

The Impact of Yang's Ideas
on the Geometry and Topology
of Smooth 4-Manifolds

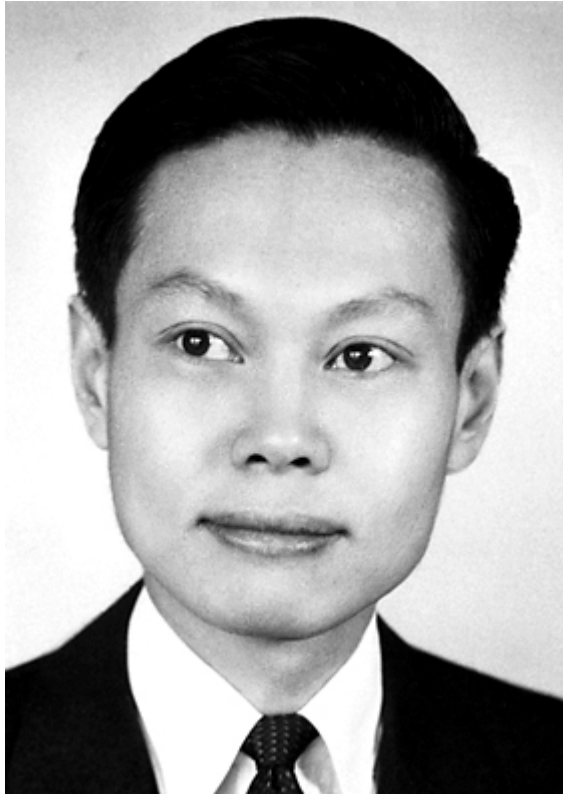
Claude LeBrun
Department of Mathematics
Stony Brook University

C.N. Yang Memorial Symposium
Simons Center for Geometry and Physics
Stony Brook University
October 13, 2026

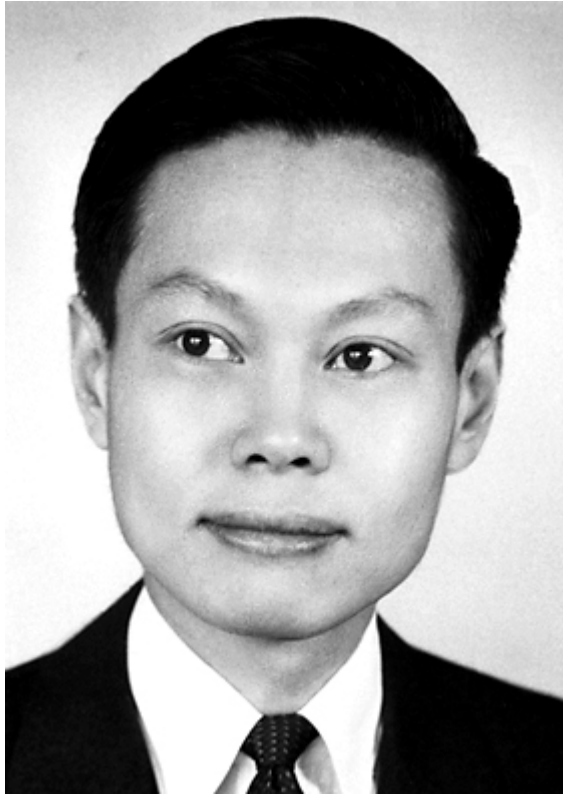


Chen-Ning “Frank” Yang (1922-2025)

Yang-Mills Theory (1954)



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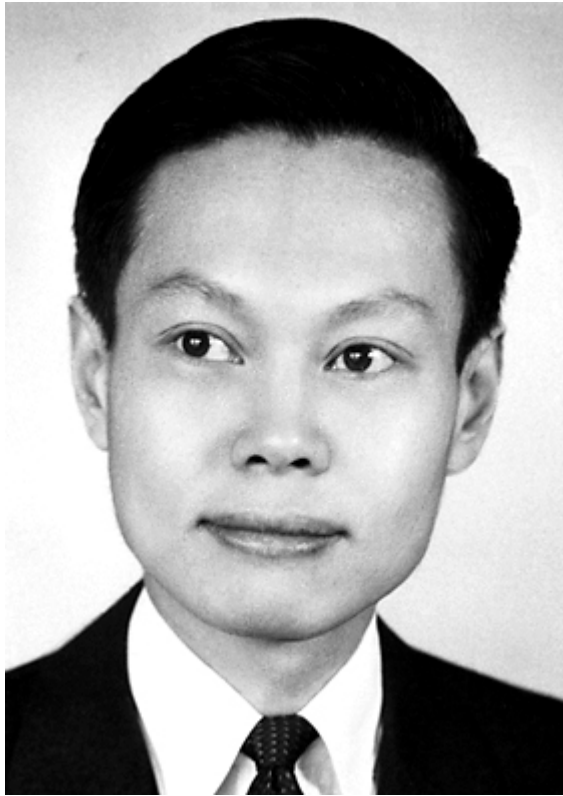


Yang



Mills

Yang-Mills Theory (1954)



Yang



Mills

Officemates at Brookhaven National Laboratories!

Conservation of Isotopic Spin and Isotopic Gauge Invariance*

C. N. YANG † AND R. L. MILLS

Brookhaven National Laboratory, Upton, New York

(Received June 28, 1954)

It is pointed out that the usual principle of invariance under isotopic spin rotation is not consistent with the concept of localized fields. The possibility is explored of having invariance under local isotopic spin rotations. This leads to formulating a principle of isotopic gauge invariance and the existence of a **b** field which has the same relation to the isotopic spin that the electromagnetic field has to the electric charge. The **b** field satisfies nonlinear differential equations. The quanta of the **b** field are particles with spin unity, isotopic spin unity, and electric charge $\pm e$ or zero.

INTRODUCTION

THE conservation of isotopic spin is a much discussed concept in recent years. Historically an isotopic spin parameter was first introduced by Heisenberg¹ in 1932 to describe the two charge states (namely neutron and proton) of a nucleon. The idea that the neutron and proton correspond to two states of the same particle was suggested at that time by the fact that their masses are nearly equal, and that the light

stable even nuclei contain equal numbers of them. Then in 1937 Breit, Condon, and Present pointed out the approximate equality of $p-p$ and $n-p$ interactions in the ¹S state.² It seemed natural to assume that this equality holds also in the other states available to both the $n-p$ and $p-p$ systems. Under such an assumption one arrives at the concept of a total isotopic spin³ which is conserved in nucleon-nucleon interactions. Experi-

* Work performed under the auspices of the U. S. Atomic Energy Commission.

