : B_1(0) \to (X_\omega, d_\omega), \quad u_\omega(p) = [(u_1(p), u_2(p), \dots)]$$

Constructing a tangent map – Ultralimits

- Regarding the ultralimit (X_ω, d_ω) (Kleiner-Leeb, '97):
  - The ultralimit (X_ω, d_ω) is a Euclidean building with rotational part of the affine Weyl group the same as that of (X, d_k).
  - A chart $\iota : \mathbb{E}^r \to A$ for X_ω is

$$\iota = (\iota_1, \iota_2, \dots)$$

  where $\iota_k : \mathbb{E}^r \to A_k$ is a chart for $X_k = (X, d_k)$.

- Regarding the ultralimit $u_\omega : B_1(0) \to X_\omega$ (Korevaar-Schoen '97):

$$d_k(u_k(p), u_k(q)) \to d_\omega(u_\omega(p), u_\omega(q)),$$

$$|\nabla u_k|^2 d\mu_{g_k} \rightharpoonup |\nabla u_\omega|^2 d\mu_0,$$

and

$$\left|\frac{\partial u_k}{\partial x_i}\right|^2 d\mu_{g_k} \rightharpoonup \left|\frac{\partial u_\omega}{\partial x_i}\right|^2 d\mu_0.$$

Constructing a tangent map – Ultralimits

- Regarding the ultralimit (X_ω, d_ω) (Kleiner-Leeb, '97):
  - The ultralimit (X_ω, d_ω) is a Euclidean building with rotational part of the affine Weyl group the same as that of (X, d_k).
  - A chart $\iota : \mathbb{E}^r \to A$ for X_ω is

$$\iota = (\iota_1, \iota_2, \dots)$$

    where $\iota_k : \mathbb{E}^r \to A_k$ is a chart for $X_k = (X, d_k)$.

- Regarding the ultralimit $u_\omega : B_1(0) \to X_\omega$ (Korevaar-Schoen '97):

$$d_k(u_k(p), u_k(q)) \to d_\omega(u_\omega(p), u_\omega(q)),$$

$$|\nabla u_k|^2 d\mu_{g_k} \rightharpoonup |\nabla u_\omega|^2 d\mu_0,$$

and

$$\left|\frac{\partial u_k}{\partial x_i}\right|^2 d\mu_{g_k} \rightharpoonup \left|\frac{\partial u_\omega}{\partial x_i}\right|^2 d\mu_0.$$

Issue 1: Approximation by a tangenet map

Assume $Ord^u(0) = 1$.

- $u_\omega : \mathbb{E}^n \to X_\omega$ is a homogeneous degree 1 harmonic map.
- u_ω is “linear", i.e. $\exists$ a linear map $L : \mathbb{E}^n \to \mathbb{E}^N = A \subset X_\omega$ such that
$$d_\omega(u_\omega(x), u_\omega(y)) = |L(x) - L(y)|$$
Here, $A = \iota(\mathbb{R}^r)$ and $\iota = (\iota_1, \iota_2, \dots)$.
- u_ω and u_k have different target spaces. But using the chart,
$$\iota_k : \mathbb{E}^N = A_k \subset X_k,$$
we construct a homogenous degree 1 map into X_k
$$L_k := \iota_k \circ L : \mathbb{R}^n \to A_k \subset X_k$$
which is close to u_k; i.e. $\boxed{\lim_{k\to\infty} d_k(u_k, L_k) = 0}$

This takes care of **Issue 1**

Issue 1: Approximation by a tangenet map

Assume $Ord^u(0) = 1$.

- $u_\omega : \mathbb{E}^n \to X_\omega$ is a homogeneous degree 1 harmonic map.
- u_ω is “linear", i.e. $\exists$ a linear map $L : \mathbb{E}^n \to \mathbb{E}^N = A \subset X_\omega$ such that
$$d_\omega(u_\omega(x), u_\omega(y)) = |L(x) - L(y)|$$
Here, $A = \iota(\mathbb{R}^r)$ and $\iota = (\iota_1, \iota_2, \dots)$.
- u_ω and u_k have different target spaces. But using the chart,
$$\iota_k : \mathbb{E}^N = A_k \subset X_k,$$
we construct a homogenous degree 1 map into X_k
$$L_k := \iota_k \circ L : \mathbb{R}^n \to A_k \subset X_k$$
which is close to u_k; i.e. $\boxed{\lim_{k\to\infty} d_k(u_k, L_k) = 0}$

This takes care of **Issue 1**

Issue 1: Approximation by a tangenet map

Assume $Ord^u(0) = 1$.

