Anti-Self-Dual 4-Manifolds,

Quasi-Fuchsian Groups, &

Almost-Kähler Geometry

Claude LeBrun
Stony Brook University

Mathematical Sciences Colloquium,
UT Dallas. October 21, 2024.

Main references:

C.J. Bishop and C. LeBrun

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Anti-Self-Dual 4-Manifolds, Quasi-Fuchsian Groups, and Almost-Kähler Geometry

Comm. An. Geom. 28 (2020) 745-780

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Anti-Self-Dual Metrics and Kähler Geometry

Proc. Int. Cong. Math. 1994, vol. 1
Birkhäuser, 1995, pp. 498–507.

Facets of Kähler Geometry:

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Complex Geometry

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Symplectic Geometry

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Metric Geometry

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$$\begin{tikzcd}
\text{Complex Geometry} \arrow[rr, shift left] \arrow[dr, shift left] & & \text{Symplectic Geometry} \arrow[ll, shift left] \arrow[dl, shift left] \\
& \text{Metric Geometry} \arrow[ul, shift left] \arrow[ur, shift left] &
\end{tikzcd}$$

Facets of Kähler Geometry:

$$\begin{tikzcd}
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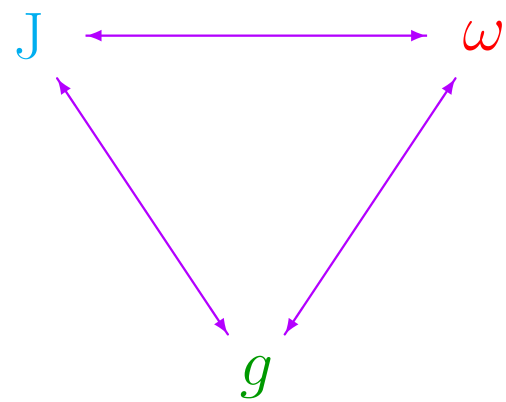

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Linked to conformal geometry in dimension 4.

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Higher dimensions are demonstrably different.

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Imitates Kähler geometry in a non-Kähler setting.

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$$\text{Kähler} \Longleftrightarrow J \text{ integrable.}$$

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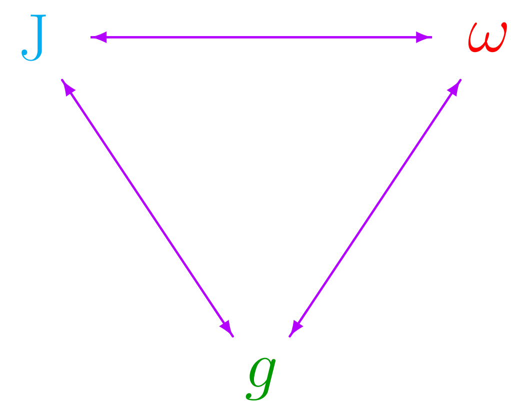

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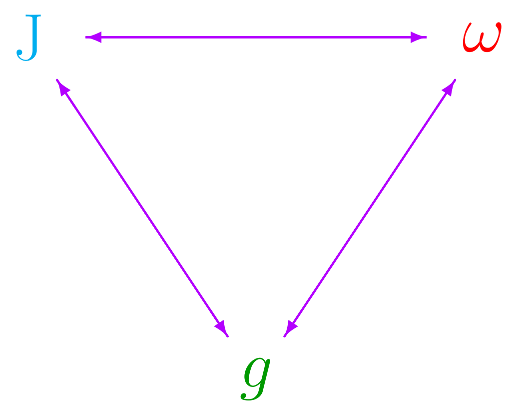

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For example, can avoid explicitly mentioning J.

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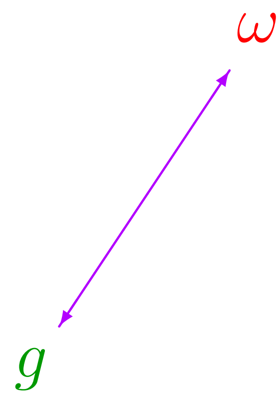
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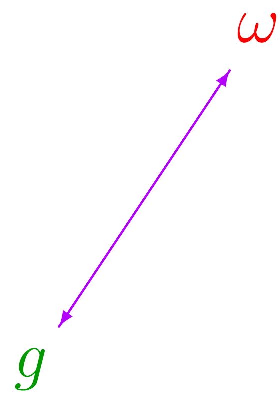
Simplifies dramatically when $m = 2$:



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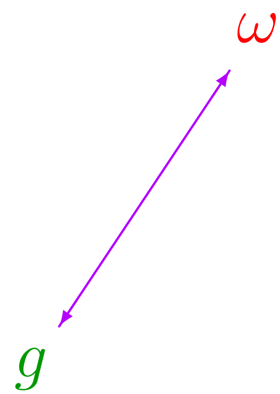

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Moreover, the set of conformal classes $[g]$ *on* M *that carry such a harmonic form* ω *is open in the* C^2 *topology.*

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In particular, the numbers

$$b_\pm(M) = \dim \mathcal{H}^\pm_g$$

are independent of g, and so are invariants of M.

$b_{\pm}(M)$?

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Diagonalize:

$$
\begin{bmatrix}
+1 & & & & & \\
& \ddots & & & & \\
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$$\left[\begin{array}{cc} \underbrace{\begin{matrix} +1 & & \\ & \ddots & \\ & & +1 \end{matrix}}_{b_+(M)} & \\ & b_-(M) \left\{ \begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix} \right. \end{array} \right] .$$

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 & & & & \ddots & \\
 & & & & & -1
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$$

$$
b_2(M) = b_+(M) + b_-(M)
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$$\tau(M) = b_+(M) - b_-(M)$$

"Signature" of M.

Signature defined in terms of intersection pairing

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(Thom-Hirzebruch Signature Formula)

Signature defined in terms of intersection pairing,

but also expressible as a curvature integral:

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Has major consequences in conformal geometry.

On oriented (M^4, g),

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

Riemann curvature of g

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 & \Lambda^{+*} & \Lambda^{-*} \\ \hline
\Lambda^+ & W_+ + \frac{s}{12} & \mathring{r} \\ \hline
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\end{array}$$

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$$\begin{array}{c|c|c|}
 & \Lambda^{+*} & \Lambda^{-*} \\
\hline
\Lambda^{+} & W_+ + \frac{s}{12} & \mathring{r} \\
\hline
\Lambda^{-} & \mathring{r} & W_- + \frac{s}{12} \\
\hline
\end{array}$$

where

$$\begin{aligned}
s &= \text{scalar curvature} \\
\mathring{r} &= \text{trace-free Ricci curvature} \\
W_+ &= \text{self-dual Weyl curvature} \\
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\end{aligned}$$

Riemann curvature of g

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s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W_+ = self-dual Weyl curvature *(conformally invariant)*

W_- = anti-self-dual Weyl curvature ″

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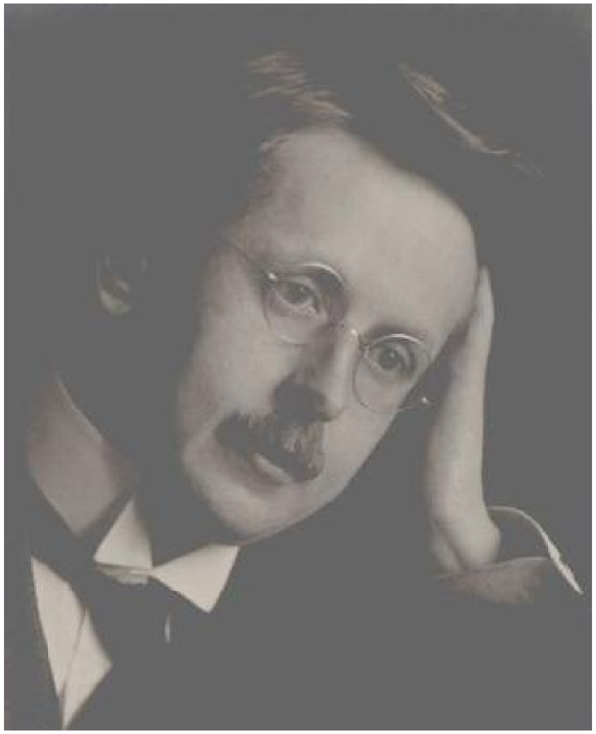

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Basic problems: For given smooth compact M^4,

- What is $\inf \mathscr{W}$?
- Do there exist minimizers?

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But we've already noted that

$$12\pi^2\tau(M) = \int_M \left(|W_+|^2 - |W_-|^2\right) d\mu_g$$

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So Weyl functional is essentially equivalent to

$$[g] \longmapsto \int_M |W_+|^2 d\mu_g$$

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(ASD)

Twistor picture of anti-self-duality condition:

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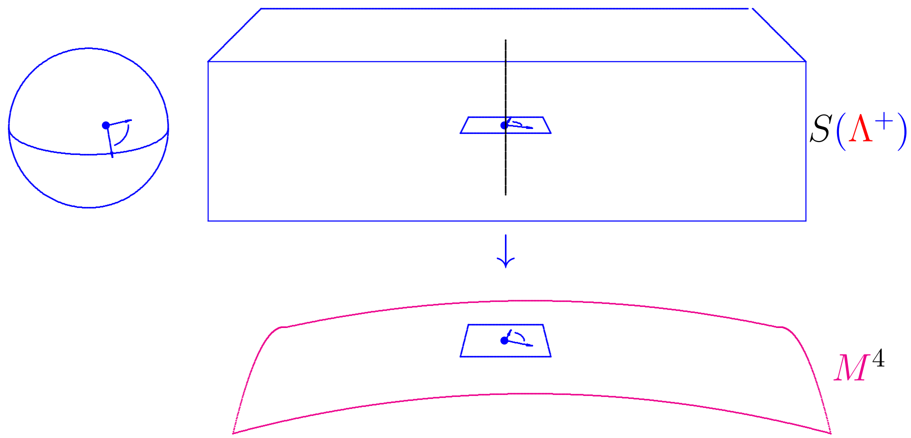

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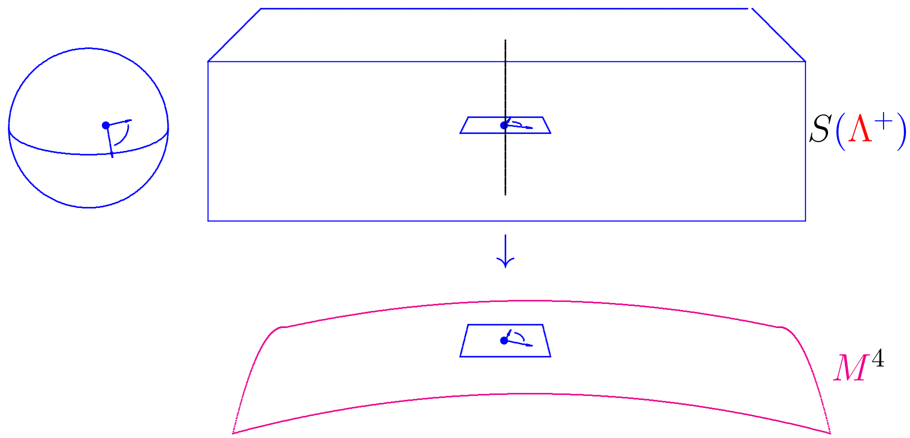

Theorem (Atiyah-Hitchin-Singer). (Z, J) *is a complex 3-manifold iff* $W_+ = 0$.

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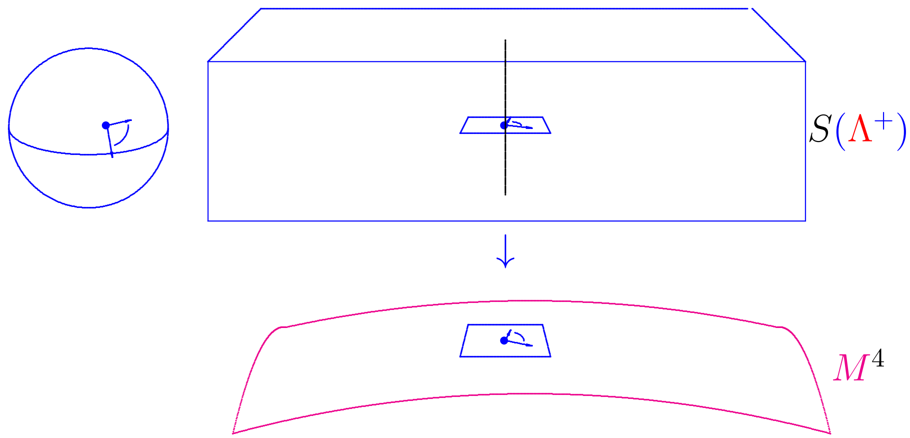

Theorem (Atiyah-Hitchin-Singer). *(Z, J) is a complex 3-manifold iff $W_+ = 0$.*

Reconceptualizes earlier work by Penrose.

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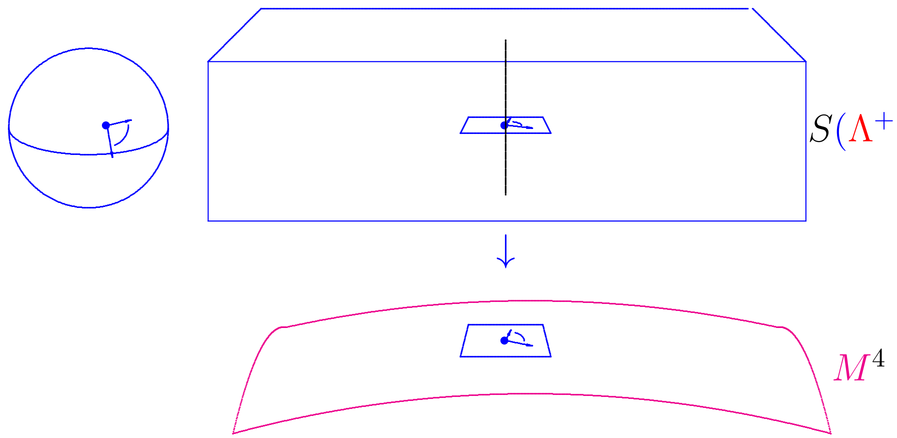

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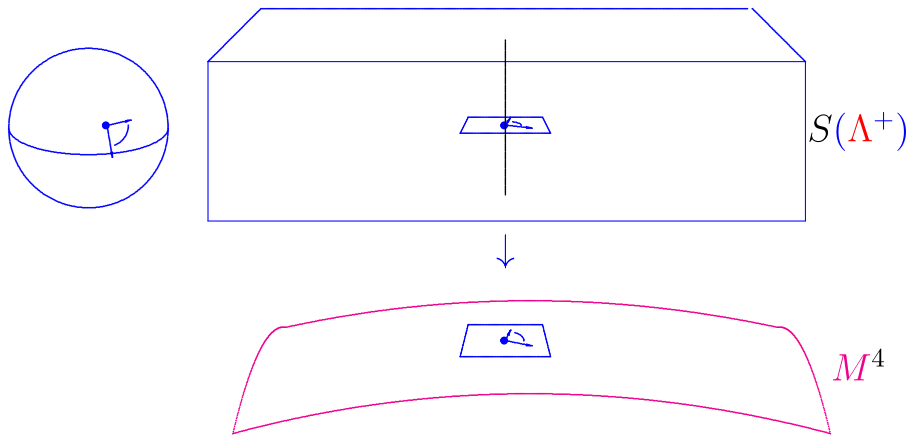

Theorem (Atiyah-Hitchin-Singer). (Z, J) *is a complex 3-manifold iff* $W_+ = 0$.

Motivates study of ASD metrics,
and yields methods for constructing them.

So ASD metrics are linked to complex geometry. . .

A different link with complex geometry:

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Results proved about SFK in '90s foreshadowed
many more recent results about general case.