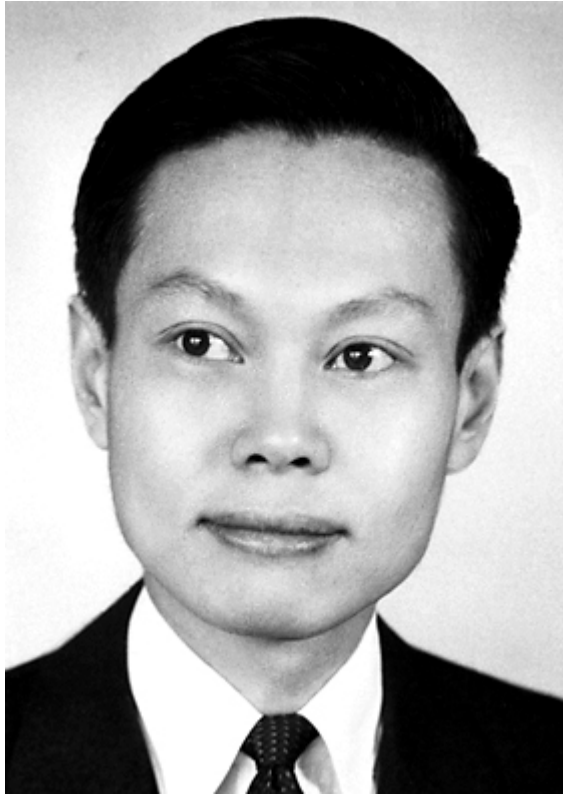
† On leave of absence from the Institute for Advanced Study, Princeton, New Jersey.

¹ W. Heisenberg, *Z. Physik* **77**, 1 (1932).

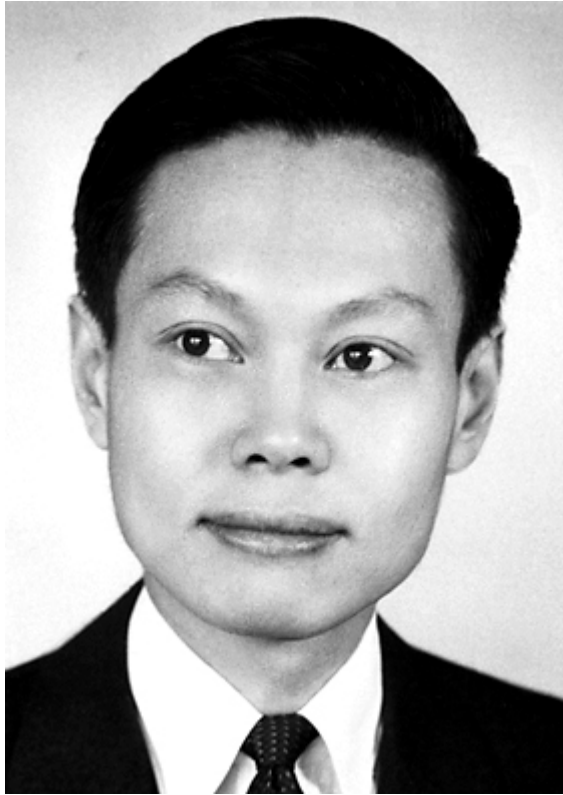
² Breit, Condon, and Present, *Phys. Rev.* **50**, 825 (1936). J. Schwinger pointed out that the small difference may be attributed to magnetic interactions [*Phys. Rev.* **78**, 135 (1950)].

³ The total isotopic spin **T** was first introduced by E. Wigner, *Phys. Rev.* **51**, 106 (1937); B. Cassen and E. U. Condon, *Phys. Rev.* **50**, 846 (1936).

Yang-Mills Theory (1954)

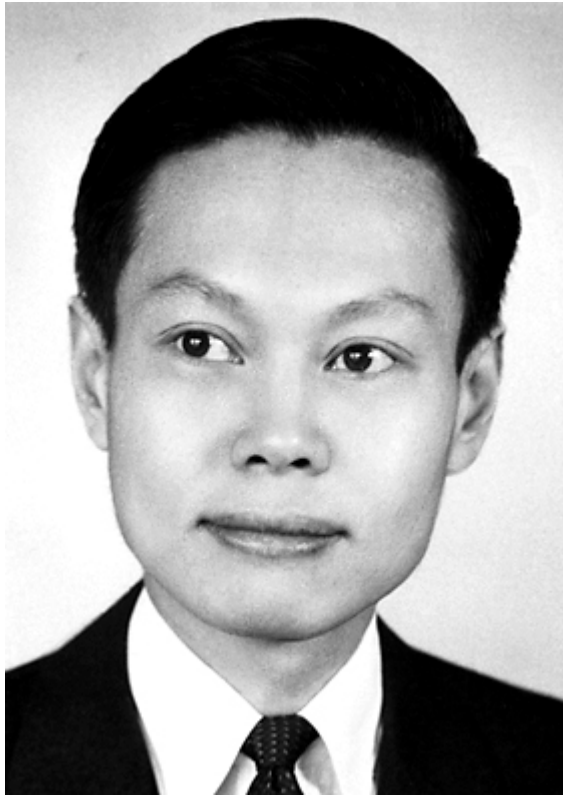


Yang-Mills Theory (1954)