- $u_\omega : \mathbb{E}^n \to X_\omega$ is a homogeneous degree 1 harmonic map.
- u_ω is "linear", i.e. $\exists$ a linear map $L : \mathbb{E}^n \to \mathbb{E}^N = A \subset X_\omega$ such that
$$d_\omega(u_\omega(x), u_\omega(y)) = |L(x) - L(y)|$$
Here, $A = \iota(\mathbb{R}^r)$ and $\iota = (\iota_1, \iota_2, \dots)$.
- u_ω and u_k have different target spaces. But using the chart,
$$\iota_k : \mathbb{E}^N = A_k \subset X_k,$$
we construct a homogenous degree 1 map into X_k
$$L_k := \iota_k \circ L : \mathbb{R}^n \to A_k \subset X_k$$
which is close to u_k; i.e. $\boxed{\lim_{k\to\infty} d_k(u_k, L_k) = 0}$

This takes care of **Issue 1**

Issue 1: Approximation by a tangenet map

Assume $Ord^u(0) = 1$.

- $u_\omega : \mathbb{E}^n \to X_\omega$ is a homogeneous degree 1 harmonic map.
- u_ω is "linear", i.e. $\exists$ a linear map $L : \mathbb{E}^n \to \mathbb{E}^N = A \subset X_\omega$ such that
$$d_\omega(u_\omega(x), u_\omega(y)) = |L(x) - L(y)|$$
Here, $A = \iota(\mathbb{R}^r)$ and $\iota = (\iota_1, \iota_2, \dots)$.
- u_ω and u_k have different target spaces. But using the chart,
$$\iota_k : \mathbb{E}^N = A_k \subset X_k,$$
we construct a homogenous degree 1 map into X_k
$$L_k := \iota_k \circ L : \mathbb{R}^n \to A_k \subset X_k$$
which is close to u_k; i.e. $\boxed{\lim_{k\to\infty} d_k(u_k, L_k) = 0}$

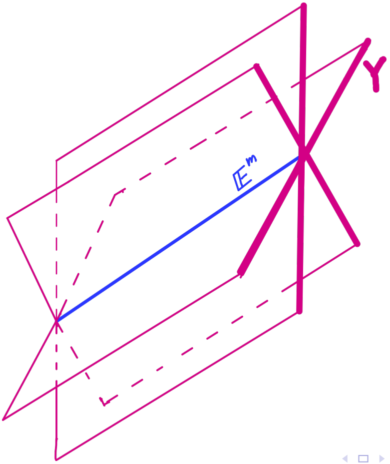
This takes care of **Issue 1**

Proof of the Regularity Theorem: Tangent Map

- (Kleiner-Leeb, '97) For $F := L(\mathbb{E}^n) \simeq \mathbb{E}^m$, the set of all parallel flats to F is a subbuilding isometric to

$$P_F := \mathbb{E}^m \times Y$$

where Y is a Euclidean building of dimension $N - m$.



Proof of the Regularity Theorem: Tangent Maps

KEY STEP: We show u maps into P_F so that we can induct on the dimension of the building.

That is, we want to show:

If $\mathrm{Ord}^u(0) = 1$, then there exists a splitting:

$$u|B_r(0) := (u_1, u_2) : B_r(0) \to \mathbb{E}^m \times Y$$

where

- $u_1 : B_r(0) \to \mathbb{E}^m$ is smooth, and
- $u_2 : B_r(0) \to Y$ has the property that $\mathrm{Ord}^{u_2}(0) > 1$.

Gromov-Schoen uses "effectively contained".

Proof of the Regularity Theorem: Tangent Maps

KEY STEP: We show u maps into P_F so that we can induct on the dimension of the building.

That is, we want to show:

If $\mathrm{Ord}^u(0) = 1$, then there exists a splitting:

$$u|B_r(0) := (u_1, u_2) : B_r(0) \to \mathbb{E}^m \times Y$$

where

- $u_1 : B_r(0) \to \mathbb{E}^m$ is smooth, and
- $u_2 : B_r(0) \to Y$ has the property that $\mathrm{Ord}^{u_2}(0) > 1$.

Gromov-Schoen uses “effectively contained".

<u>u maps into P_F</u>

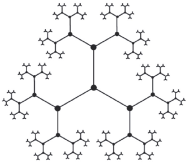
THE KEY IDEA: Too much energy at points not mapping into P_F.

The key lemma is the following:

Lemma

$$\mathcal{B}_k := \{x \in B_1(0) : u_k(x) \notin P_F\} \quad \Rightarrow \quad \lim_{k\to\infty} \mu(\mathcal{B}_k) = 0.$$

To illustrate the idea of the proof of the lemma, we will first consider the easiest case when the target is an $\mathbb{R}$-tree T.



Loss of energy argument for harmonic maps into an $\mathbb{R}$-tree.