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Convention:

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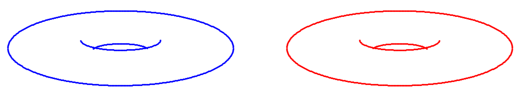

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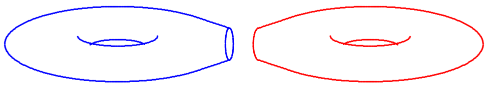

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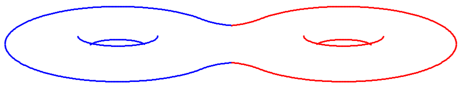

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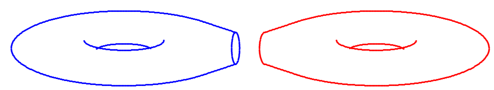

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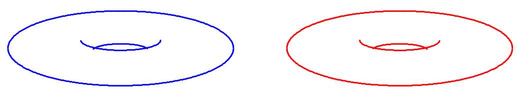

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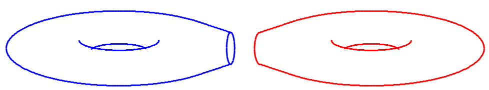

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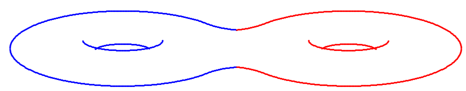

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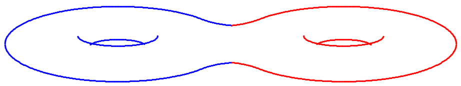

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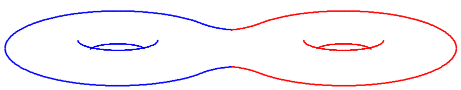

Blowing Up: $M \rightsquigarrow M\#\overline{\mathbb{CP}}_2$

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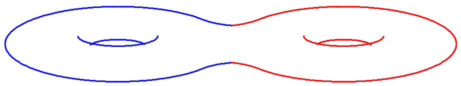

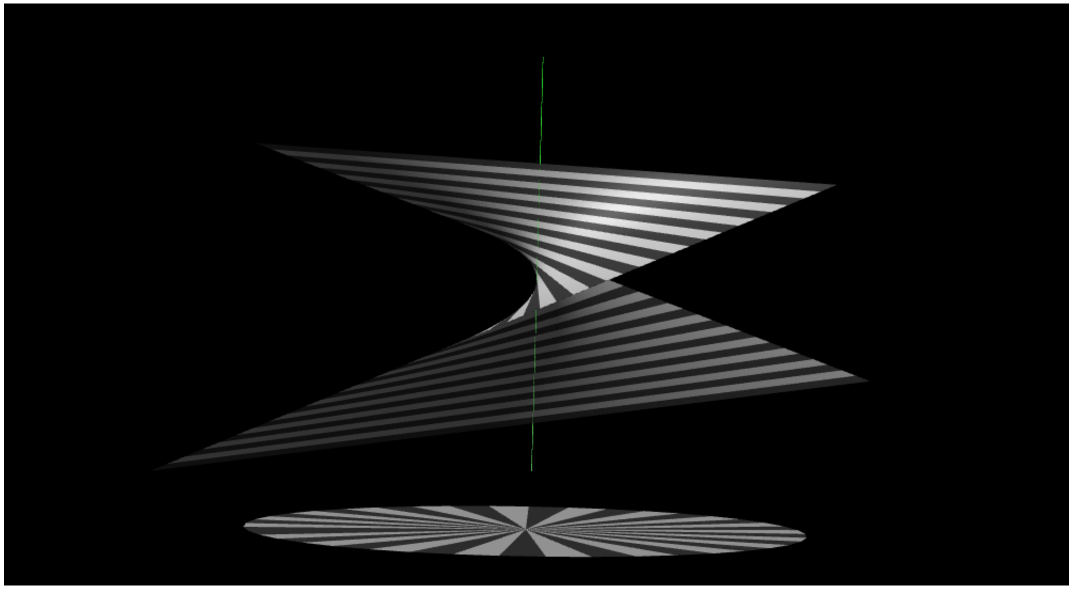

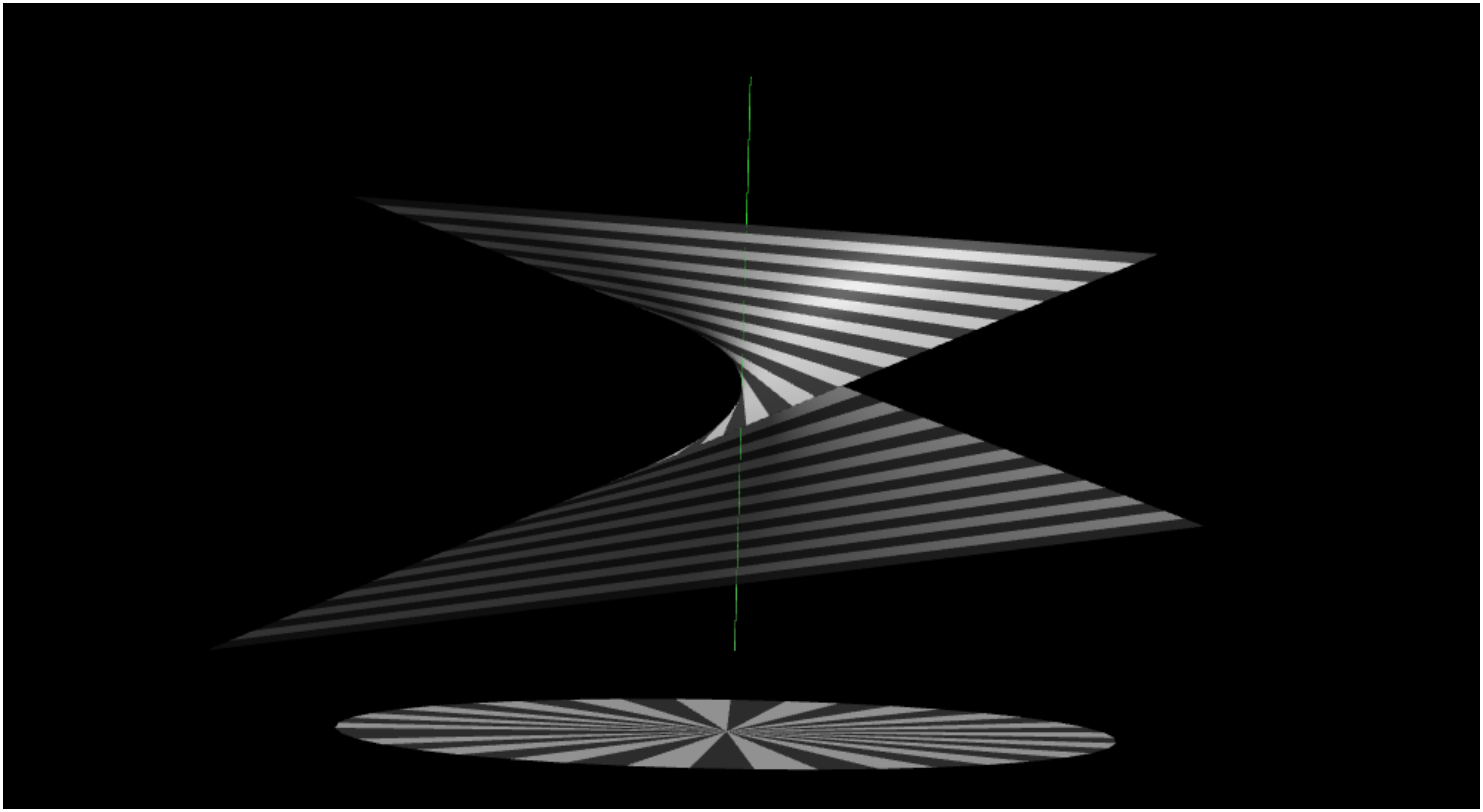

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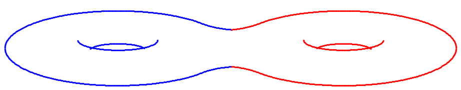

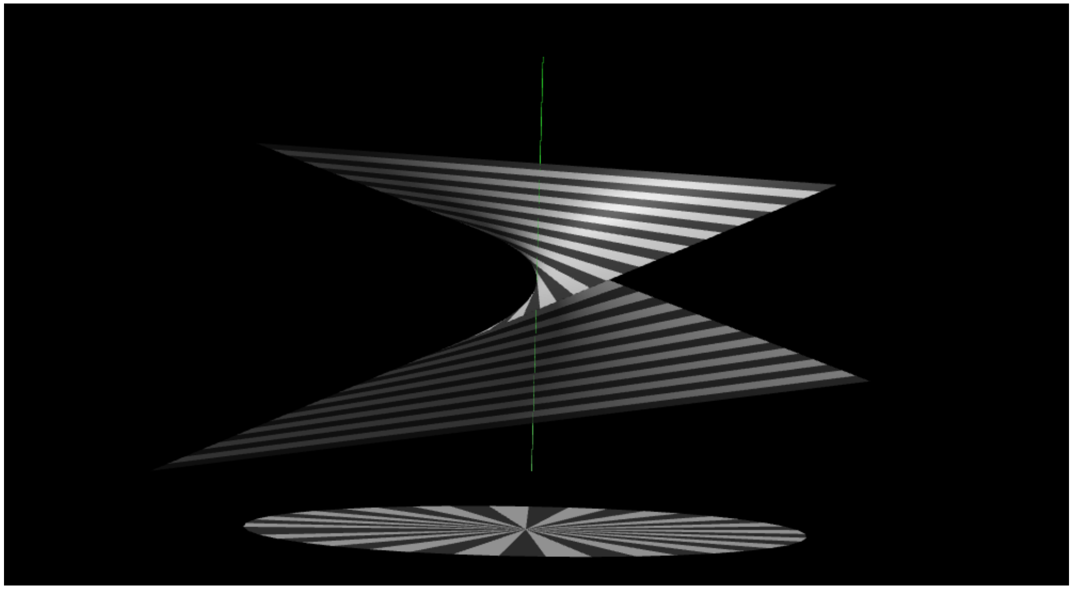

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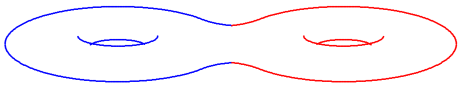

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L '15, I. Kim '16: classification of almost-Kähler ASD roughly the same as in scalar-flat Kähler case.

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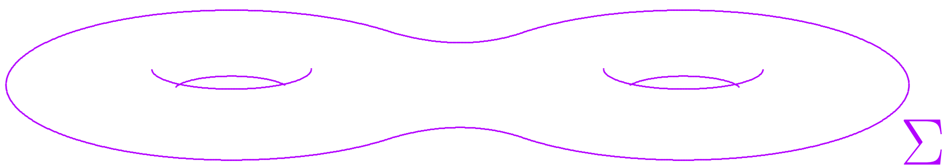
Almost-Kähler ASD metrics sweep out an open set in the ASD moduli space.

Example.



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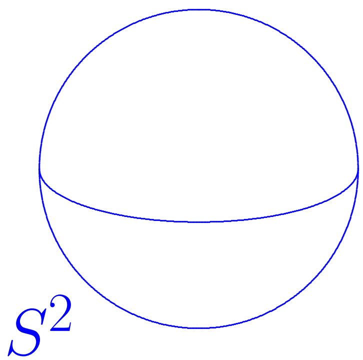
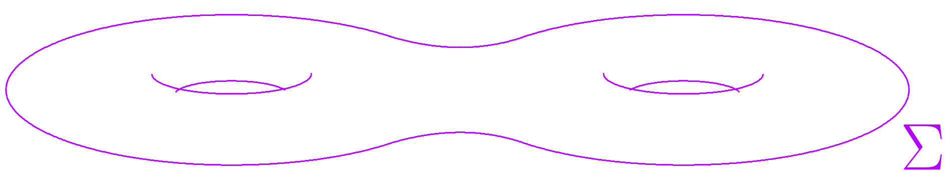
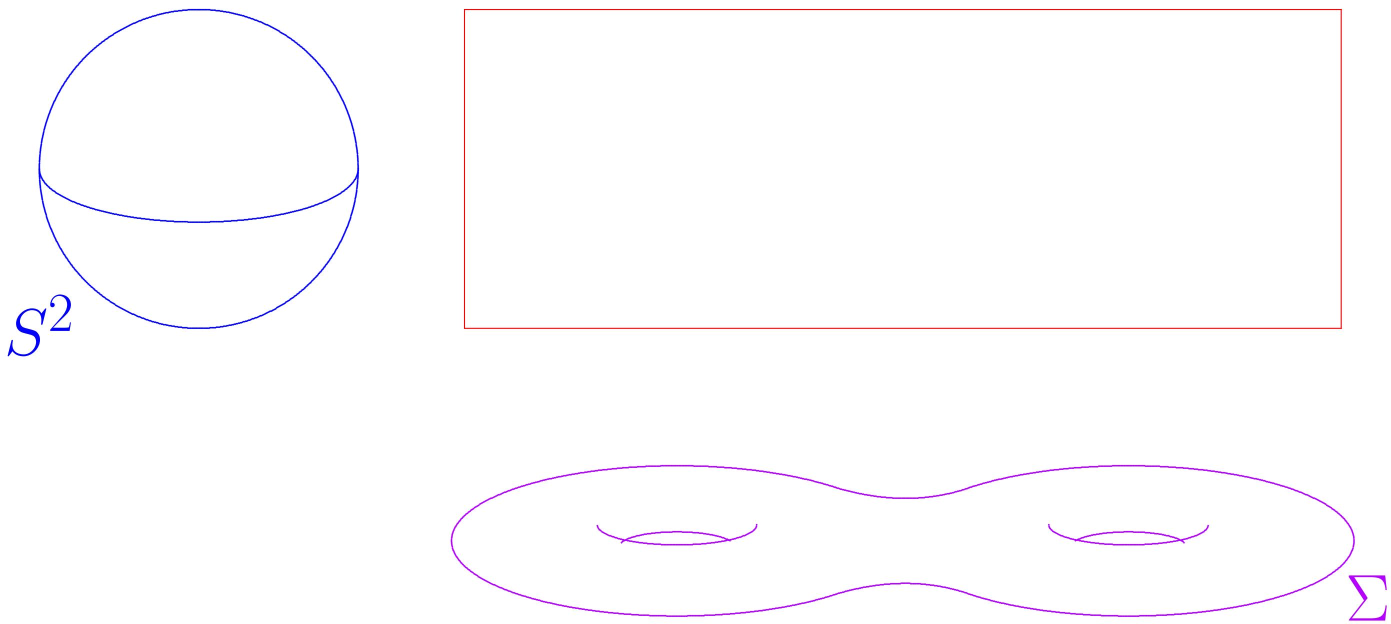




Example.