Introduction of non-Abelian gauge theory

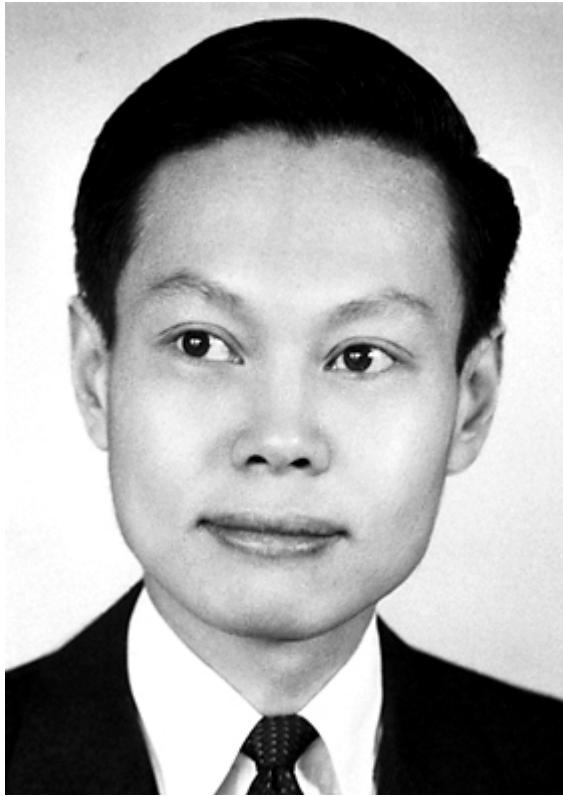
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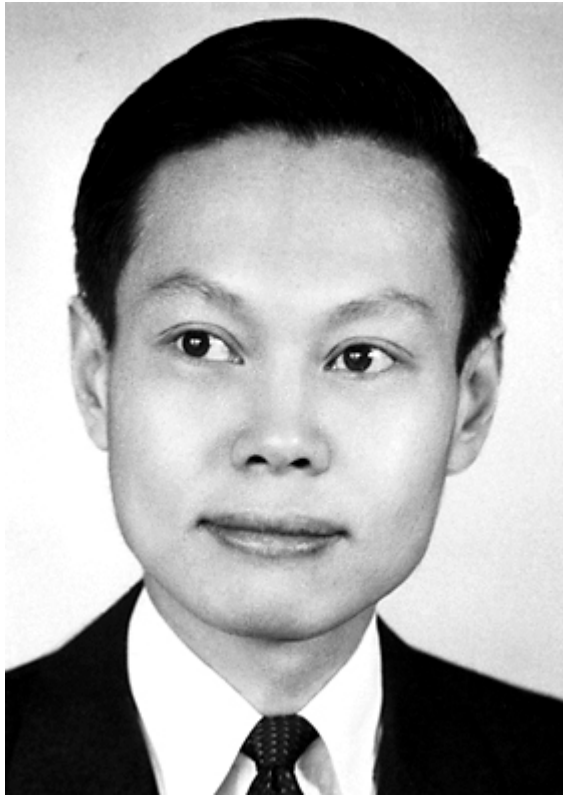


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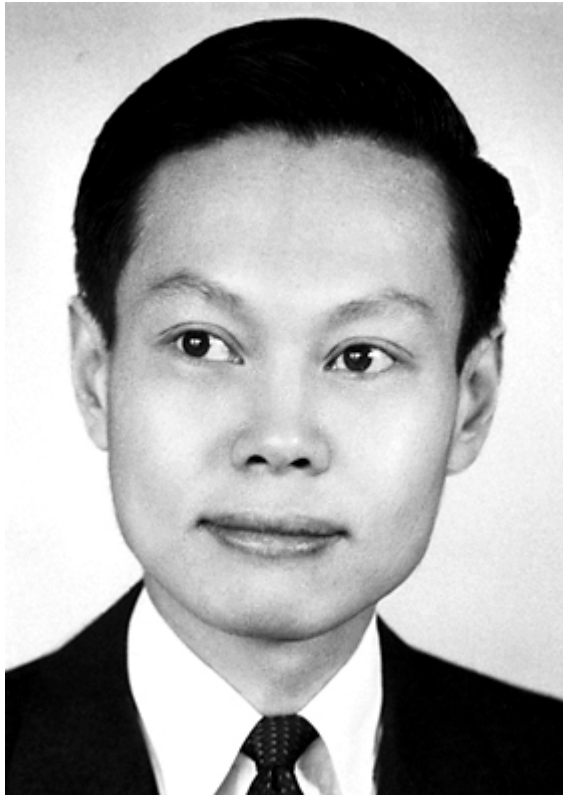
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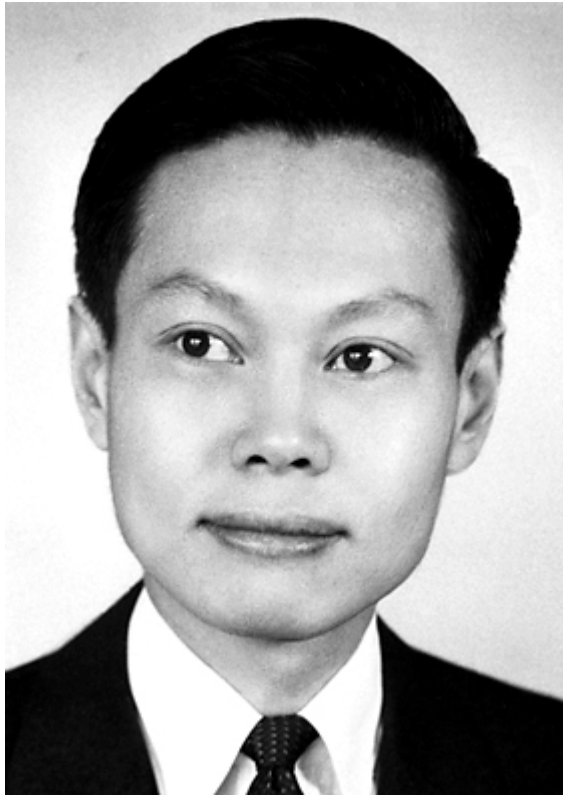
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Introduction of non-Abelian gauge theory

But geometry was missing!

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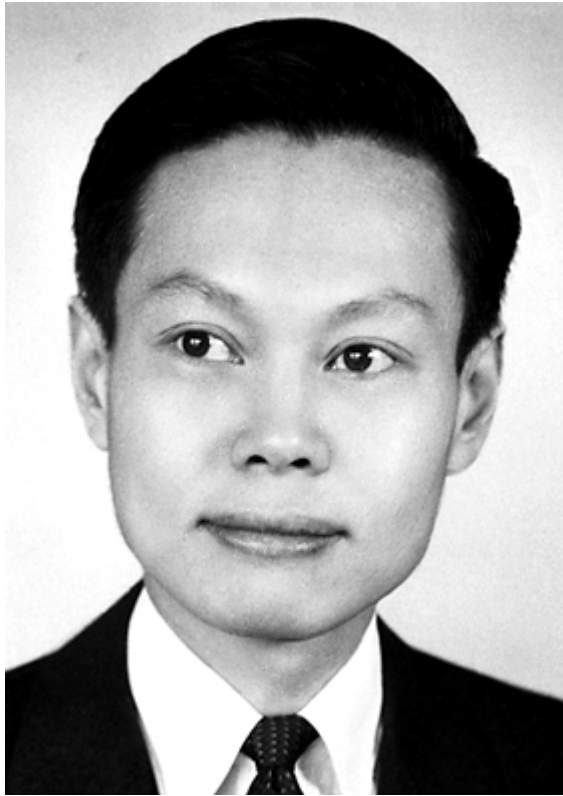


Introduction of non-Abelian gauge theory

But geometry was missing!