Assume

- $u : \Omega \to T$ is a harmonic map into an $\mathbb{R}$-tree.
- $x_0 = 0 \in \Omega$.
- $u_k : B_1(0) \to X_k$ and $L_k : B_1(0) \to A_k$ as before.

Lemma (X. Sun, 2003)

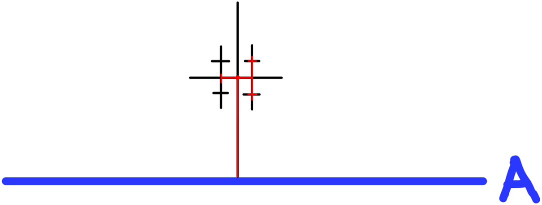
Assume $\dim X = 1$*.*

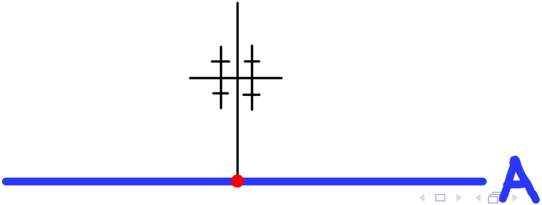
$$\mathcal{B}_k := \{x \in B_1(0) : u_k(x) \notin A_k\} \quad \Rightarrow \quad \lim_{k\to\infty} \mu(\mathcal{B}_k) = 0.$$

- **Nearest point projection map.**

$$\pi_k : X_k \to A_k$$

- $|\nabla(\pi_k \circ u_k)|^2(x) \leqslant |\nabla u_k|^2(x), \quad \forall x \in B_1(0)$
- $|\nabla(\pi_k \circ u_k)|^2(x) = 0$ for $x \in \mathcal{B}_k$.



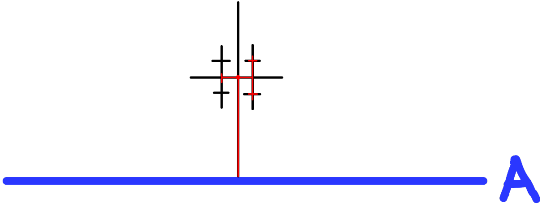


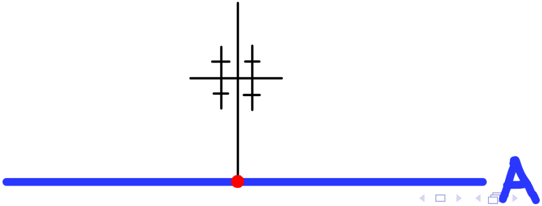
Proof of Regularity when $\dim X = 1$

- **Nearest point projection map.**

$$\pi_k : X_k \to A_k$$

- $|\nabla(\pi_k \circ u_k)|^2(x) \leqslant |\nabla u_k|^2(x), \quad \forall x \in B_1(0)$
- $|\nabla(\pi_k \circ u_k)|^2(x) = 0$ for $x \in \mathcal{B}_k$.





Proof of Regularity when $\dim X = 1$

Harmonic replacement map:

$$h_k : B_1(0) \to X_k$$
$$h_k|_{\partial B_1(0)} = \pi_k \circ u_k|_{\partial B_1(0)}$$

Lemma

The tangent map u_ω is also the ω-limit of the sequence h_k.

PROOF

$$\begin{aligned}
\sup_{B_1(0)} d_k(h_k, u_k) &\leqslant C \sup_{\partial B_1(0)} d_k(h_k, u_k) && \text{(subharmonicity)} \\
&= C \sup_{\partial B_1(0)} d_k(\pi_k \circ u_k, u_k) && (h_k = \pi_k \circ u_k \text{ on } \partial B_1(0)) \\
&\leqslant C \sup_{\partial B_1(0)} d_k(L_k, u_k). && (L_k \text{ maps into } A_k)
\end{aligned}$$

$$\Rightarrow \quad \omega\text{-}\lim h_k = u_\omega$$

Proof of Regularity when $\dim X = 1$

PROOF OF THAT $\mu(\mathcal{B}_k) \to 0$.