$$M = \Sigma \times S^2$$



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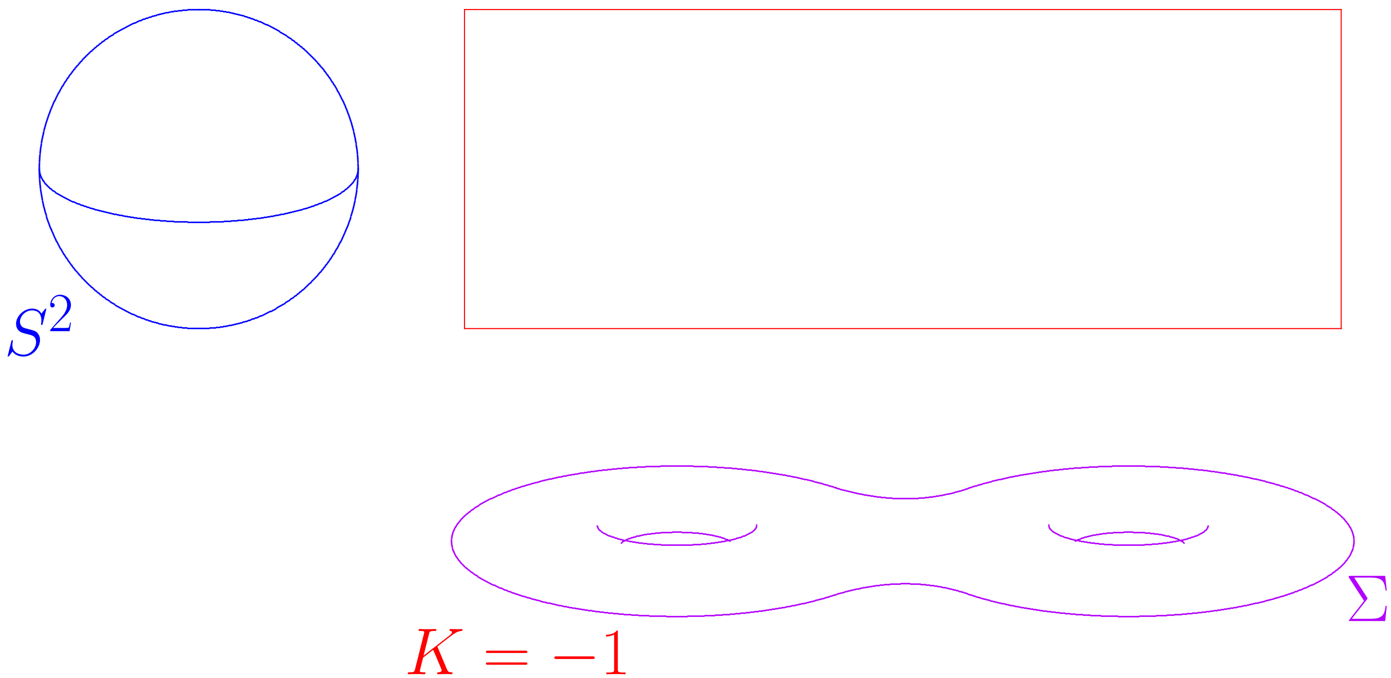

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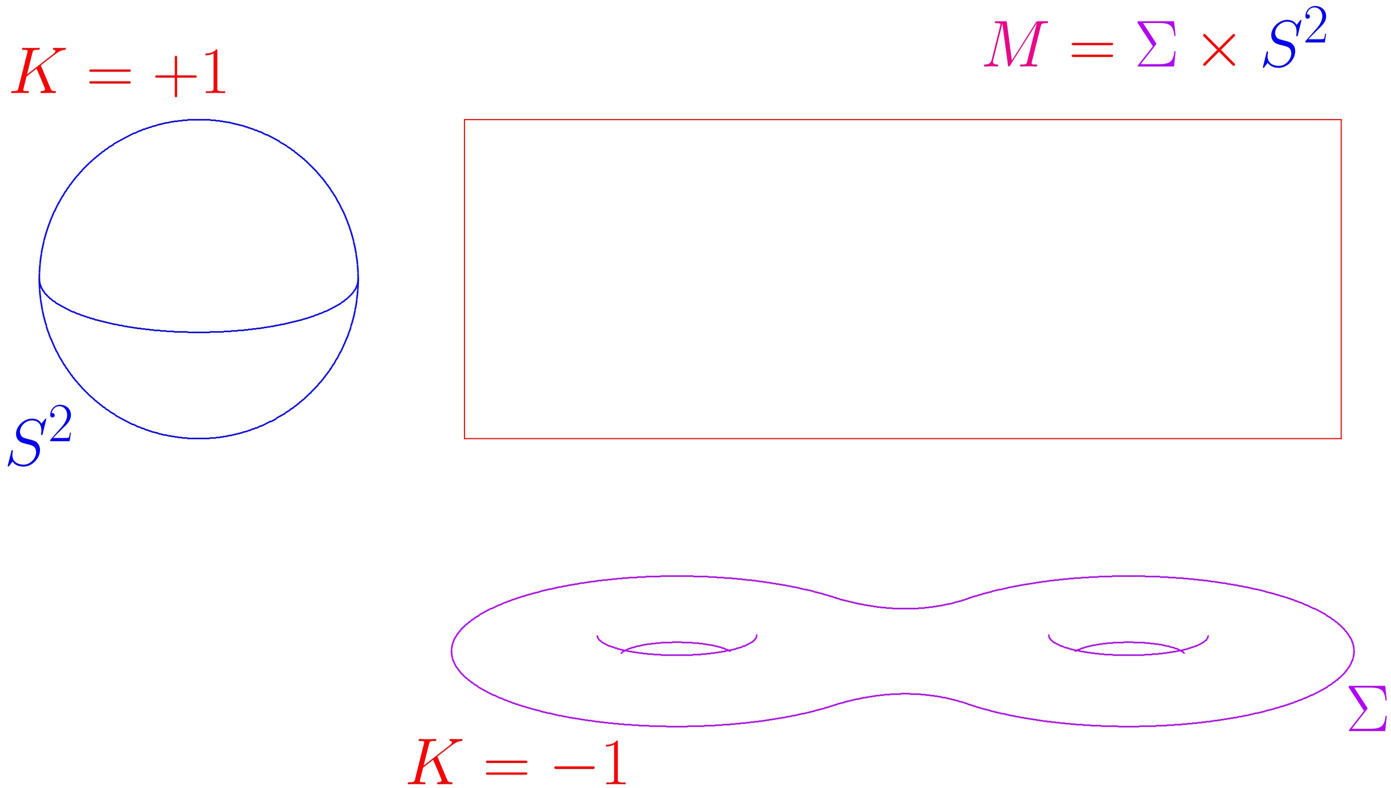

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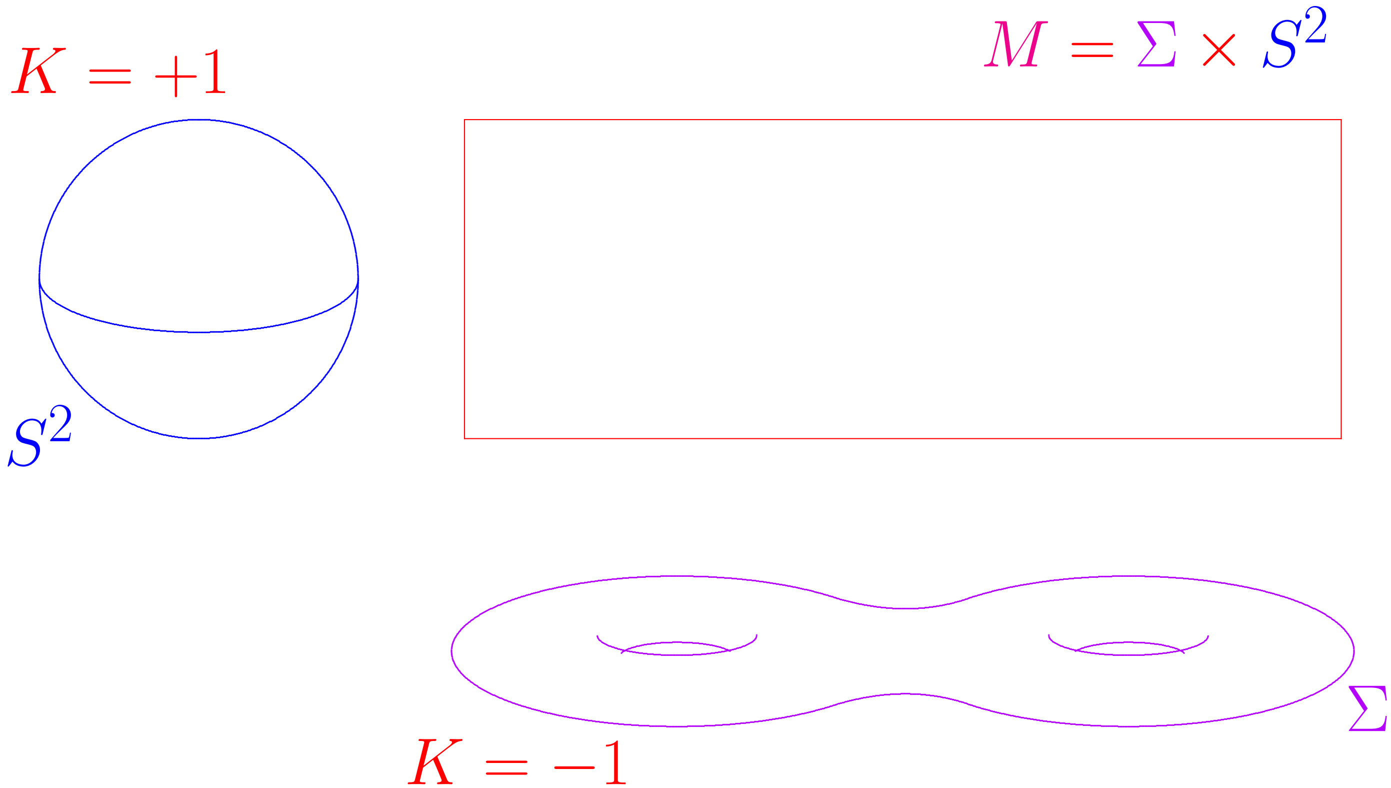
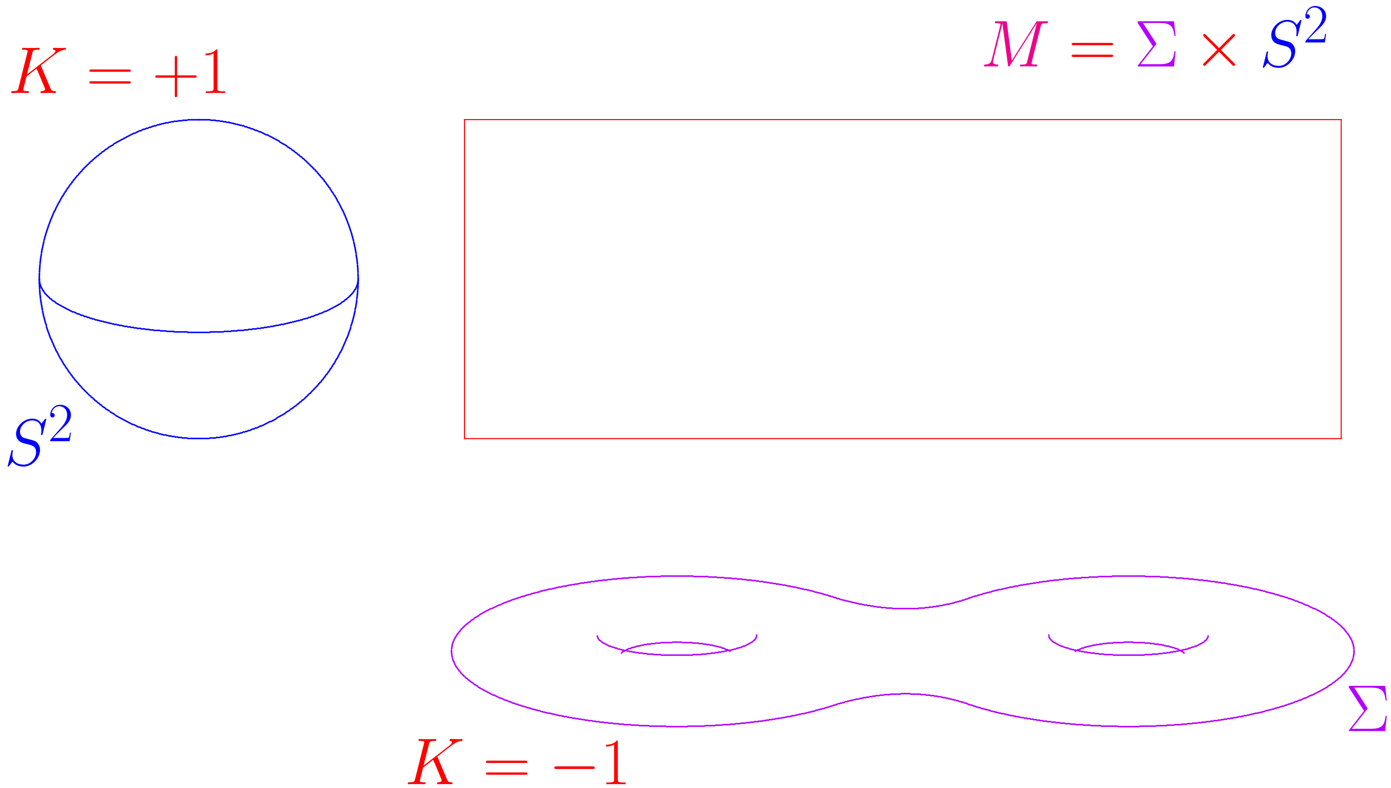


Product is scalar-flat

Example.



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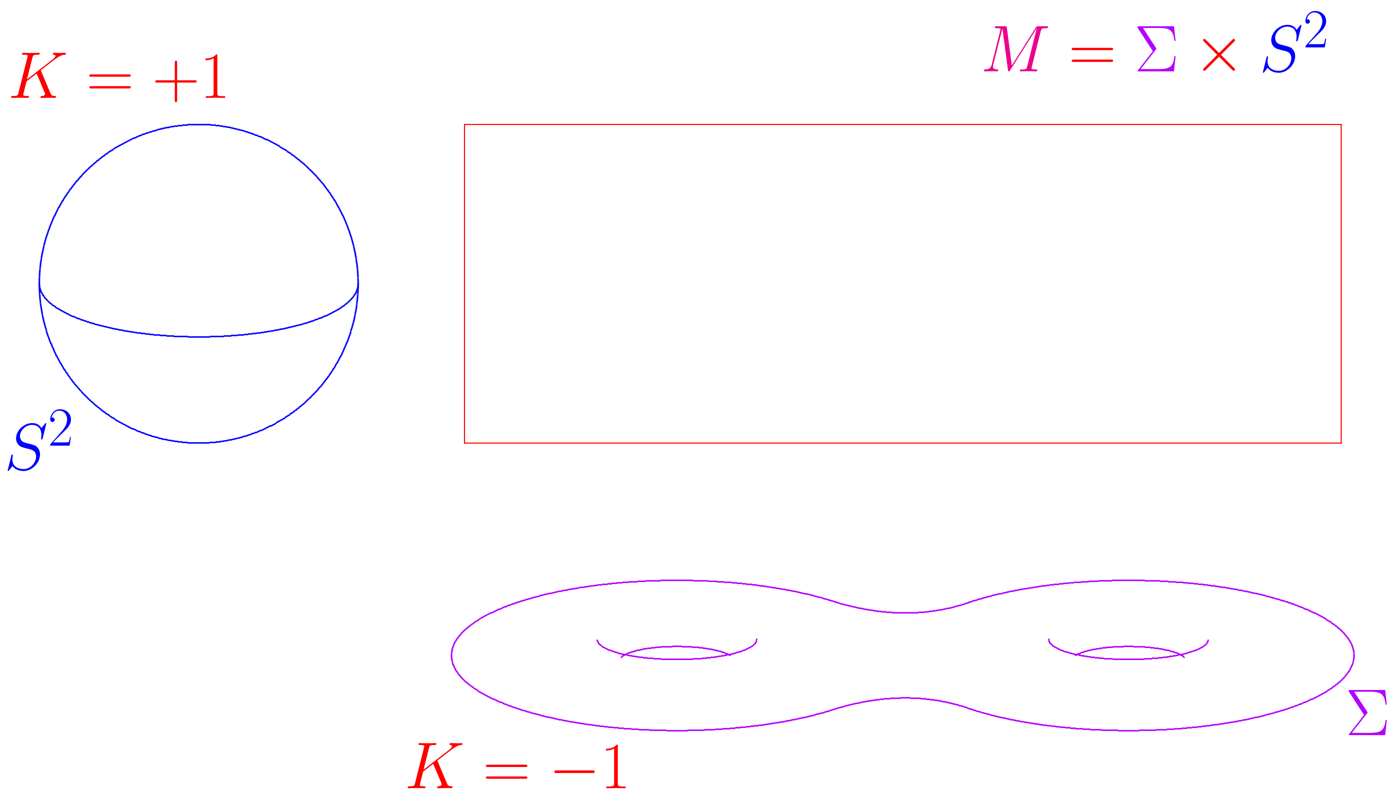

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For both orientations!