Just phrased in terms of “vector potential”

Yang-Mills Theory (1954)



Introduction of non-Abelian gauge theory

$$\textcolor{blue}{F} = d\textcolor{red}{A} + \frac{1}{2}[\textcolor{red}{A}, \textcolor{red}{A}]$$

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(Simons had just been fired by IDA in Princeton
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By 1970, it began to emerge that gauge fields are actually connections in fiber bundles!

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This series of observations later became known as the Wu-Yang dictionary, after they appeared in a joint paper that Yang wrote with T.T. Wu.

Concept of nonintegrable phase factors and global formulation of gauge fields

Tai Tsun Wu*

Gordon McKay Laboratory, Harvard University, Cambridge, Massachusetts 02138

Chen Ning Yang[†]

Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11794

(Received 8 September 1975)

Through an examination of the Bohm-Aharonov experiment an intrinsic and complete description of electromagnetism in a space-time region is formulated in terms of a nonintegrable phase factor. This concept, in its global ramifications, is studied through an examination of Dirac's magnetic monopole field. Generalizations to non-Abelian groups are carried out, and result in identification with the mathematical concept of connections on principal fiber bundles.

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TABLE I. Translation of terminology.

Gauge field terminology	Bundle terminology
gauge (or global gauge)	principal coordinate bundle
gauge type	principal fiber bundle
gauge potential b_μ^h	connection on a principal fiber bundle
S_{ba} (see Sec. V)	transition function
phase factor Φ_{QP}	parallel displacement
field strength $f_{\mu\nu}^h$	curvature
source ^a J_μ^K	?
electromagnetism	connection on a $U_1(1)$ bundle
isotopic spin gauge field	connection on a SU_2 bundle
Dirac's monopole quantization	classification of $U_1(1)$ bundle according to first Chern class
electromagnetism without monopole	connection on a trivial $U_1(1)$ bundle
electromagnetism with monopole	connection on a nontrivial $U_1(1)$ bundle

^a I.e., electric source. This is the generalization (see Ref. 3) of the concept of electric charges and currents.

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Deformations of instantons

M. F. ATIYAH[†], N. J. HITCHIN[†], AND I. M. SINGER^{†‡}

[†] Mathematical Institute, Oxford, OX1 3LB, England; and [‡] Department of Mathematics, University of California, Berkeley, California 94720; and Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Contributed by I. M. Singer, April 20, 1977

ABSTRACT A study is made of the self-dual Yang-Mills fields in Euclidean 4-space. For $SU(2)$ gauge theory it is rigorously shown that the solutions depend on $8k - 3$ parameters, where k is the Pontrjagin index.

There has been considerable interest recently in the instanton or pseudo-particle solutions of the classical Yang-Mills equations in Euclidean 4-space (refs. 1, 2, and 3). In geometrical terms, these equations are the variational equations for the norm-square $\|F\|^2$ of the curvature F of a fiber-bundle with group G and connection A over R^4 . In physics terminology, $\|F\|^2$ is the action, F the gauge field, A the gauge potential, and G the gauge group. The cases studied in most detail are for $G = SU(n)$ and, particularly, $G = SU(2)$.

The connection A is assumed to be asymptotically flat in an appropriate sense so that $F \rightarrow 0$ at ∞ and $\|F\|^2 < \infty$. Since the variational equations are conformally invariant with respect to change in the metric on R^4 , the most natural geometrical restriction to impose on A at ∞ is that it extends to a connection for a bundle over the 4-sphere S^4 . The topological type of such a bundle is then determined by a homotopy class of maps $S^3 \rightarrow G$, which is given by an integer k when $G = SU(n)$ (and more generally for any simple compact nonabelian Lie group): this is referred to by physicists as the Pontrjagin index (differing from the topologist's terminology by a factor of 2).

Using the duality $\star$ -operator on R^4 or S^4 , we can decompose F into $F^+ \oplus F^-$, where $\star F^+ = F^+$ and $\star F^- = -F^-$. Clearly, $\|F\|^2 = \|F^+\|^2 + \|F^-\|^2$, while the Pontrjagin index k is given by[§]

$$k = \frac{1}{8\pi^2} (\|F^+\|^2 - \|F^-\|^2).$$

Hence, $\|F^+\|^2 \geq 8\pi^2 k$, and the minimum is attained only if $F^- = 0$ or $F^+ = 0$. Solutions with $F^- = 0$ are called self-dual solutions and have been constructed for all $k \geq 0$. For $k = 0$ we have the trivial solution $F = 0$, for $k = 1$ we have the "instanton," and for $k > 1$ we have "multi-instantons." The most general explicit solutions constructed so far are those of Jackiw *et al.* (ref. 3), which depend on $5k + 4$ parameters.[¶] Our main result is that the complete set of solutions depends on $8k - 3$ parameters. This confirms some preliminary results of Jackiw and Rebbi (ref. 4) and Schwartz (ref. 5).

RESULTS

If a connection A yields a self-dual Yang-Mills field F , then so does any connection $g(A)$ where g is a bundle automorphism (or gauge transformation). The space of all solutions A modulo the action of this gauge group will, as usual in such geometric problems, be called the space of moduli. Our main result can now be formulated as a precise theorem:

THEOREM. *The space of moduli of self-dual $SU(2)$ -Yang-Mills fields over S^4 , with Pontrjagin index $k \geq 1$, is a manifold of dimension $8k - 3$.*

The standard deformation theory approach to such problems is to consider the linearized equations modulo the infinitesimal gauge transformations. Here this leads to a three-step elliptic complex

$$0 \rightarrow \mathcal{G} \xrightarrow{D_0} \mathcal{G} \otimes \Omega^1 \xrightarrow{D_1} \mathcal{G} \otimes \Omega^2 \rightarrow 0$$

where Ω^1 denotes 1-forms, Ω^2 denotes anti-self-dual 2-forms on S^4 , and $\mathcal{G}$ is the Lie algebra of G . The operator D_0 is the covariant derivative and D_1 is the anti-self-dual part of the covariant derivative ($D_1 D_0 = 0$ because we are using a self-dual connection). The index theorem of Atiyah-Singer (ref. 6) yields the alternating sum formula $h^0 - h^1 + h^2 = 3 - 8k$. Here h^0 is just the dimension of the null space of D_0 , and this is zero unless the $SU(2)$ -bundle is trivial (which is excluded for $k \geq 1$). h^1 is the potential number of moduli and h^2 is the dimension of the null space of D_1^* . Fortunately, in our case, a Bochner type vanishing theorem works very well and we find $h^2 = 0$. This gives $h^1 = 8k - 3$, showing that this is the dimension of the solutions of the linearized problem.