On the contrary, assume $m(\mathcal{B}_{k'}) \geqslant \epsilon$. $\Rightarrow \int_{\mathcal{B}_{k'}} |\nabla u_{k'}|^2 d\mu \geqslant \delta$.

$$
\begin{aligned}
E^{h_k} &\leqslant E^{\pi_k \circ u_k} && (h_k \text{ is minimizing}) \\
&= \int_{B_1(0)} |\nabla(\pi_k \circ u_k)|^2 d\mu \\
&= \int_{B_1(0)\backslash\mathcal{B}_k} |\nabla(\pi_k \circ u_k)|^2 d\mu && (|\nabla(\pi_k \circ u_k)|^2(x) = 0 \text{ for } x \in \mathcal{B}_k) \\
&\leqslant \int_{B_1(0)\backslash\mathcal{B}_k} |\nabla u_k|^2 d\mu && (u_k = \pi_k \circ u_k \text{ for } x \notin \mathcal{B}_k) \\
&= E^{u_k} - \int_{\mathcal{B}_k} |\nabla u_k|^2 d\mu \\
&\leqslant E^{u_k} - \delta
\end{aligned}
$$

$$\omega\text{-}\lim h_k = u_\omega = \omega\text{-}\lim u_k \quad \Rightarrow \quad E^{u_\omega} \leqslant E^{u_\omega} - \delta, \text{ a contradiction!}$$

Proof of the Regularity Theorem

Generalizing the Gromov-Schoen argument boils down to finding a way to prove $\lim_{k\to\infty} \mu(\mathcal{B}_k) = 0$ for higher dimensional buildings.

KEY STEPS IN THE PROOF:

- If $u(x) \notin P_F$ then $\pi \circ u(x) \in A$ lands in a wall of A which has at least one direction not parallel to F.
- Let $\pi_F^i : P_F \to \mathbb{R}$ denote the projection onto the i-th component function of $\mathbb{R}^m \simeq F$.

  There exists $\theta_0 \in (0, \pi/2]$ depending on F and W_{aff} such that for every x such that $u(x) \notin P_F$, there exists at least one $i \in \{1, \ldots, m\}$ such that

$$\left| \frac{\partial(\pi_F^i \circ \pi \circ u)}{\partial x^i} \right|^2 (x) \leqslant \cos^2 \theta_0 \left| \frac{\partial u}{\partial x^i} \right|^2 (x).$$

Proof of the Regularity Theorem

Generalizing the Gromov-Schoen argument boils down to finding a way to prove $\lim_{k\to\infty} \mu(\mathcal{B}_k) = 0$ for higher dimensional buildings.

KEY STEPS IN THE PROOF:

- If $u(x) \notin P_F$ then $\pi \circ u(x) \in A$ lands in a wall of A which has at least one direction not parallel to F.
- Let $\pi_F^i : P_F \to \mathbb{R}$ denote the projection onto the i-th component function of $\mathbb{R}^m \simeq F$.

  There exists $\theta_0 \in (0, \pi/2]$ depending on F and W_{aff} such that for every x such that $u(x) \notin P_F$, there exists at least one $i \in \{1, \ldots, m\}$ such that

$$\left| \frac{\partial(\pi_F^i \circ \pi \circ u)}{\partial x^i} \right|^2 (x) \leqslant \cos^2 \theta_0 \left| \frac{\partial u}{\partial x^i} \right|^2 (x).$$

Proof of the Regularity Theorem

Generalizing the Gromov-Schoen argument boils down to finding a way to prove $\lim_{k\to\infty} \mu(\mathcal{B}_k) = 0$ for higher dimensional buildings.

KEY STEPS IN THE PROOF:

- If $u(x) \notin P_F$ then $\pi \circ u(x) \in A$ lands in a wall of A which has at least one direction not parallel to F.
- Let $\pi_F^i : P_F \to \mathbb{R}$ denote the projection onto the i-th component function of $\mathbb{R}^m \simeq F$.

  There exists $\theta_0 \in (0, \pi/2]$ depending on F and W_{aff} such that for every x such that $u(x) \notin P_F$, there exists at least one $i \in \{1, \ldots, m\}$ such that

$$\left|\frac{\partial(\pi_F^i \circ \pi \circ u)}{\partial x^i}\right|^2 (x) \leqslant \cos^2\theta_0 \left|\frac{\partial u}{\partial x^i}\right|^2 (x).$$

Proof of the Regularity Theorem

Generalizing the Gromov-Schoen argument boils down to finding a way to prove $\lim_{k\to\infty} \mu(\mathcal{B}_k) = 0$ for higher dimensional buildings.

KEY STEPS IN THE PROOF:

- If $u(x) \notin P_F$ then $\pi \circ u(x) \in A$ lands in a wall of A which has at least one direction not parallel to F.
- Let $\pi_F^i : P_F \to \mathbb{R}$ denote the projection onto the i-th component function of $\mathbb{R}^m \simeq F$.

  There exists $\theta_0 \in (0, \pi/2]$ depending on F and W_{aff} such that for every x such that $u(x) \notin P_F$, there exists at least one $i \in \{1, \ldots, m\}$ such that

$$\left| \frac{\partial(\pi_F^i \circ \pi \circ u)}{\partial x^i} \right|^2 (x) \leqslant \cos^2 \theta_0 \left| \frac{\partial u}{\partial x^i} \right|^2 (x).$$

Proof of the Regularity Theorem

Proof that $u(\Omega) \subset P_F$.