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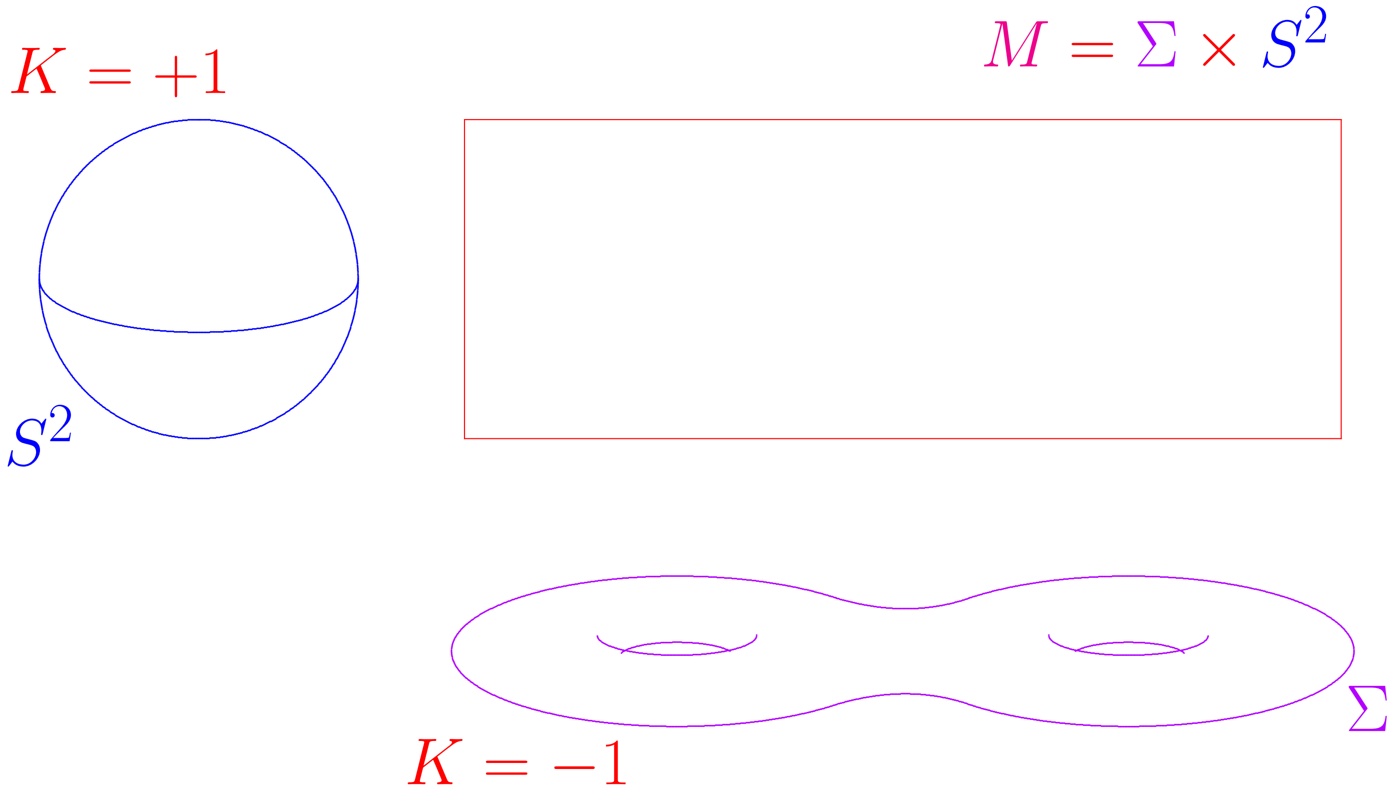

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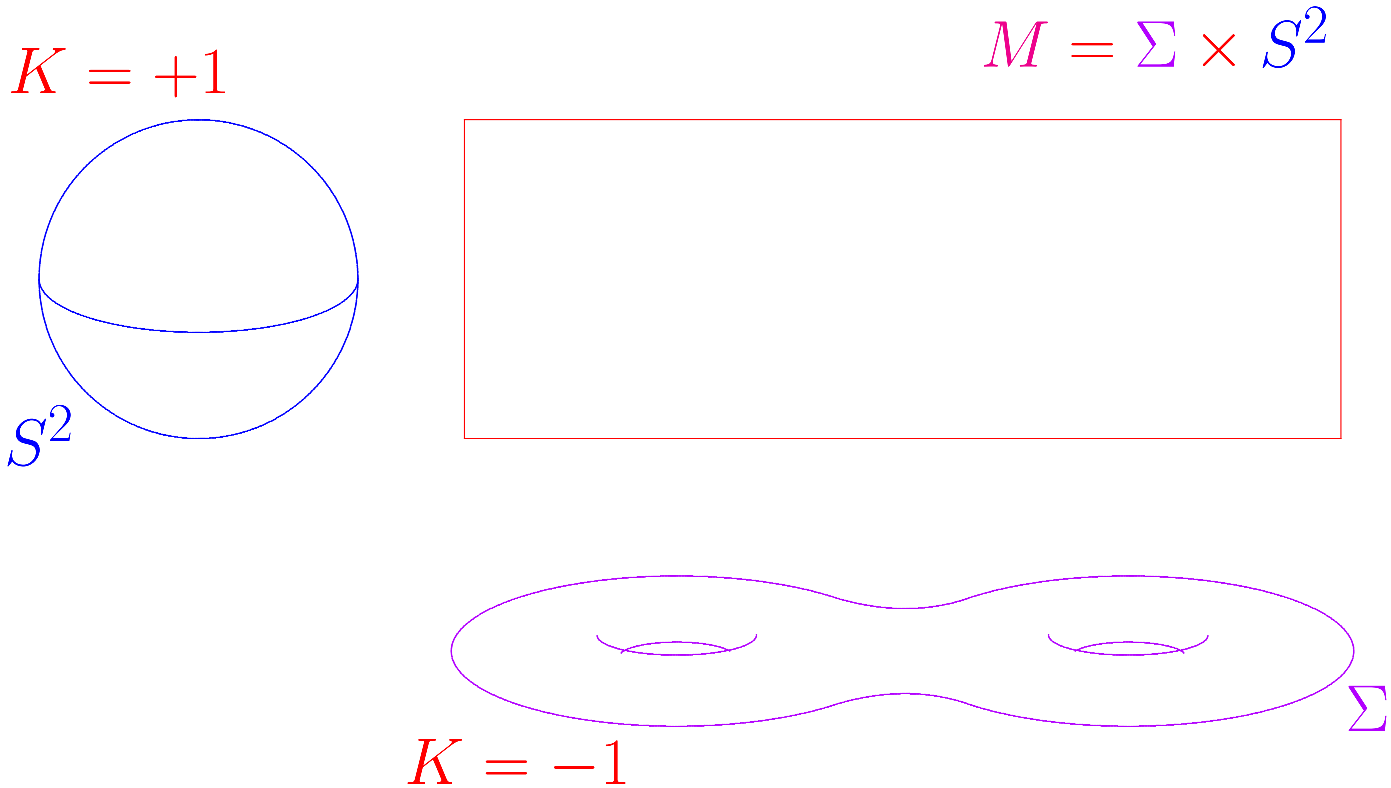

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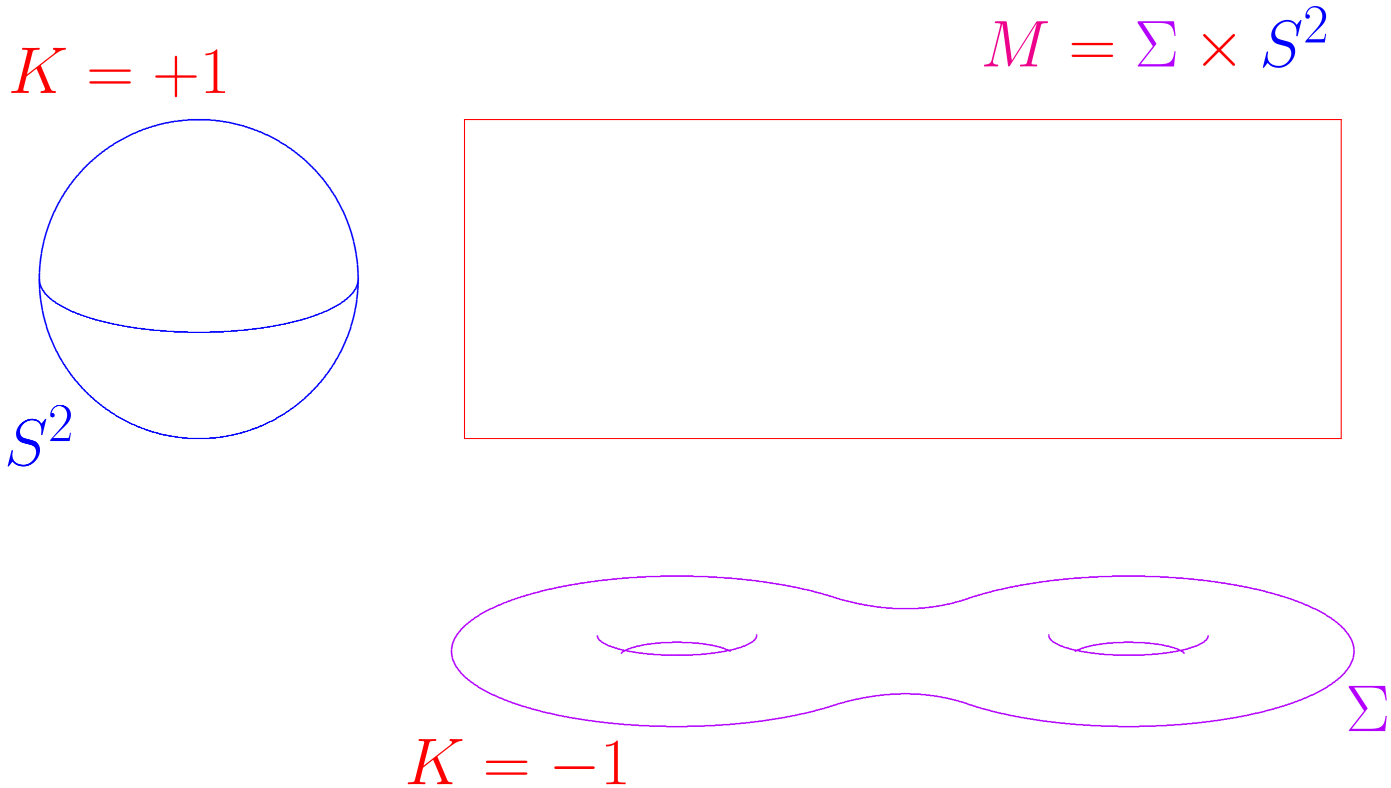

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For both orientations!

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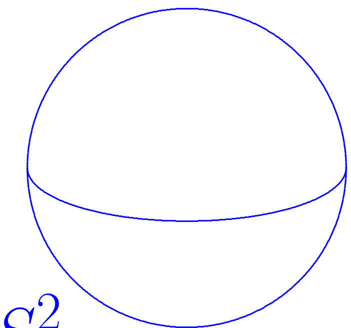
Locally conformally flat!

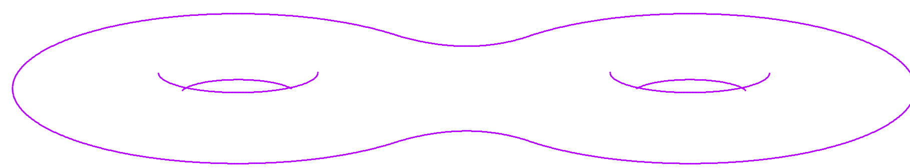
Example.

$K = +1$

$M = \Sigma \times S^2$



S^2



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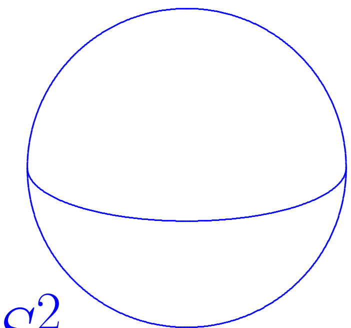

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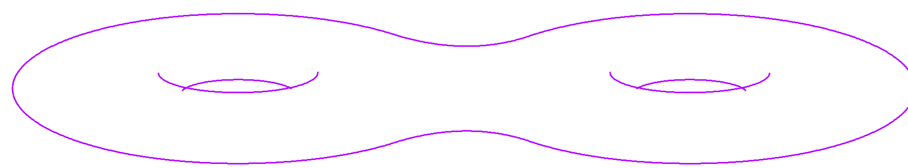

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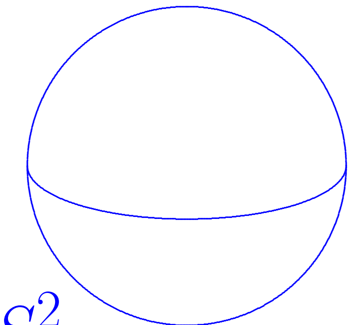

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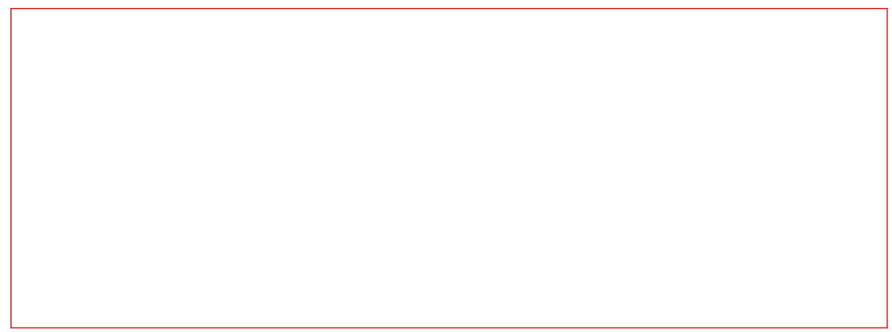