Now we appeal to the general theorem of Kuranishi (ref. 7), which, when $h^2 = 0$, guarantees that the infinitesimal variations really integrate to give genuine local variations. Moreover, the Kuranishi theorem asserts that the family of solutions thus obtained is (locally) complete and effective (non-redundant). This leads to the theorem as stated above.

Note. The theorem does not assert that, for each k , the space of moduli is connected: In principle it may have several components. For $k = 1$ it is in fact connected and is the hyperbolic 5-space. See Yang (ref. 8) for a different discussion of the case $k = 1$.

FURTHER REMARKS

The above arguments apply equally to $SU(n)$, provided we have an irreducible connection (so that $h^0 = 0$), i.e., one that does not come trivially from $SU(n-1)$. We then find $h^1 = 4nk - n^2 + 1$. Moreover, the existence of irreducible $SU(n)$ solutions can be deduced from this formula provided $k \geq (n-1)/2$. In the opposite direction one can deduce nonexistence for $k < n/4$.

Similar methods, i.e., index theorem plus vanishing theorem, yield a formula for the dimension d of the space of zero-eigenvalue fermions (harmonic spinors): one finds $d = k$.

The problem of explicitly constructing the $(8k-3)$ -parameter families of solutions, whose existence is asserted by our theorem, can be treated by converting it into a problem in algebraic geometry (M. F. Atiyah and R. Ward, unpublished).

The 4-sphere can be replaced by other 4-manifolds M (compact, oriented, Riemannian). If M is also a spin-manifold, then we have the two spin $SU(2)$ -bundles, P^+ , P^- . One can show that P^+ (with the Riemannian connection) is self-dual if

[§] Physicists use a different norm and get a factor $1/(16\pi^2)$.

[¶] The formula is a little different for $k = 1, 2$.

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PHYSICAL REVIEW LETTERS

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Condition of Self-Duality for SU(2) Gauge Fields on Euclidean Four-Dimensional Space*

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(Received 15 April 1977)

The condition of self-duality for a SU(2) gauge field on Euclidean four-dimensional flat space is integrated once to obtain Laplace-like equations for three real variables or for variables one real and one complex. A special case of the latter reduces to the Corrigan-Fairlie-Wilczek-'t Hooft Ansatz.

There has been great interest in recent years in sourceless gauge fields. A self-dual gauge field is sourceless. We shall consider a flat space $ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$ and consider a self-dual SU(2) gauge field. The main purpose of this Letter is to show that the condition for self-duality of the field can be integrated once, resulting in a set of Laplace-like equations for three unknown functions. All considerations are local in character, and do not refer to global properties.

Consider a self-dual gauge potential b_μ^i with field strengths

$$f_{\mu\nu}^i = b_{\mu,\nu}^i - b_{\nu,\mu}^i - C_{jk}^i b_\mu^j b_\nu^k. \quad (1)$$

I shall write $X_i = -\frac{1}{2}i\sigma_i$, where σ_i are the 2×2 Pauli matrices. Define $F_{\mu\nu} = f_{\mu\nu}^i X_i$ and $B_\mu = b_\mu^i X_i$. Then

$$F_{\mu\nu} = B_{\mu,\nu} - B_{\nu,\mu} - B_\mu B_\nu + B_\nu B_\mu. \quad (2)$$

Now consider an analytic continuation of b_μ^i into complex space where x_1, x_2, x_3 , and x_4 are complex. The condition of self-duality

$$2F_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta} \quad (3)$$

is then valid also in complex space, in a region containing real space where the x 's are real.

Now consider four new complex variables $y, \bar{y},$

z , and $\bar{z}$ defined by

$$\begin{aligned} \sqrt{2}y &= x_1 + ix_2, & \sqrt{2}\bar{y} &= x_1 - ix_2, \\ \sqrt{2}z &= x_3 - ix_4, & \sqrt{2}\bar{z} &= x_3 + ix_4. \end{aligned} \quad (4)$$

The Jacobian of the transformation $(x_1, x_2, x_3, x_4) \rightarrow (y, \bar{y}, z, \bar{z})$ is 1. The metric in the new variables has only the following nonvanishing elements:

$$g_{y\bar{y}} = g_{\bar{y}y} = g_{z\bar{z}} = g_{\bar{z}z} = 1. \quad (5)$$

It is simple to check that the condition of self-duality reduces to

$$F_{y\bar{z}} = F_{\bar{y}z} = 0, \quad (6)$$

$$F_{y\bar{y}} + F_{z\bar{z}} = 0. \quad (7)$$

Equation (6) implies that for fixed $\bar{y}$ and $\bar{z}$, b_y^i and b_z^i form a fieldless gauge potential in the (complex) two-dimensional space of y and z . In that space then, the nonintegrable phase factor¹ becomes integrable. i.e.,

$$\begin{aligned} D_{QP} &= (1 + B_y dy + B_z dz)_Q \\ &\quad \cdots (1 + B_y dy + B_z dz)_P \end{aligned} \quad (8)$$

is independent of the path from P to Q and is only dependent on the endpoints Q and P . Fixing Q and differentiating with respect to the coordinates y and z of P , I obtain

$$B_y = D^{-1} D_y, \quad B_z = D^{-1} D_z, \quad (9)$$

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By the [Atiyah-Singer Index Theorem](#) for elliptic operators on compact manifolds,

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The proof reduces the problem to classifying suitable holomorphic vector bundles on $\mathbb{CP}_3$, using a twistor correspondence discovered by Richard Ward, inspired by ideas of Roger Penrose.