- On the contrary, suppose there exists $y \notin u^{-1}(P_F)$.
- Choose $r > 0$ so that $B_r(y) \subset \mathcal{B}_k = \mathcal{B}_k$ and there exists $x_0 \in \partial B_r(y) \cap B_1(0) \backslash \mathcal{B}_k$.
- Consider blow up maps $u_k : B_1(0) \to X_k$ at x_0. About half of $B_1(0)$ should lie in $\mathcal{B}_k$.
- This is a contradiction because $\mu\{x \in B_1(0) : u_k(x) \notin P_F\} \to 0$.

Proof of the Regularity Theorem

Proof that $u(\Omega) \subset P_F$.

- On the contrary, suppose there exists $y \notin u^{-1}(P_F)$.
- Choose $r > 0$ so that $B_r(y) \subset \mathcal{B}_k = \mathcal{B}_k$ and there exists $x_0 \in \partial B_r(y) \cap B_1(0) \backslash \mathcal{B}_k$.
- Consider blow up maps $u_k : B_1(0) \to X_k$ at x_0. About half of $B_1(0)$ should lie in $\mathcal{B}_k$.
- This is a contradiction because $\mu\{x \in B_1(0) : u_k(x) \notin P_F\} \to 0$.

Proof of the Regularity Theorem

Proof that $u(\Omega) \subset P_F$.

- On the contrary, suppose there exists $y \notin u^{-1}(P_F)$.
- Choose $r > 0$ so that $B_r(y) \subset \mathcal{B}_k = \mathcal{B}_k$ and there exists $x_0 \in \partial B_r(y) \cap B_1(0) \backslash \mathcal{B}_k$.
- Consider blow up maps $u_k : B_1(0) \to X_k$ at x_0. About half of $B_1(0)$ should lie in $\mathcal{B}_k$.
- This is a contradiction because $\mu\{x \in B_1(0) : u_k(x) \notin P_F\} \to 0$.

Proof of the Regularity Theorem

Proof that $u(\Omega) \subset P_F$.

- On the contrary, suppose there exists $y \notin u^{-1}(P_F)$.
- Choose $r > 0$ so that $B_r(y) \subset \mathcal{B}_k = \mathcal{B}_k$ and there exists $x_0 \in \partial B_r(y) \cap B_1(0) \backslash \mathcal{B}_k$.
- Consider blow up maps $u_k : B_1(0) \to X_k$ at x_0. About half of $B_1(0)$ should lie in $\mathcal{B}_k$.
- This is a contradiction because $\mu\{x \in B_1(0) : u_k(x) \notin P_F\} \to 0$.

Proof of the Regularity Theorem

Proof that $u(\Omega) \subset P_F$.

- On the contrary, suppose there exists $y \notin u^{-1}(P_F)$.
- Choose $r > 0$ so that $B_r(y) \subset \mathcal{B}_k = \mathcal{B}_k$ and there exists $x_0 \in \partial B_r(y) \cap B_1(0) \backslash \mathcal{B}_k$.
- Consider blow up maps $u_k : B_1(0) \to X_k$ at x_0. About half of $B_1(0)$ should lie in $\mathcal{B}_k$.
- This is a contradiction because $\mu\{x \in B_1(0) : u_k(x) \notin P_F\} \to 0$.

Proof of the Regularity Theorem

- If x_0 is an order 1 point, then $\exists r > 0$ such that $u = (u_1, u_2)$ where

$$u_1 : B_r(0) \to \mathbb{R}^m$$
$$u_2 : B_r(0) \to Y.$$

- By an induction on the dimension, the singular set $\mathcal{S}(u_2)$ is of Hausdorff codimension 2.

- The set of higher order points is of Hausdorff codimension 2.

Q.E.D.

Proof of the Regularity Theorem

- If x_0 is an order 1 point, then $\exists r > 0$ such that $u = (u_1, u_2)$ where

$$u_1 : B_r(0) \to \mathbb{R}^m$$
$$u_2 : B_r(0) \to Y.$$

- By an induction on the dimension, the singular set $\mathcal{S}(u_2)$ is of Hausdorff codimension 2.

- The set of higher order points is of Hausdorff codimension 2.

Q.E.D.

Proof of the Regularity Theorem

- If x_0 is an order 1 point, then $\exists r > 0$ such that $u = (u_1, u_2)$ where

$$u_1 : B_r(0) \to \mathbb{R}^m$$
$$u_2 : B_r(0) \to Y.$$

- By an induction on the dimension, the singular set $\mathcal{S}(u_2)$ is of Hausdorff codimension 2.