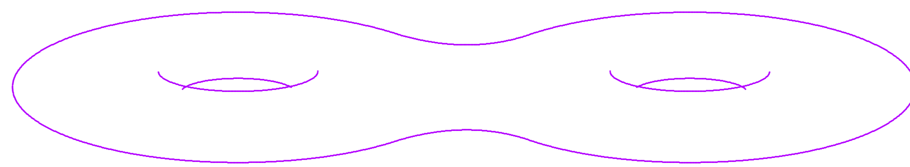

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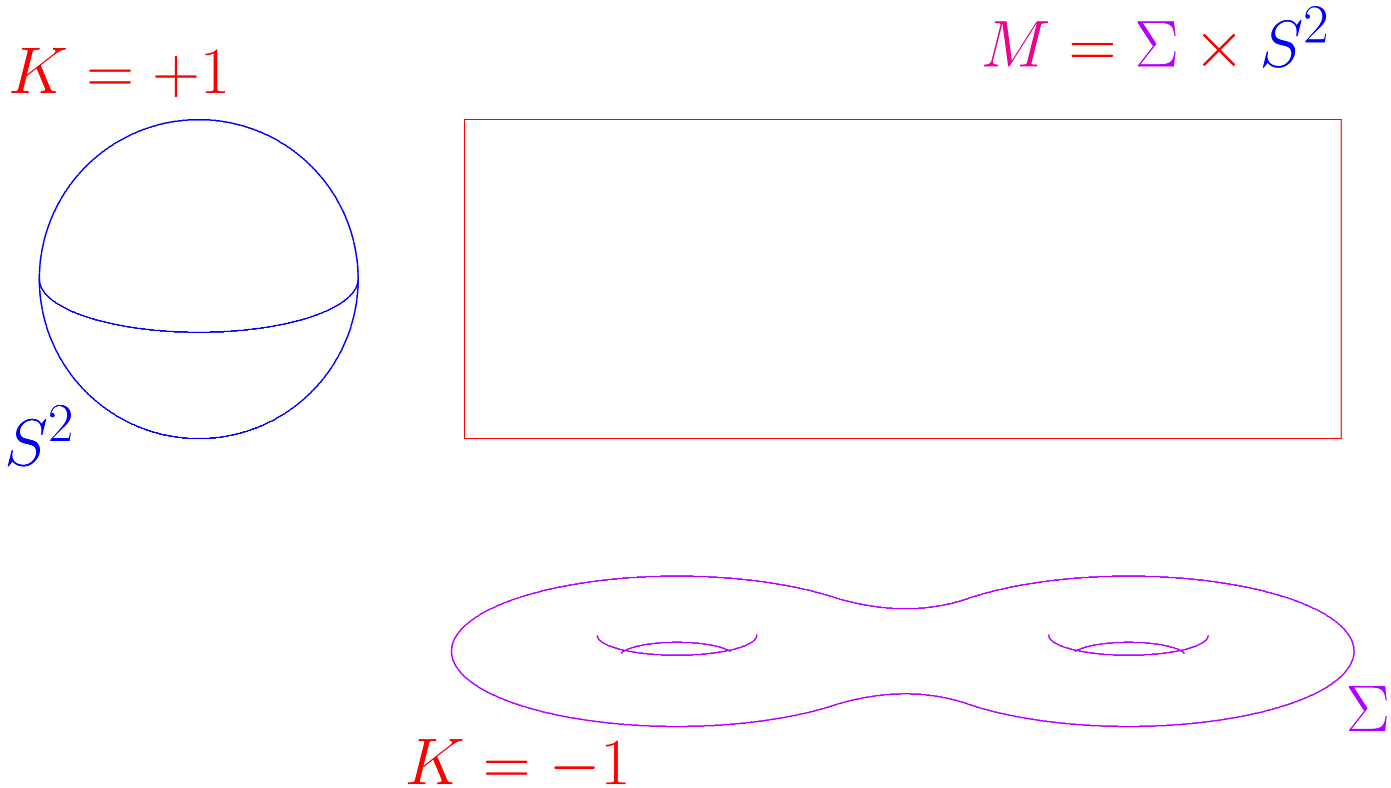

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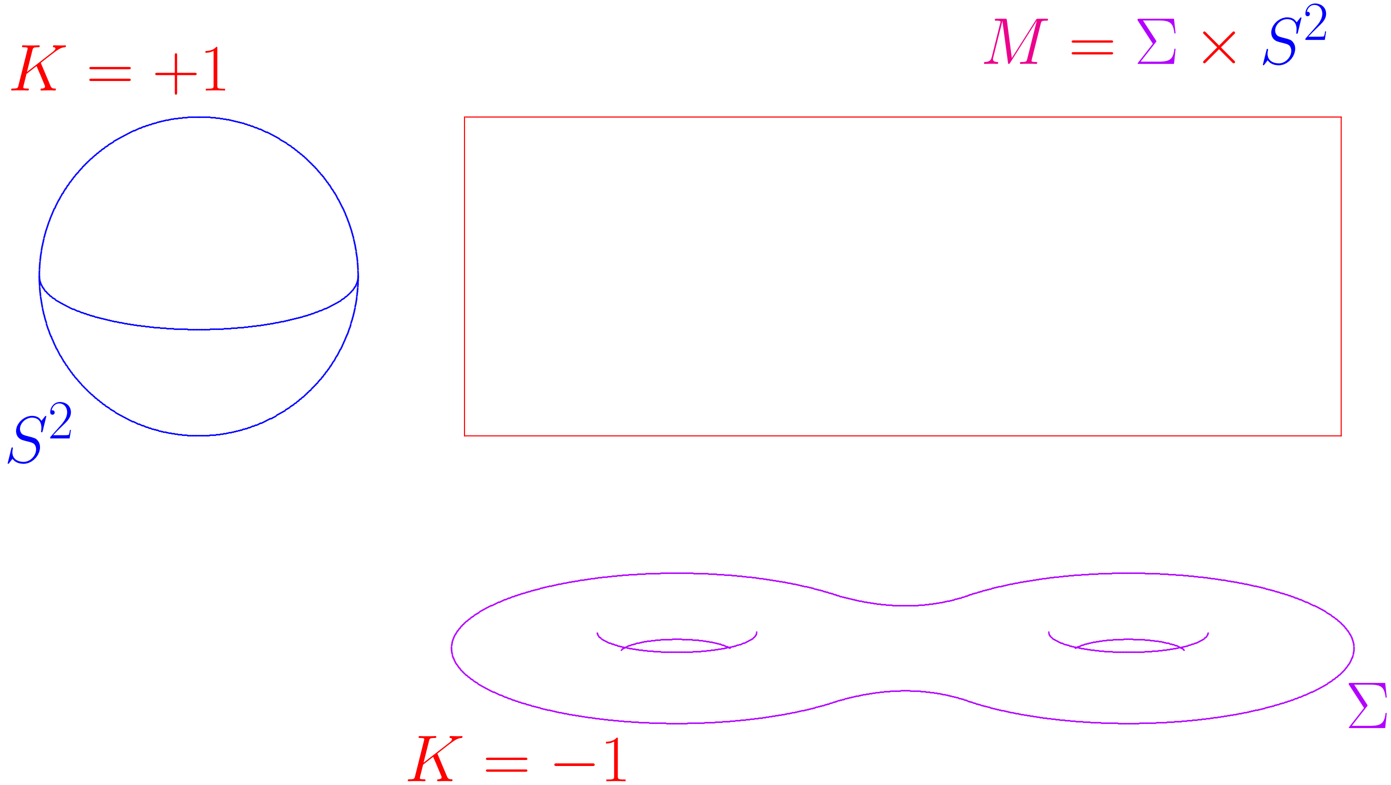

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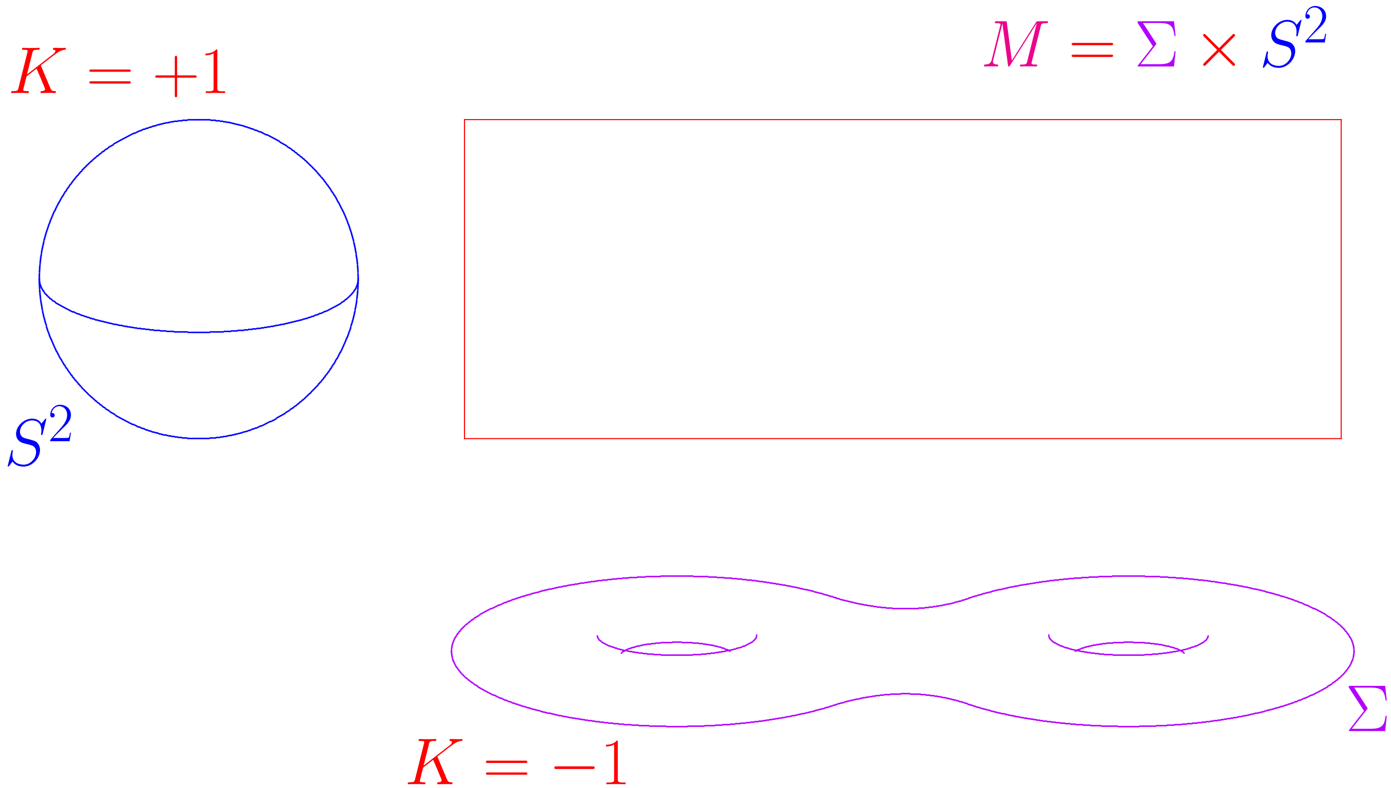

Scalar-flat Kähler deformations: $12(\mathbf{g} - 1)$ moduli

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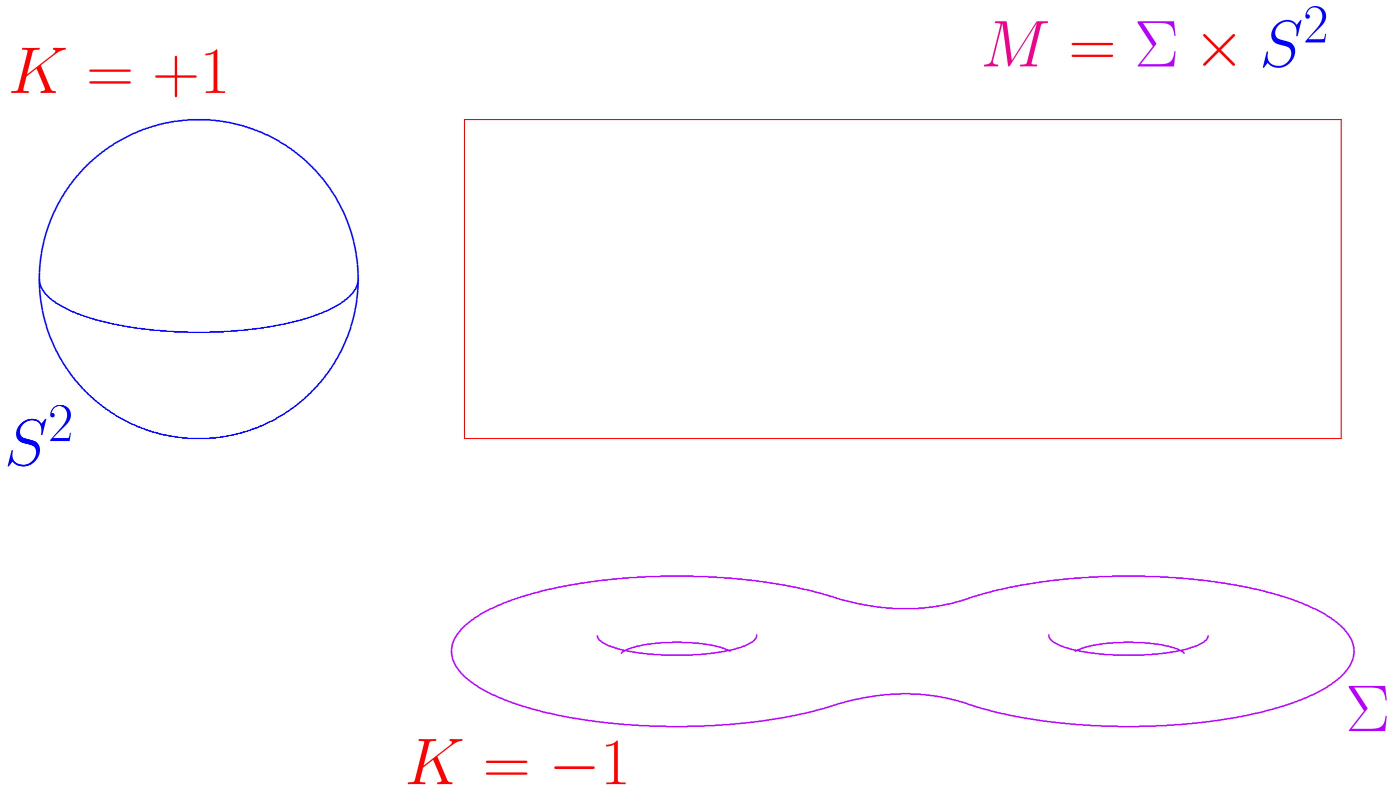

Scalar-flat Kähler deformations: $12(g-1)$ moduli
Locally conformally flat def'ms: $30(g-1)$ moduli

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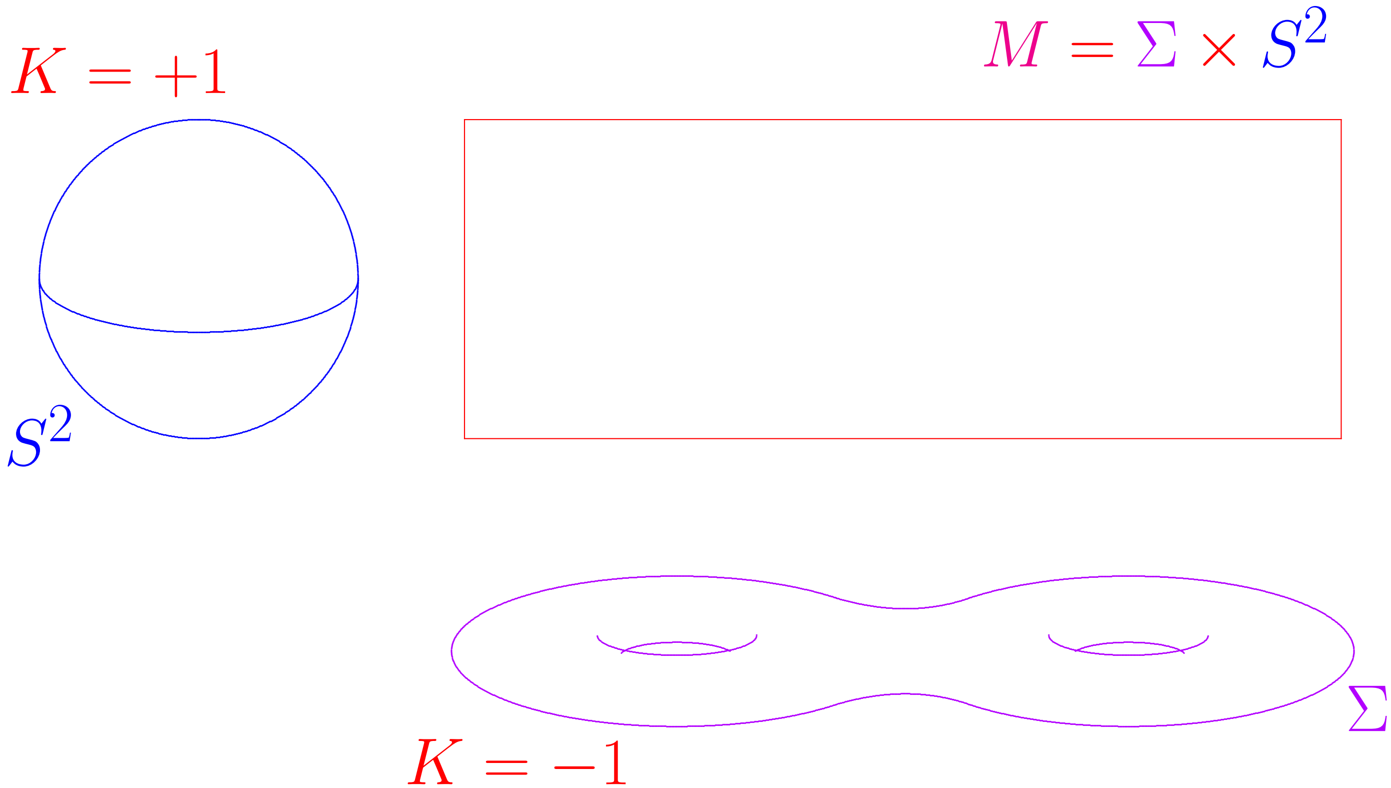

Scalar-flat Kähler deformations: $12(\mathbf{g}-1)$ moduli

almost-Kähler ASD deformat'ns: $30(\mathbf{g}-1)$ moduli

The almost-Kähler condition gives extra control on ASD conformal geometry.