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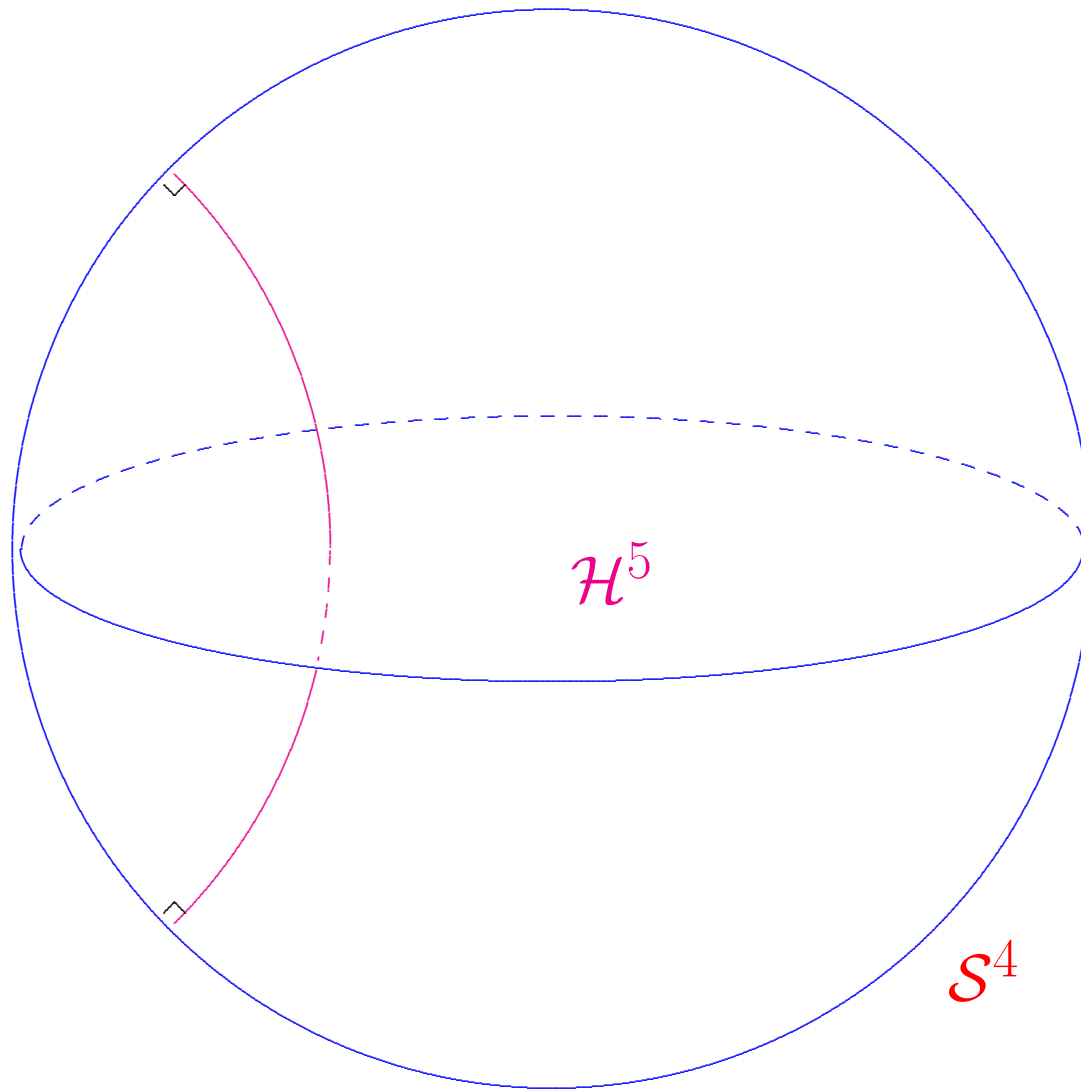
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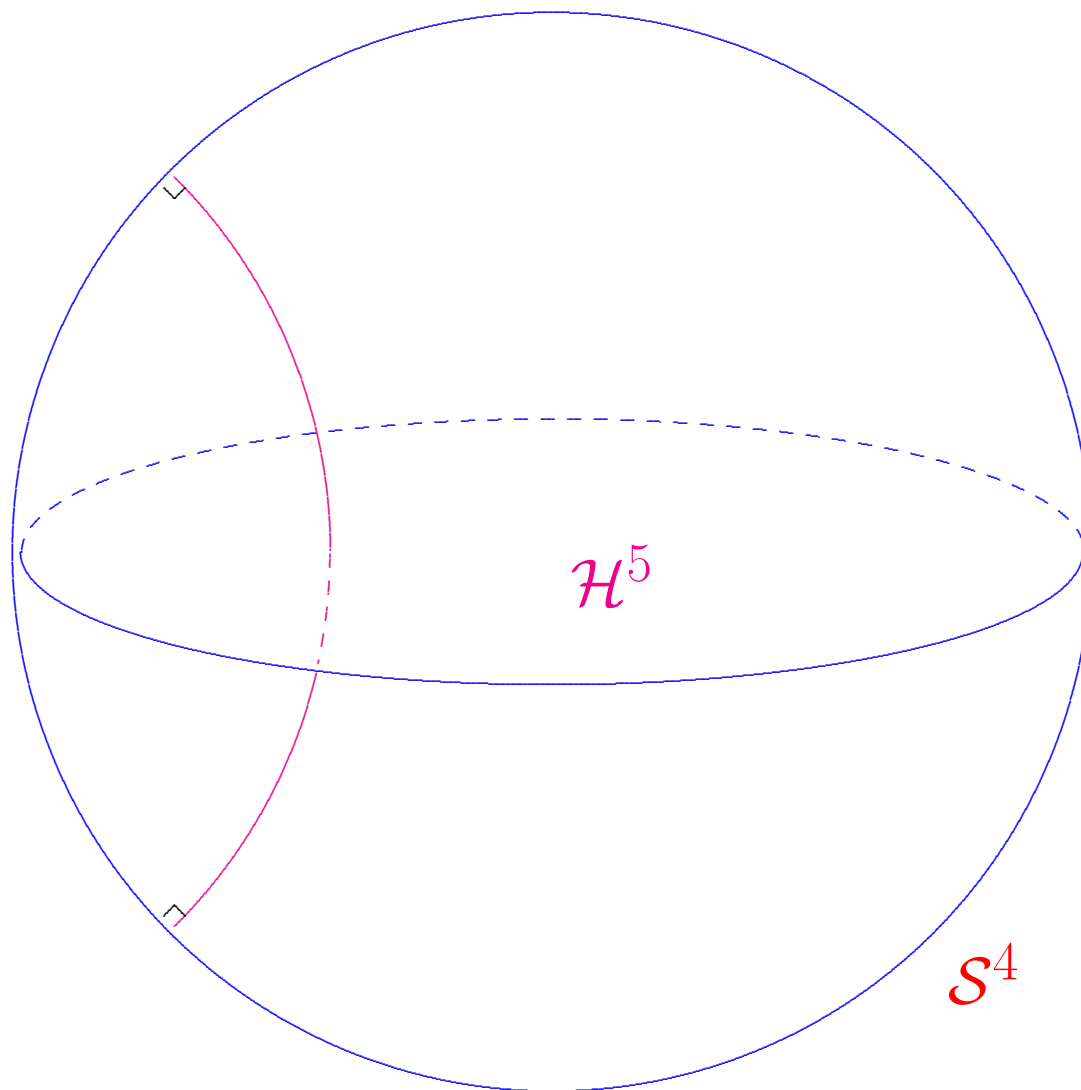
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This Ward correspondence is in fact foreshadowed by some of Yang’s comments about integrability.