- The set of higher order points is of Hausdorff codimension 2.

Q.E.D.

Application: Rigidity Problems

Rigidity in group theory means that a homomorphism

$$\rho : G' \to G$$

is determined by its restriction to a discrete subgroup $\Gamma \subset G'$.

In other words, rigidity means:

$$\boxed{\Gamma \to G \text{ uniquely extends to } \rho : G' \to G.}$$

Application: Rigidity Problems

Given a lattice $\Gamma \subset G'$ and a representation $\rho : \Gamma \to G$:

Strong rigidity (Mostow 1960's)

- $G' = G$ are semi-simple Lie group with no compact factors.
- $G, G' \neq PSL(2, \mathbb{R})$

If ρ is **faithful**, then ρ is **rigid**.

Superrigidity (Margulis 1970's)

- G, G' are reductive algebraic groups in GL_n
- The Lie group of its real points has real rank at least 2 and no compact factors.

If $\rho(\Gamma)$ is **Zariski dense**, then ρ is **rigid**.

Geometric rigidity

Observation: G acts isometrically on a geometric space.

Mostow's original strong rigidity:

$$G' = G = PO(n,1) \subset \mathrm{Isom}(\mathbb{H}^n), \quad n \geqslant 3$$

Geometric interpretation

The geometry of a compact hyperbolic manifold of dim $\geqslant 3$ is uniquely determined by its fundamental group.

Geometric rigidity

- M, Riemannian manifold with universal cover $\tilde{M}$
- X, NPC space
- $\rho : \pi_1(M) \to \mathrm{Isom}(X)$

Definition

ρ is *geodesically rigid* if there exists a ρ-equivariant totally geodesic map $u : \tilde{M} \to X$.

Harmonic maps approach to rigidity

Pioneered by Siu in the 1980's:

Step 1. Show $\exists$ a ρ-equivariant harmonic map $u : \widetilde{M} \to X$.

Step 2. Use the geometry of M and X to prove that u is totally geodesic.

Harmonic maps approach to rigidity

Theorem (Siu 1980)

- *M, compact Kähler manifold*
- *N, compact Kähler manifolds with strongly negative in the sense of Siu*

If $u : M \to N$ is a harmonic map and the $\mathrm{rank}_{\mathbb{R}} du \geqslant 4$ at some point, then u is either holomorphic or conjugate holomorphic.

Remark: If M, N are also locally symmetric, then u is isometric.

Bochner method

Proof.

- Kähler form ω on M is parallel $\Rightarrow$ Siu's Bochner formula:

$$\partial\bar{\partial}\{\bar{\partial}u,\bar{\partial}u\}\omega^{n-2} = 2(|\partial_E\bar{\partial}u|^2 + Q_0)\omega^n$$

where

$$\{\bar{\partial}u,\bar{\partial}u\} = \left\langle \frac{\partial u}{\partial \bar{z}^\alpha}, \frac{\partial u}{\partial z^\beta} \right\rangle d\bar{z}^\alpha \wedge dz^\beta$$

- Integrate:

$$0 = \int_M \partial\bar{\partial}\{\bar{\partial}u,\bar{\partial}u\} = 2\int_M (|\partial_E\bar{\partial}u|^2 + Q_0)\omega^2$$

- Assuming $Q_0 \geqslant 0$, conclude:

$$|\partial_E\bar{\partial}u|^2 = 0 = Q_0.$$

$|\partial_E\bar{\partial}u|^2 = 0$ means u is pluriharmonic, i.e. u is harmonic restricted to any complex curve of M.

Bochner method

Proof.

- Kähler form ω on M is parallel $\Rightarrow$ Siu's Bochner formula:

$$\partial\bar{\partial}\{\bar{\partial}u, \bar{\partial}u\}\omega^{n-2} = 2(|\partial_E\bar{\partial}u|^2 + Q_0)\omega^n$$

  where

$$\{\bar{\partial}u, \bar{\partial}u\} = \left\langle \frac{\partial u}{\partial \bar{z}^\alpha}, \frac{\partial u}{\partial z^\beta} \right\rangle d\bar{z}^\alpha \wedge dz^\beta$$

- Integrate:

$$0 = \int_M \partial\bar{\partial}\{\bar{\partial}u, \bar{\partial}u\} = 2\int_M (|\partial_E\bar{\partial}u|^2 + Q_0)\omega^2$$

- Assuming $Q_0 \geqslant 0$, conclude:

$$|\partial_E\bar{\partial}u|^2 = 0 = Q_0.$$

$|\partial_E\bar{\partial}u|^2 = 0$ means u is pluriharmonic, i.e. u is harmonic restricted to any complex curve of M.