L '15, I. Kim '16: classification of almost-Kähler ASD roughly the same as in scalar-flat Kähler case.

Does this say anything about general ASD metrics?

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Alas, **No!**

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Same method simultaneously proves. . .

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Proof hinges on a construction of hyperbolic 3-manifolds.

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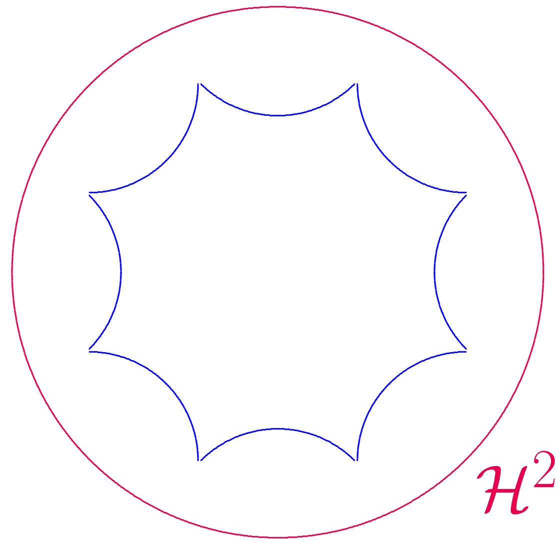
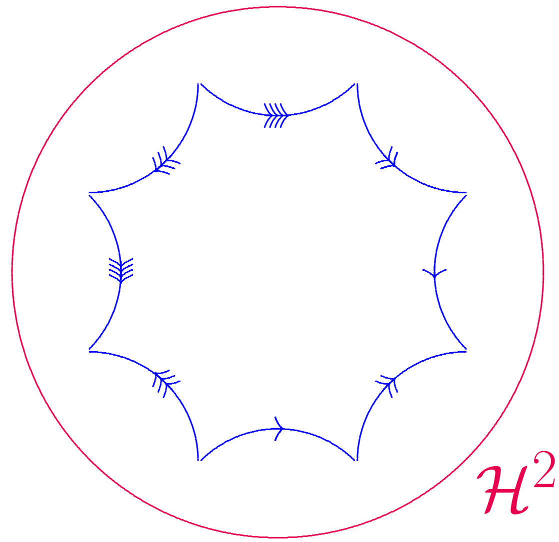
We begin by revisiting hyperbolic metrics on Σ.



$\mathcal{H}^2$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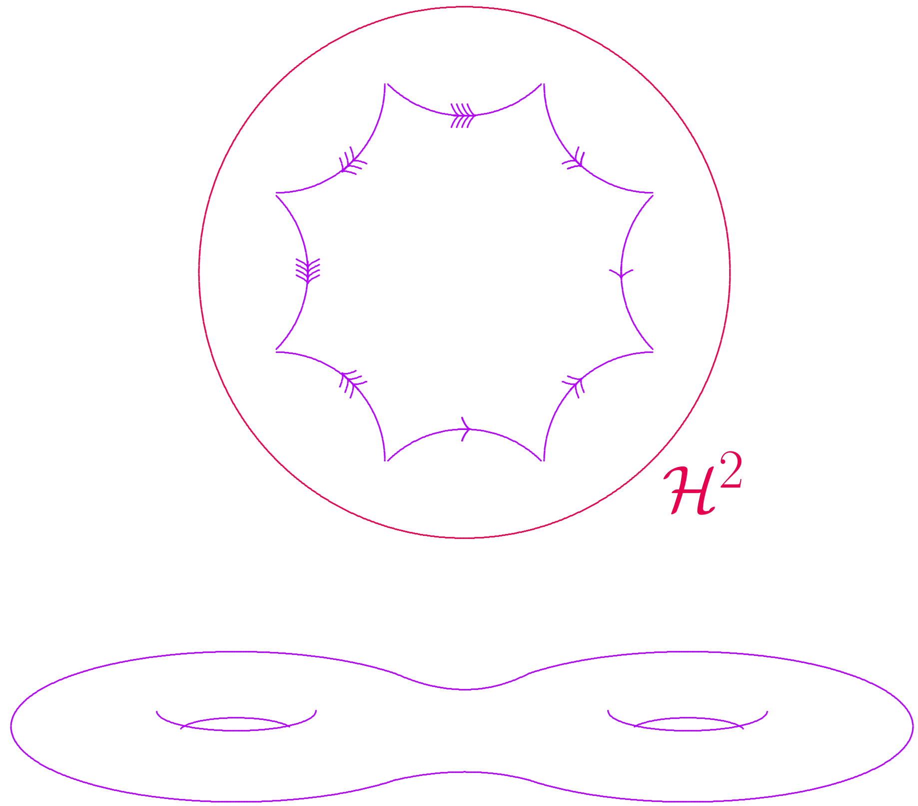
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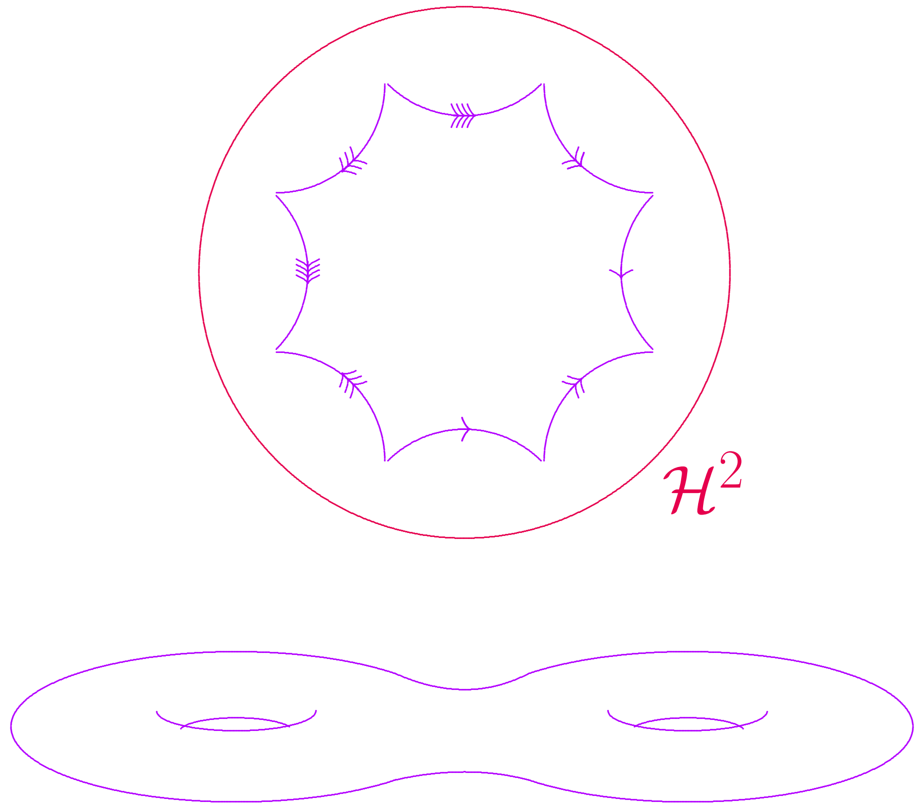
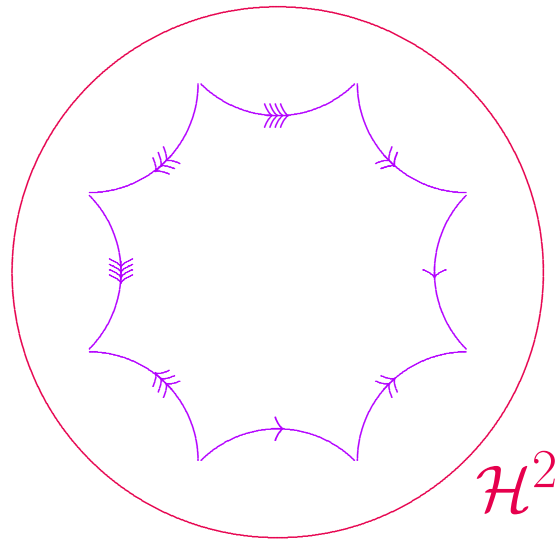
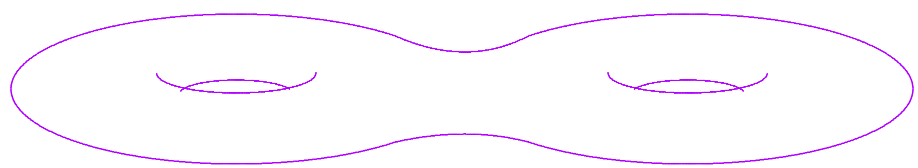
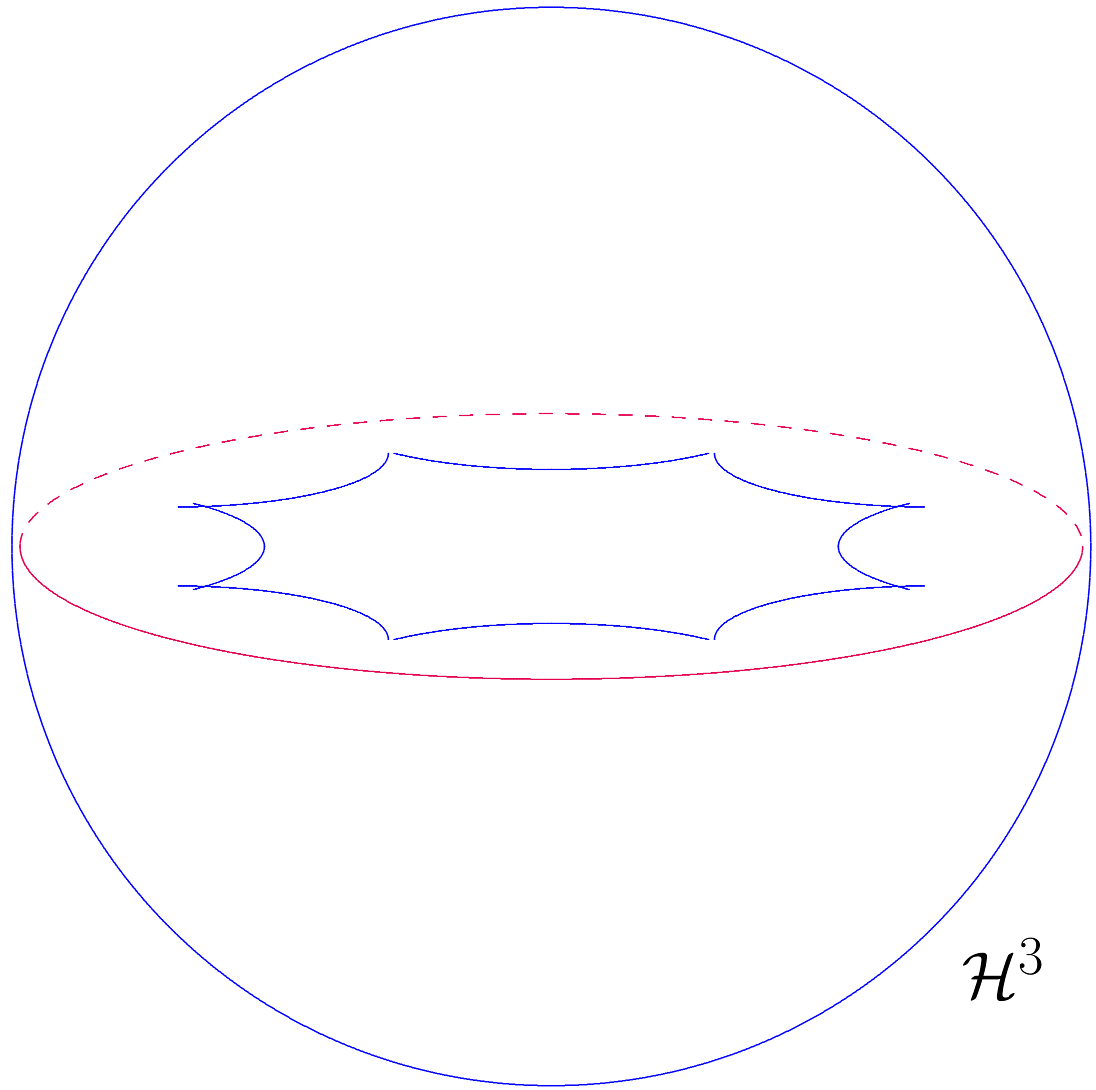


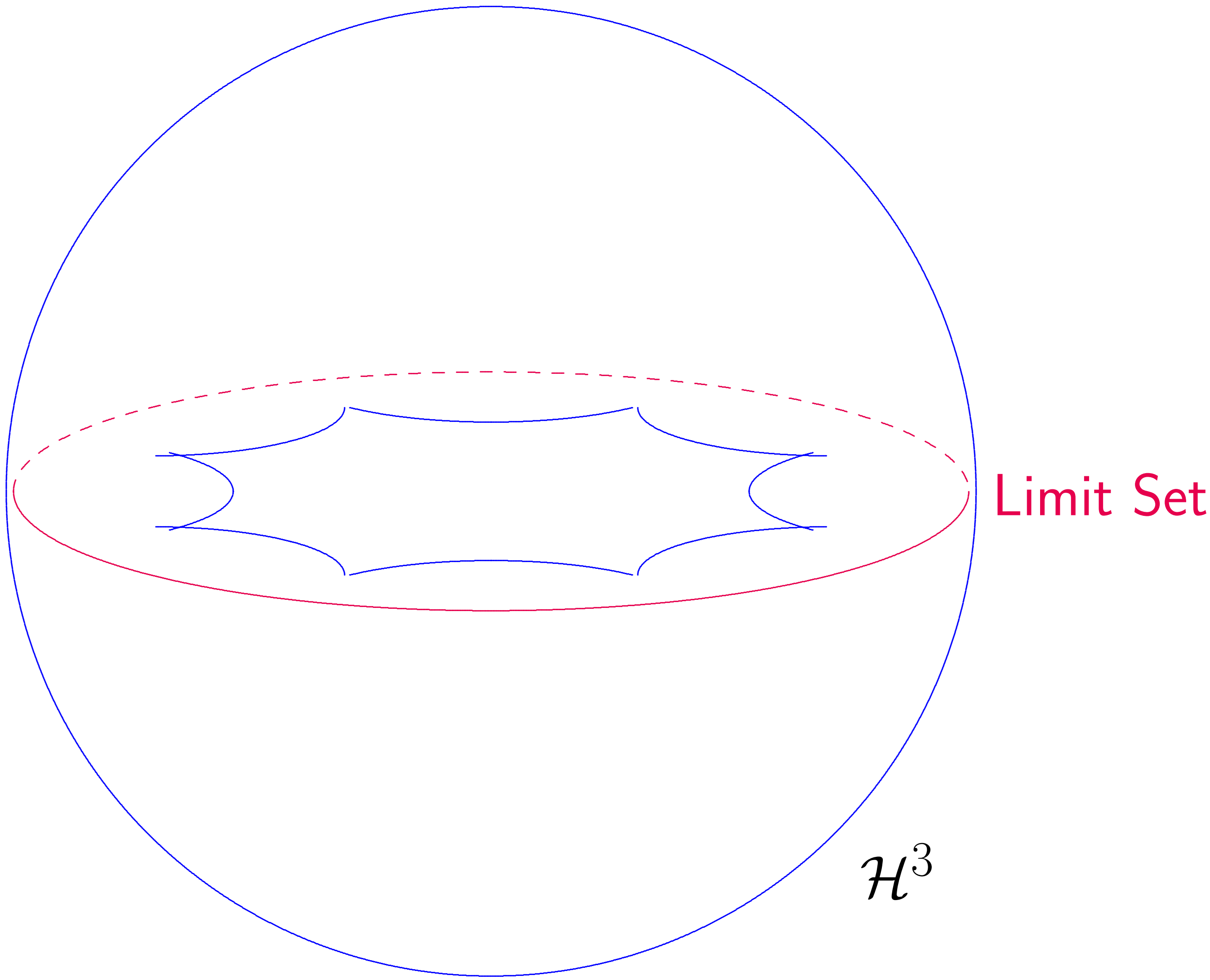
$$\pi_1(\Sigma) \hookrightarrow \mathbf{SO}_+(1,2) = \mathbf{PSL}(2,\mathbb{R})$$