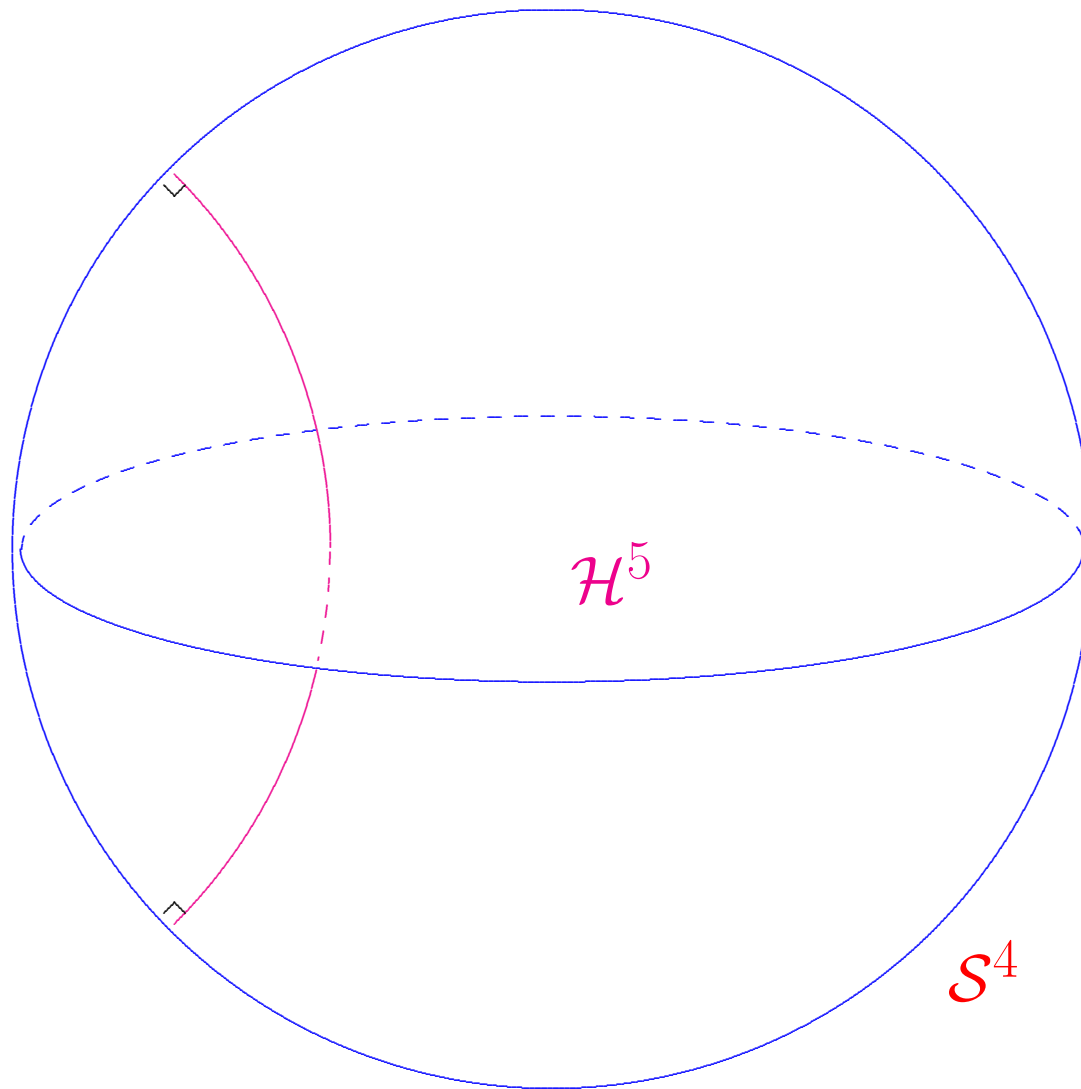
Poincaré Model of Hyperbolic 5-Space



Naturally parameterizes $k = 1$ Instantons



But this only works for standard (S^4, g_0) !



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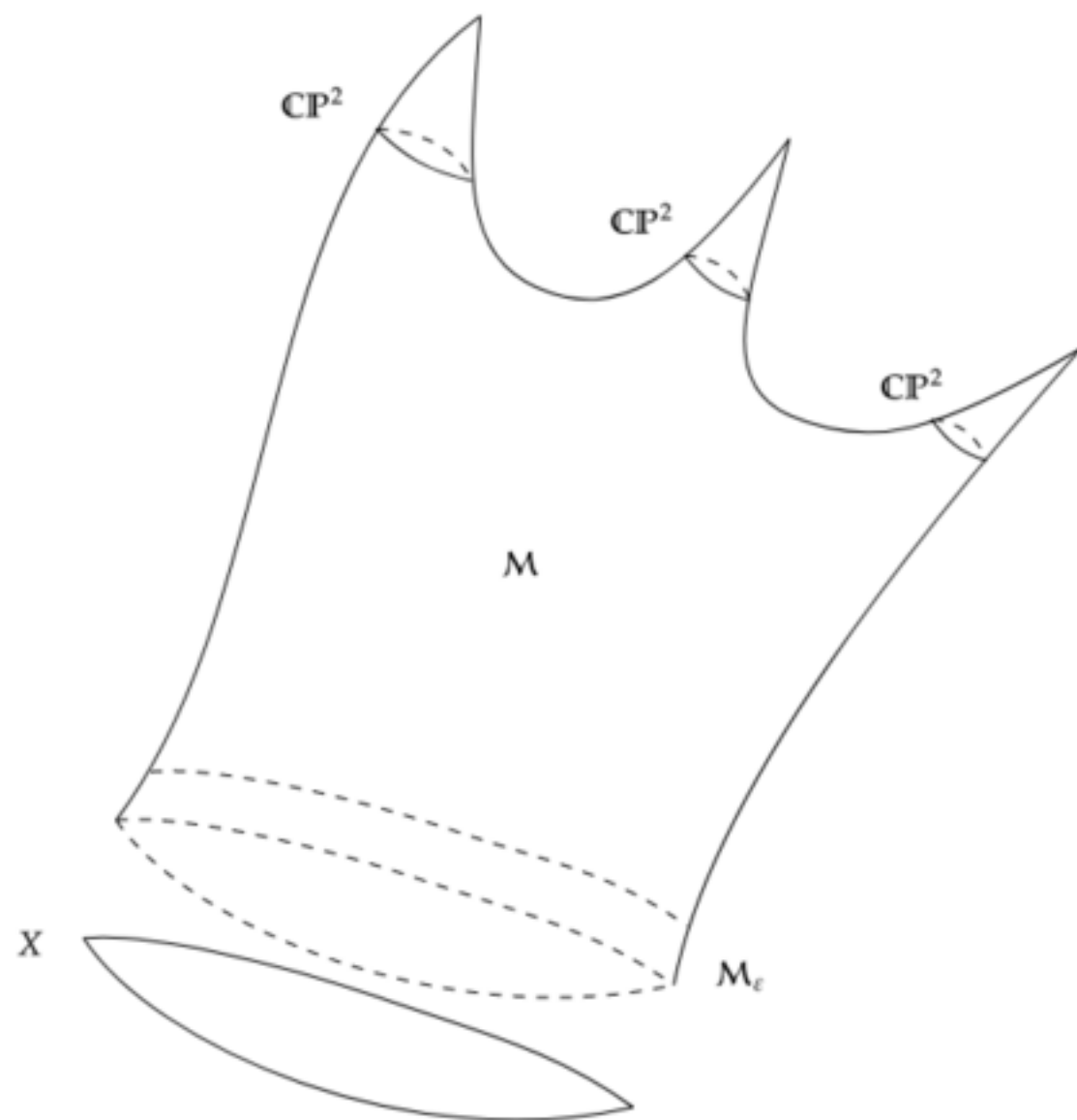
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This allowed him to prove...