Bochner method

Proof.

- Kähler form ω on M is parallel $\Rightarrow$ Siu's Bochner formula:

$$\partial\bar\partial\{\bar\partial u, \bar\partial u\}\omega^{n-2} = 2(|\partial_E\bar\partial u|^2 + Q_0)\omega^n$$

  where

$$\{\bar\partial u, \bar\partial u\} = \left\langle \frac{\partial u}{\partial \bar z^\alpha}, \frac{\partial u}{\partial z^\beta} \right\rangle d\bar z^\alpha \wedge dz^\beta$$

- Integrate:

$$0 = \int_M \partial\bar\partial\{\bar\partial u, \bar\partial u\} = 2\int_M (|\partial_E\bar\partial u|^2 + Q_0)\omega^2$$

- Assuming $Q_0 \geqslant 0$, conclude:

$$|\partial_E\bar\partial u|^2 = 0 = Q_0.$$

$|\partial_E\bar\partial u|^2 = 0$ means u is pluriharmonic, i.e. u is harmonic restricted to any complex curve of M.

Bochner method

Other variations of the Bochner formula:

- J.H. Sampson 1986
- K. Corlette 1992
- N. Mok, S.-T. Siu, S.-K. Yeung 1993
- J. Jost, S.-T.-Yau 1993

Geometric interpretation of superrigidity

Archimedean superrigidity:

- M, locally symmetric space
- $\tilde{N} = G/K$, symmetric space of non-compact type
- $\rho : \pi_1(M) \to G$ does not fix a point at infinity

$\Rightarrow \exists\ \rho$-equivariant totally geodesic map $u : \tilde{M} \to \tilde{N}$.

Non-Archimedean superrigidity:

- M, locally symmetric space
- X, Bruhat-Tits building for a reductive algebraic G groups over a local non-Archimedean field.
- $\rho : \pi_1(M) \to G$ does not fix a point at infinity

$\Rightarrow \exists\ \rho$-equivariant constant map.

Archimedian and *p*-adic superrigidity

Rank $\geqslant 2$

Theorem (Margulis 1970's)

If rank$(\widetilde{M}) \geqslant 2$, then Archimedian superrigidity and p-adic superrigidity hold.

Rank 1

Theorem (Corlette 1991)

If $\widetilde{M}$ is a quaternionic hyperbolic space or a Caley hyperbolic space, then Archimedian superrigidity holds.

Theorem (Gromov-Schoen 1992)

If $\widetilde{M}$ is a quaternionic hyperbolic space or a Caley hyperbolic space, then p-adic superrigidity hold.

Non-Archimedian superrigidity

Rank $\geqslant 2$

Theorem (Margulis 1970's, Bader-Furman 2018)

If rank$(\widetilde{M}) \geqslant 2$, then non-Archimedian superrigidity holds.

Rank 1

Theorem (Gromov-Schoen 1993, Breiner-Dees-M– 2025)

If $\widetilde{M}$ is a quaternionic hyperbolic space or a Caley hyperbolic space, then non-Archimedian superrigidity holds.

ρ-equivariant harmonic map - Existence

Existence Theorem: Donaldson 1987, Corlette 1992, Labourie 1991, Gromov-Schoen 1992, Korevaar-Schoen 1993

Theorem (Korevaar-Schoen 1990's)

- *M, Riemannian manifold of finite volume*
- *X, finite rank NPC space*
- *$\rho : \pi(M) \to Isom(X)$ does not fix a point on the visual boundary*

If there exists a ρ-equivariant finite energy map, then $\exists$ ρ-equivariant harmonic map $\tilde{u} : \widetilde{M} \to X$.

For geodesic rays $c, c' : [0, \infty) \to X$,

$$c \simeq c' \quad \Leftrightarrow \quad \exists \kappa > 0 \text{ such that } d(c(t), c'(t)) < \kappa, \ \forall t \in [0, \infty).$$

The visual boundary ∂X is the equivalent class of geodesic rays.

Regularity implies rigidity

Key to the proof of the rigidity results:

Harmonic maps are regular enough to apply Bochner methods.

Regularity implies rigidity

Example (Generalization of Siu): Harmonic map $u : M \to X$ from a compact Kähler manifold into a Euclidean building is pluriharmonic.