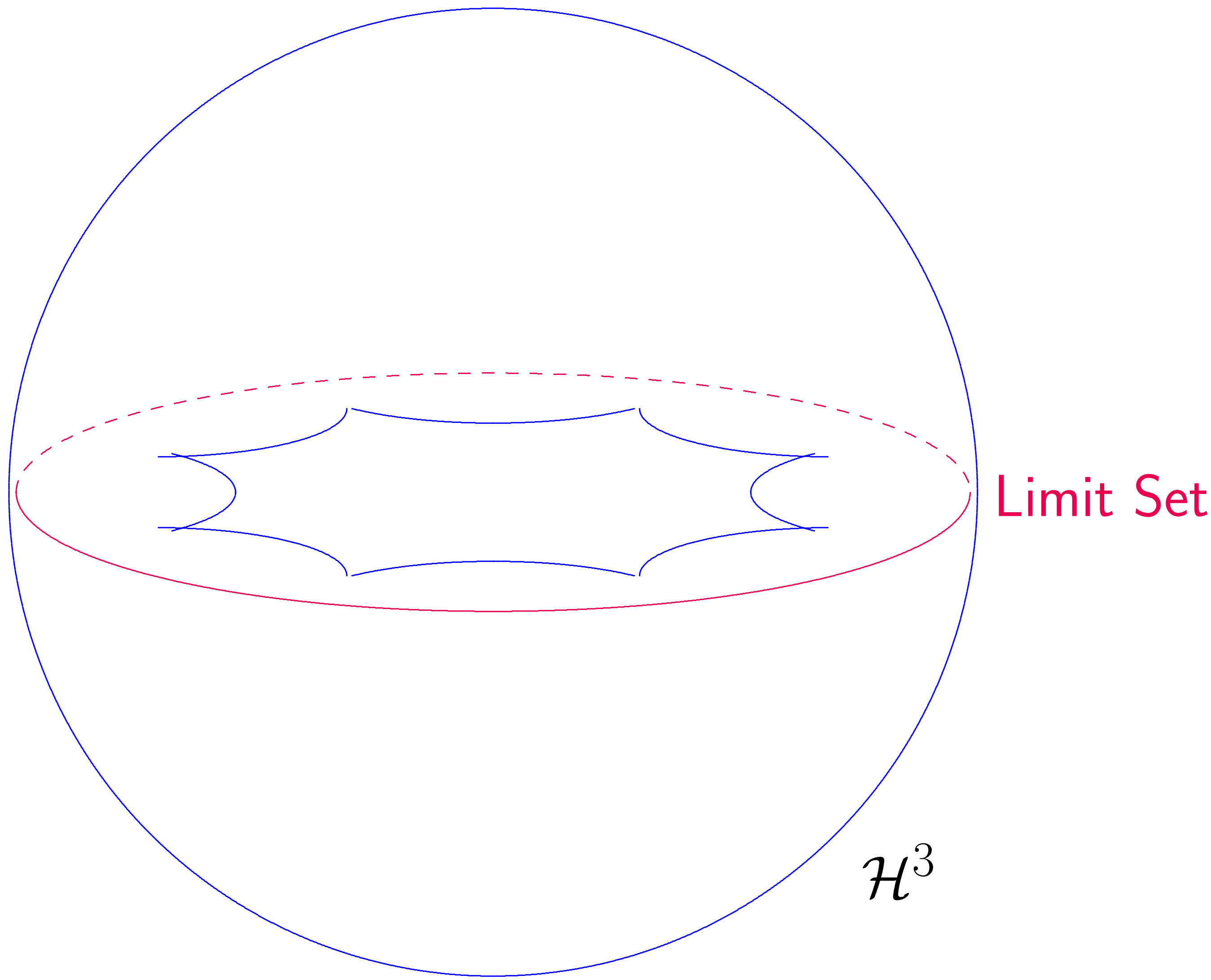




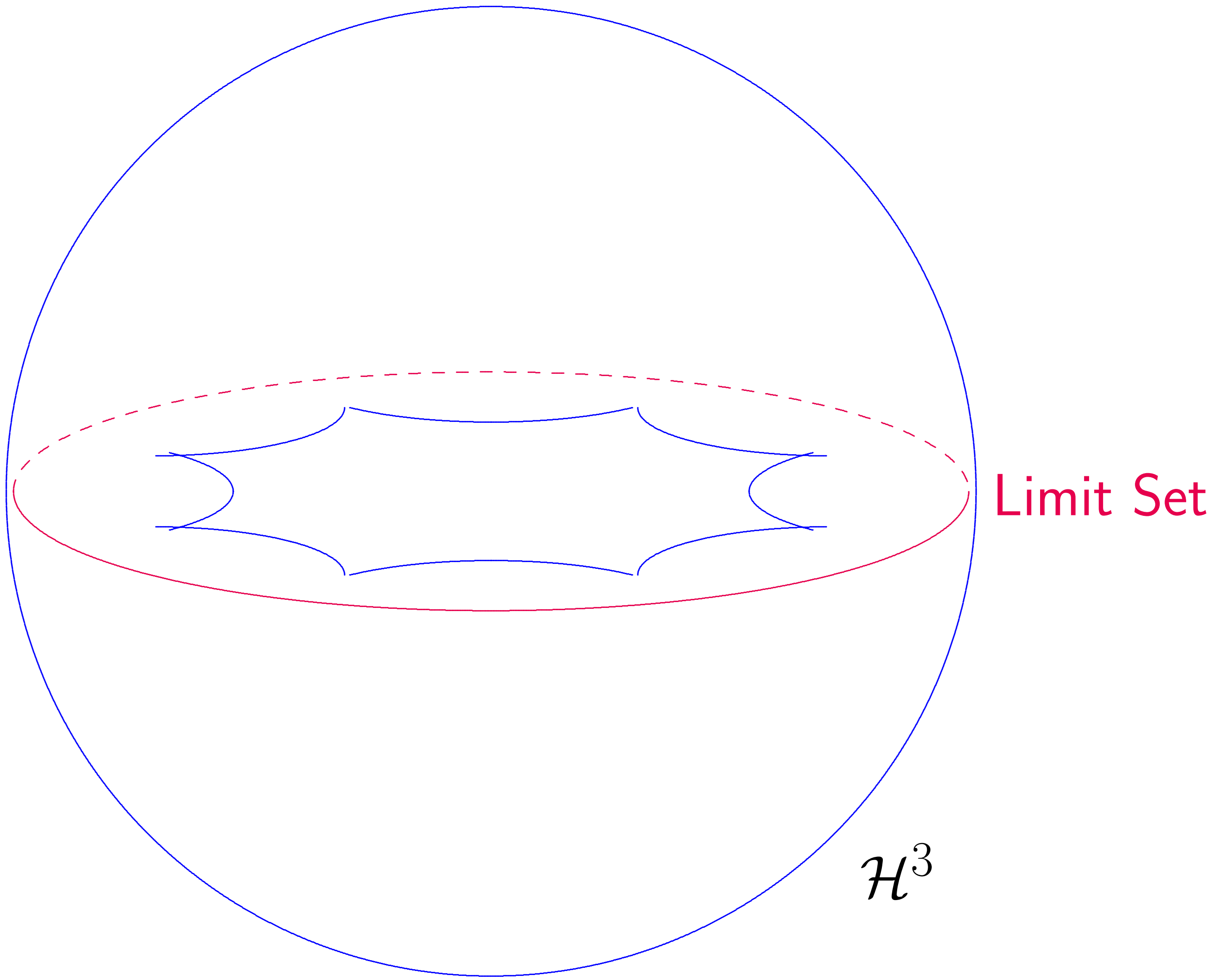
$$\begin{array}{ccccc}
\pi_1(\Sigma) & \hookrightarrow & \mathbf{SO}_+(1,2) & = & \mathbf{PSL}(2,\mathbb{R}) \\
 & & \cap & & \cap \\
 & & \mathbf{SO}_+(1,3) & = & \mathbf{PSL}(2,\mathbb{C})
\end{array}$$


$\mathcal{H}^3$


Limit Set
$\mathcal{H}^3$



$\pi_1(\Sigma) \xrightarrow{\cong} \Gamma \subset \mathbf{PSL}(2,\mathbb{R})$ Fuchsian group



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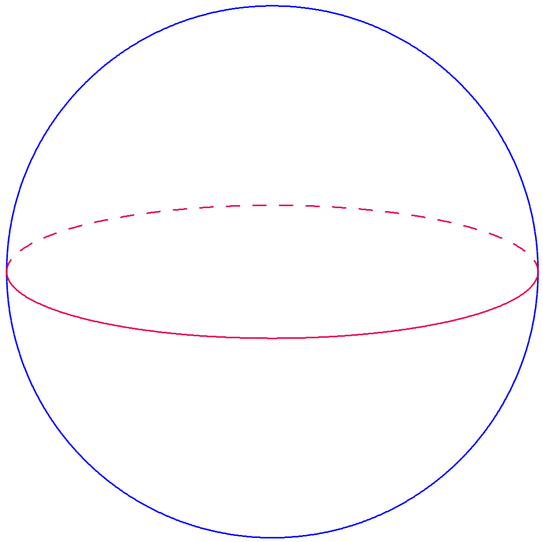
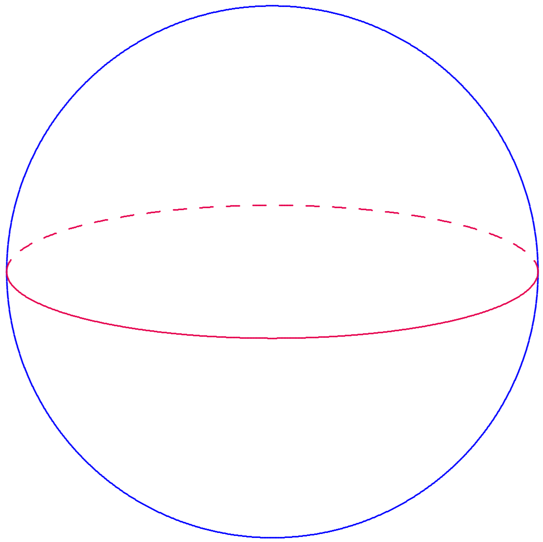
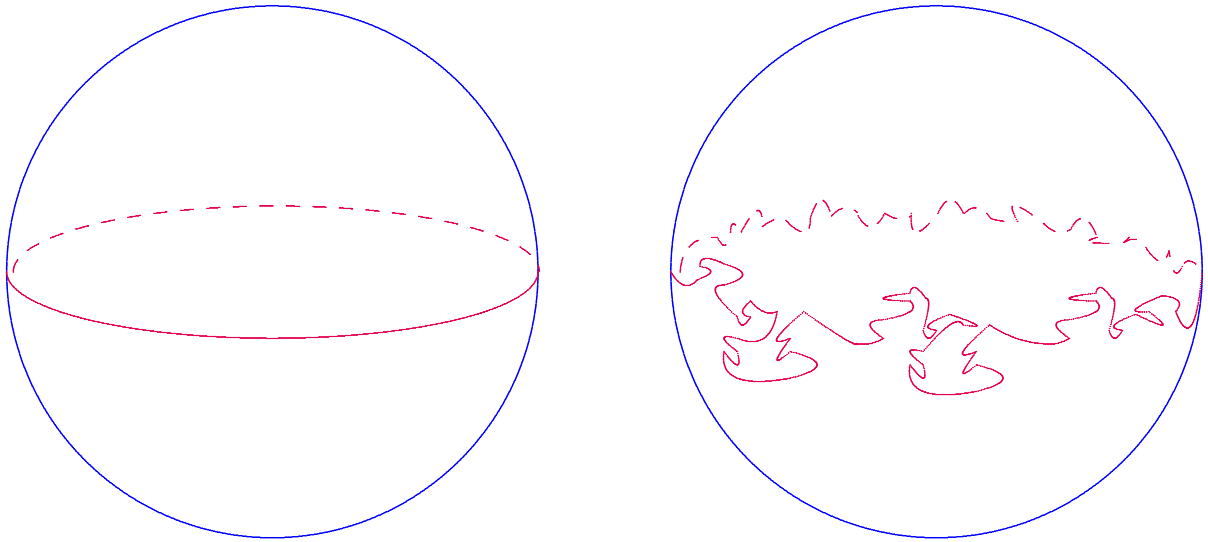




Fuchsian



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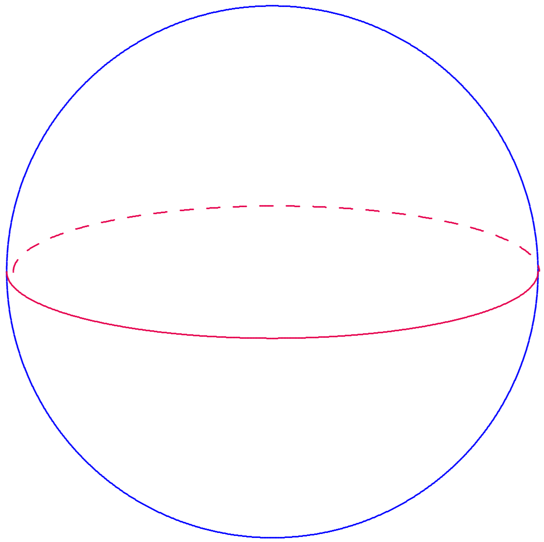