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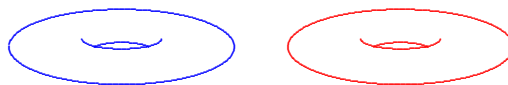
Then X is *homeomorphic* to a connected sum

$$m\mathbb{CP}^2 = \underbrace{\mathbb{CP}^2 \# \cdots \# \mathbb{CP}^2}_m,$$

$\mathbb{CP}^2$ = complex projective plane.

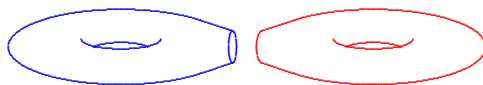
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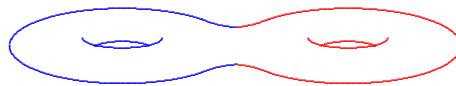
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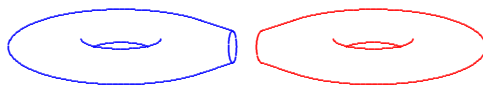
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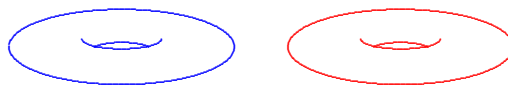
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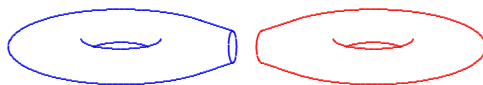
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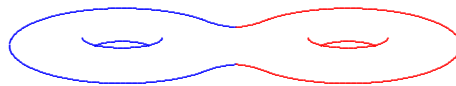
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Warning: homeomorphic $\not\Rightarrow$ diffeomorphic!

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who showed $\Rightarrow \mathbb{R}^4$ has an exotic smooth structure!

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Then used algebraic geometry to calculate these
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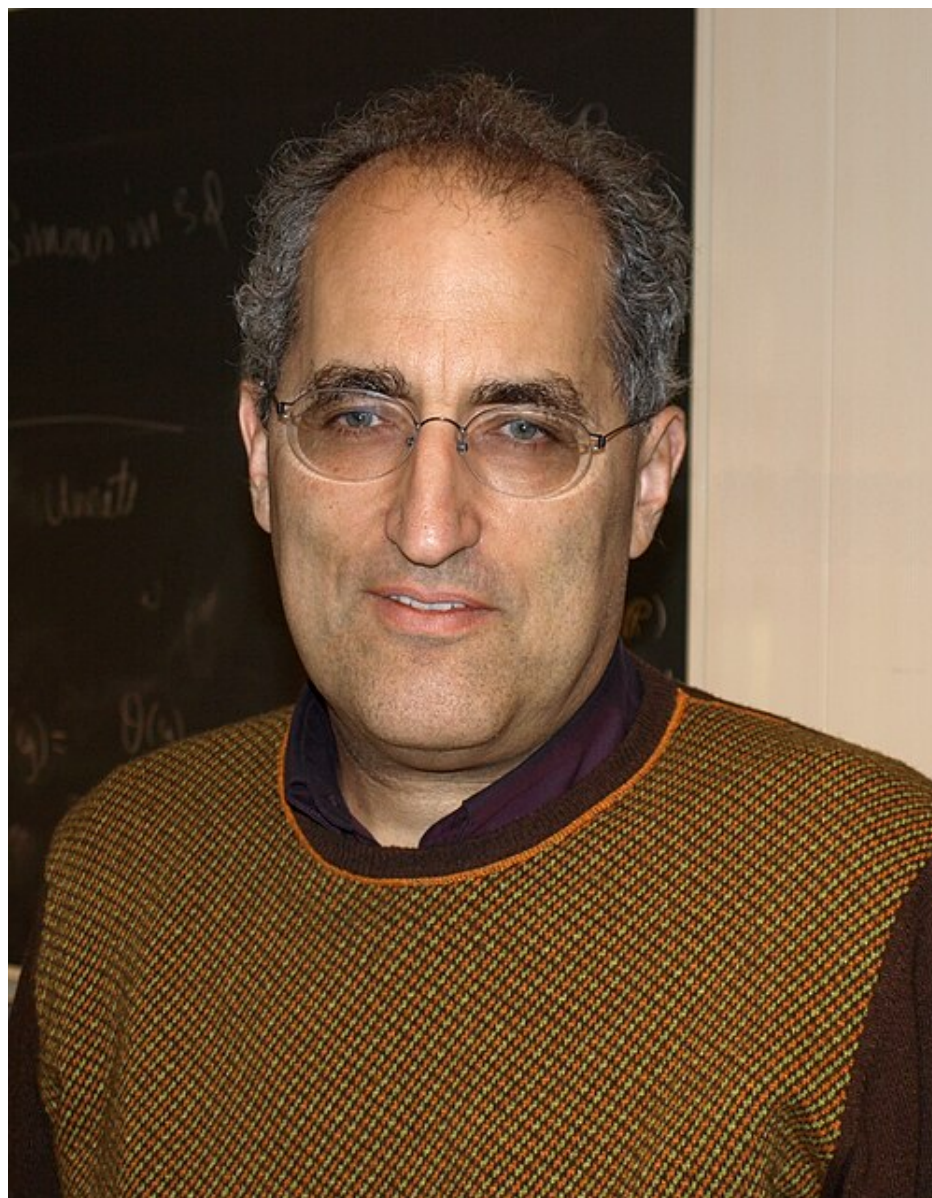
This does not happen in any dimension $\neq 4$!!!!

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