- Define cut-off functions $\{\varphi_i\}$ with support contained in the regular set $\mathcal{R}(u)$ and with $\lim_{i\to\infty} \varphi_i(x) = 1$ for all $x \in \mathcal{R}(u)$
- Multiply the Bochner formula by φ_i

$$\varphi_i \partial\bar{\partial}\{\bar{\partial}u, \bar{\partial}u\}\omega^{n-2} = 2\varphi_i(|\partial_E\bar{\partial}u|^2 + Q_0)\omega^n$$

- Justfy integration by parts and take limit:

$$\begin{aligned} 0 &= \lim_{i\to\infty}\int_M d(\varphi_i\bar{\partial}\{\bar{\partial}u, \bar{\partial}u\}\omega^{n-2}) \\ &= \lim_{i\to\infty}\int_M \varphi_i\partial\bar{\partial}\{\bar{\partial}u, \bar{\partial}u\}\omega^{n-2} + \lim_{i\to\infty}\int_M \partial\varphi_i \wedge \bar{\partial}\{\bar{\partial}u, \bar{\partial}u\}\omega^{n-2} \\ &= 2\lim_{i\to\infty}\int_M \varphi_i(|\partial_E\bar{\partial}u|^2 + Q_0)\omega^n = \int_M (|\partial_E\bar{\partial}u|^2 + Q_0)\omega^n \end{aligned}$$

- Conclude $|\partial_E\bar{\partial}u|^2 = 0$; i.e. u is pluriharmonic

Regularity implies rigidity

Example (Generalization of Siu): Harmonic map $u : M \to X$ from a compact Kähler manifold into a Euclidean building is pluriharmonic.

- Define cut-off functions $\{\varphi_i\}$ with support contained in the regular set $\mathcal{R}(u)$ and with $\lim_{i\to\infty} \varphi_i(x) = 1$ for all $x \in \mathcal{R}(u)$
- Multiply the Bochner formula by φ_i

$$\varphi_i \partial\bar\partial\{\bar\partial u, \bar\partial u\}\omega^{n-2} = 2\varphi_i(|\partial_E \bar\partial u|^2 + Q_0)\omega^n$$

- Justfy integration by parts and take limit:

$$\begin{aligned} 0 &= \lim_{i\to\infty} \int_M d(\varphi_i \bar\partial\{\bar\partial u, \bar\partial u\}\omega^{n-2}) \\ &= \lim_{i\to\infty} \int_M \varphi_i \partial\bar\partial\{\bar\partial u, \bar\partial u\}\omega^{n-2} + \lim_{i\to\infty} \int_M \partial\varphi_i \wedge \bar\partial\{\bar\partial u, \bar\partial u\}\omega^{n-2} \\ &= 2\lim_{i\to\infty} \int_M \varphi_i(|\partial_E \bar\partial u|^2 + Q_0)\omega^n = \int_M (|\partial_E \bar\partial u|^2 + Q_0)\omega^n \end{aligned}$$

- Conclude $|\partial_E \bar\partial u|^2 = 0$; i.e. u is pluriharmonic

Regularity implies rigidity

Non-Archimedean geometric rigidity

Theorem (Gromov-Schoen 1992, Breiner-Dees-M– 2025)

- $\tilde{M} = G/K$, *irreducible symmetric space of noncompact type other than the real and complex hyperbolic space*
- Γ, *discrete subgroup of G such that* $M = \tilde{M}/\Gamma$ *is compact.*
- X, *Bruhat-Tits building of an algebraic group G*
- $\rho : \Gamma \to G$, *representation such that* $\rho(\Gamma)$ *is Zariski dense*

Any ρ*-equivariant harmonic map* $u : \tilde{M} \to X$ *is a constant map.*

Proof of the rank 1 non-Archimedean superrigidity

- $\rho(\Gamma)$ Zariski dense $\Rightarrow$ ρ-equivariant harmonic map u.
- Apply regularity theorem and Bochner formula to prove $u \equiv x \in X$.
    - Mok-Siu-Yeung or Jost-Yau Bochner formula for $\mathrm{rank}(\widetilde{M}) \geqslant 2$
    - Corlette Bochner formula for $\mathrm{rank}(\widetilde{M}) = 1$

Further question

Let $u : \Omega \to X$ be a harmonic map into a Euclidean building.

We showed that if $Ord^u(x) = 1$, then there exists a neighborhood U of x such that $u(U) \subset A$.

Conjecture. For $x \in \Omega$ with $\mathrm{Ord}^u(x) > 1$, there exist a neighborhood U of x and a finite number of apartments $A_1, \ldots, A_k$ of X such that $u(U) \subset A_1 \cup \cdots \cup A_k$.

Further question

Let $u : \Omega \to X$ be a harmonic map into a Euclidean building.

We showed that if $Ord^u(x) = 1$, then there exists a neighborhood U of x such that $u(U) \subset A$.

Conjecture. For $x \in \Omega$ with $\mathrm{Ord}^u(x) > 1$, there exist a neighborhood U of x and a finite number of apartments $A_1, \dots, A_k$ of X such that $u(U) \subset A_1 \cup \cdots \cup A_k$.