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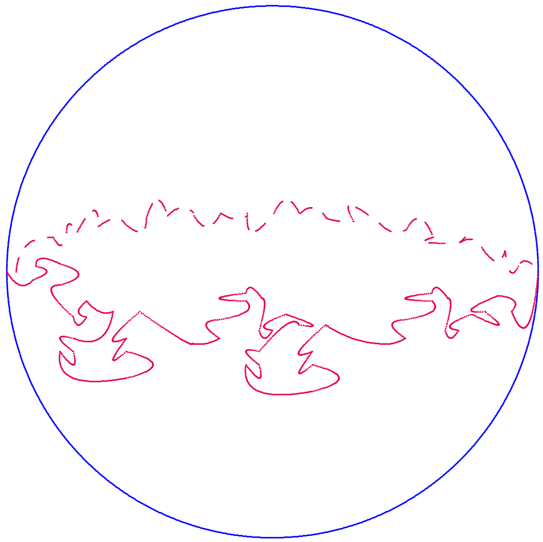
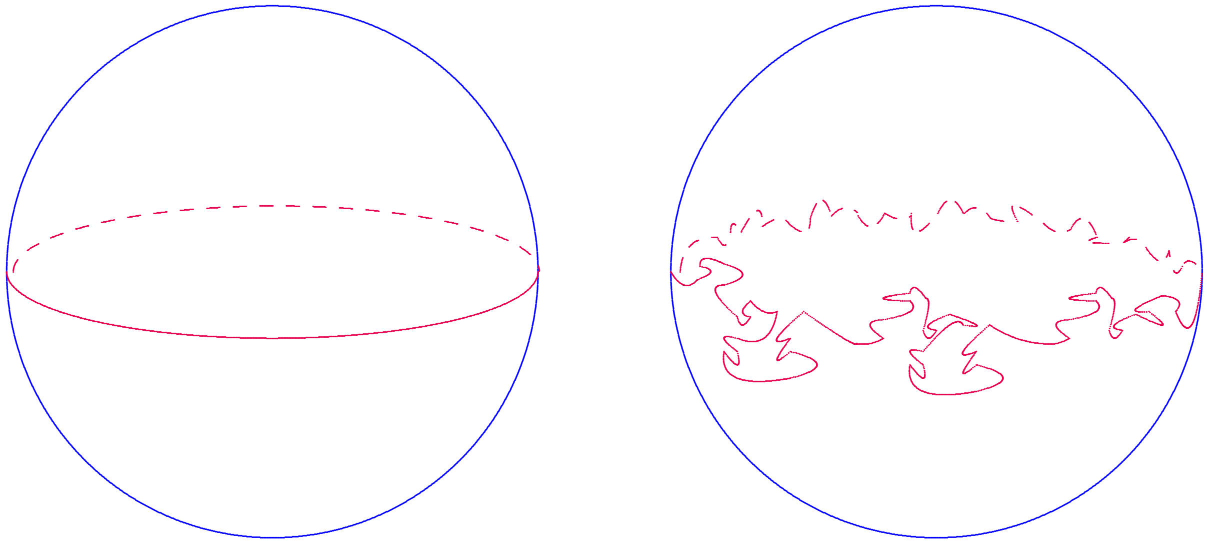
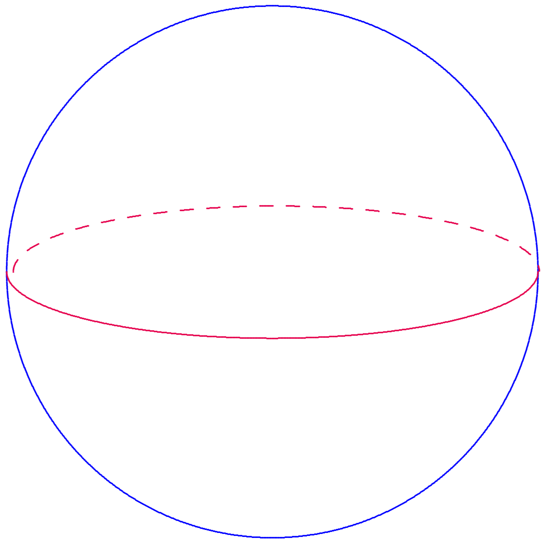


quasi-Fuchsian



Fuchsian quasi-Fuchsian

$$\pi_1(\Sigma) \xrightarrow{\cong} \Gamma \subset \mathbf{PSL}(2,\mathbb{C})$$ quasi-Fuchsian group



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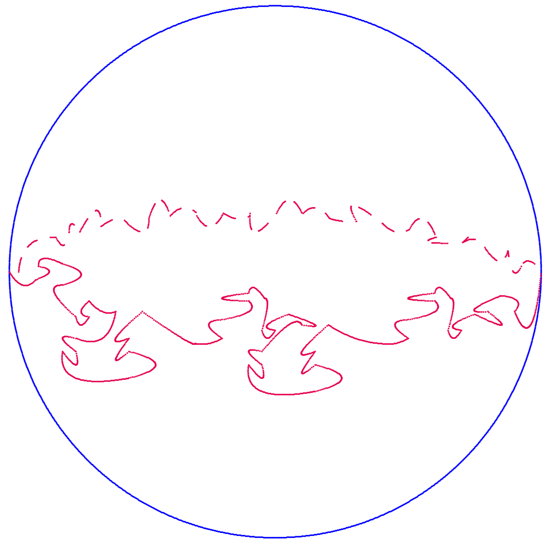
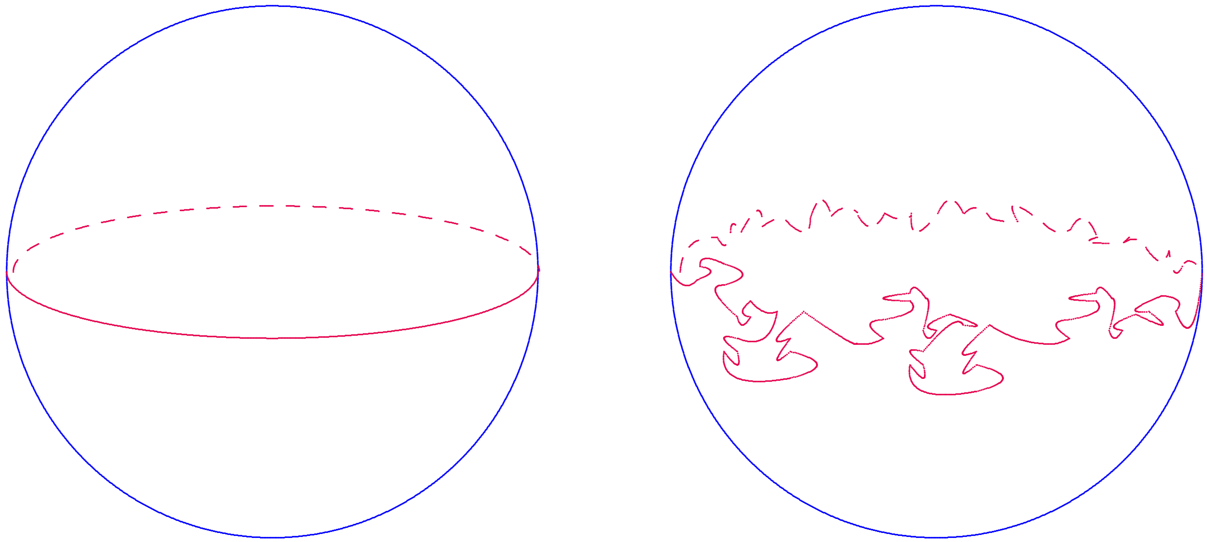


quasi-Fuchsian

$$\pi_1(\Sigma) \xrightarrow{\cong} \Gamma \subset \mathbf{PSL}(2, \mathbb{C})$$ quasi-Fuchsian group of Bers type



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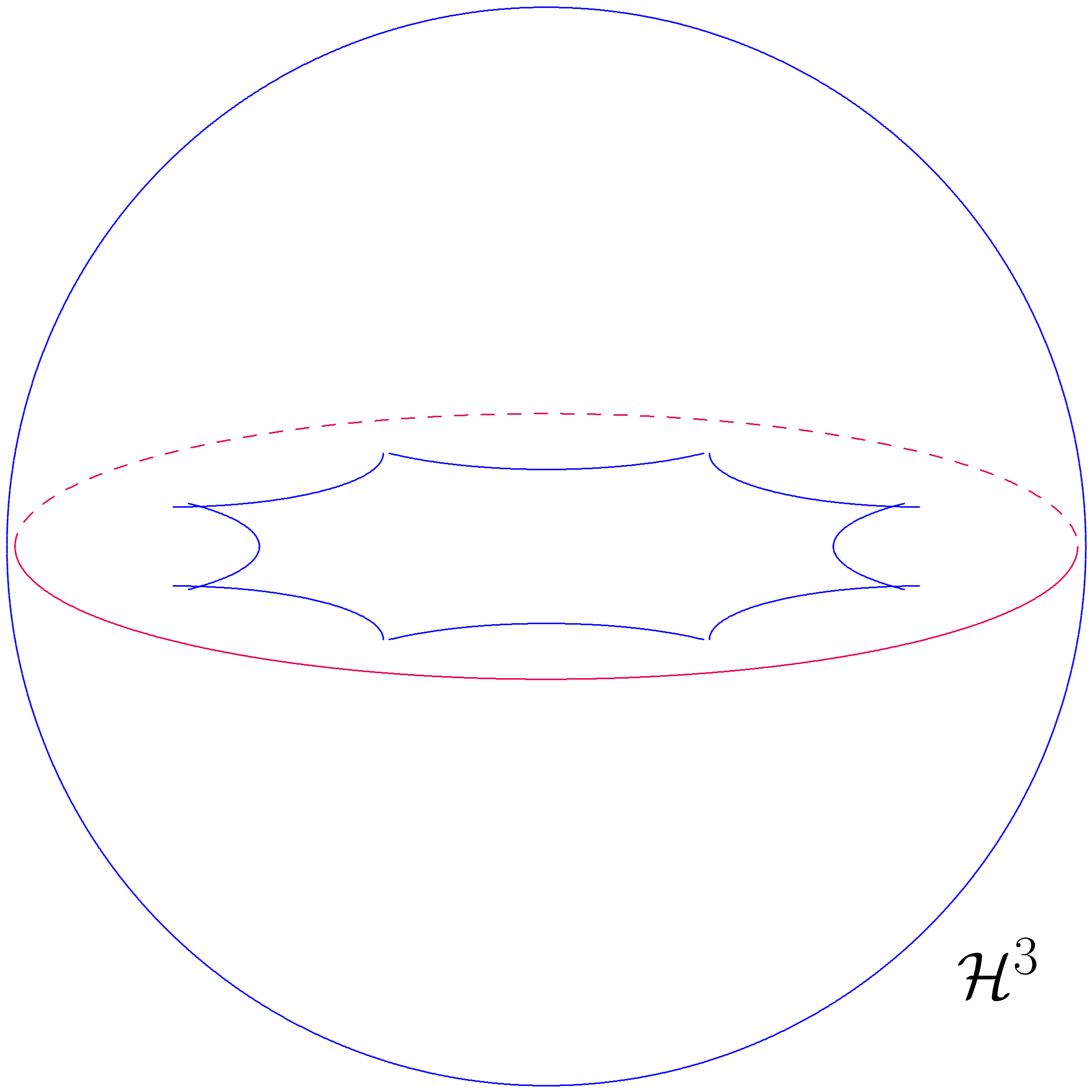
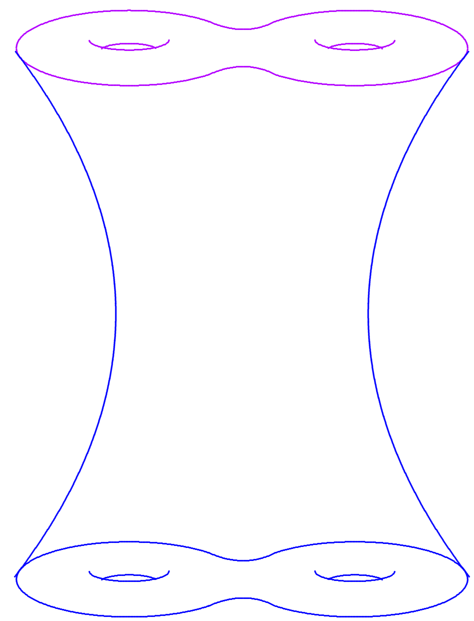
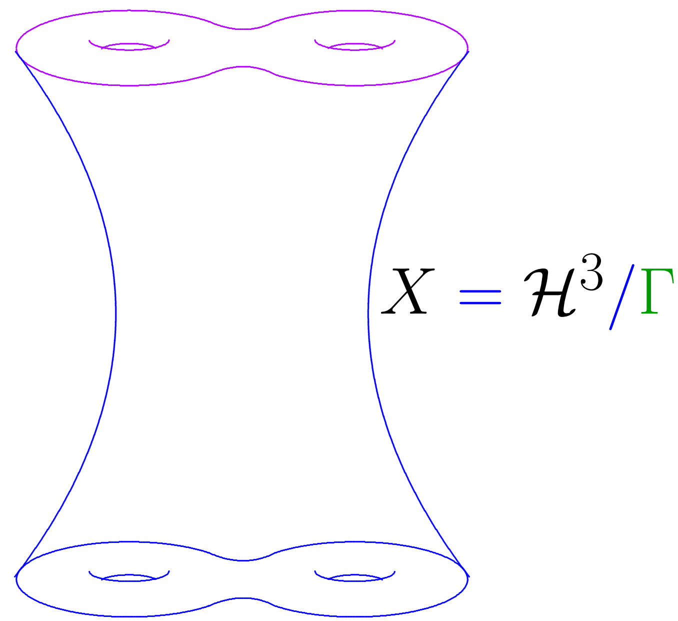
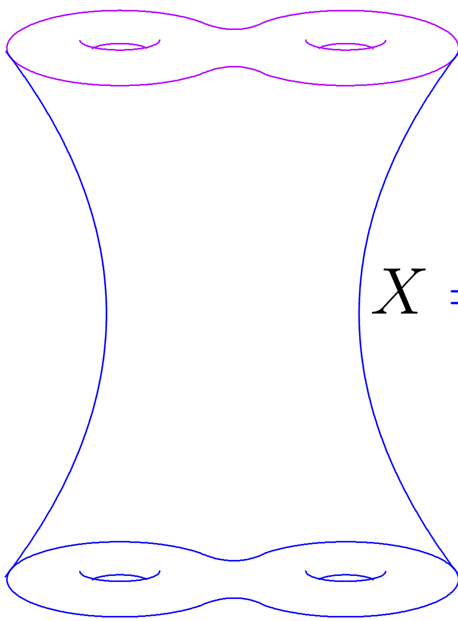
Quasi-conformally conjugate to Fuchsian.


$\mathcal{H}^3$




$X = \mathcal{H}^3/\Gamma$



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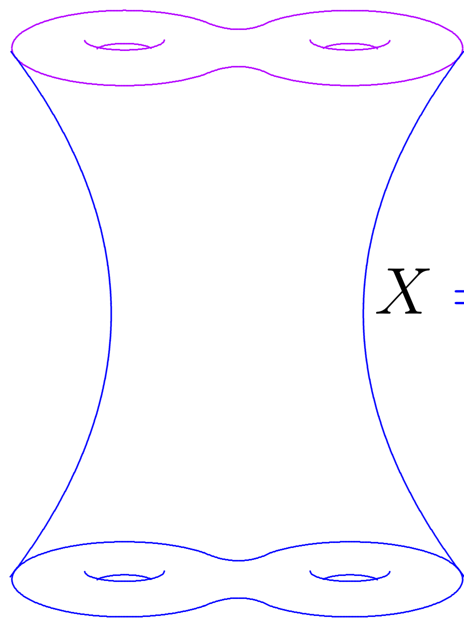
Γ Fuchsian



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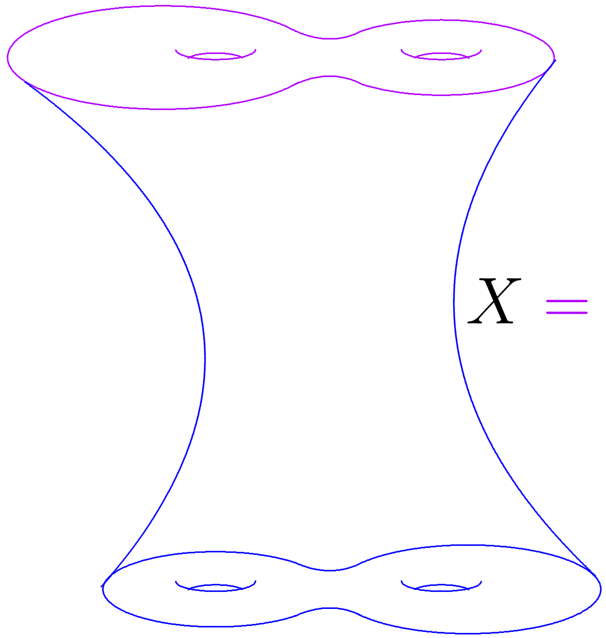

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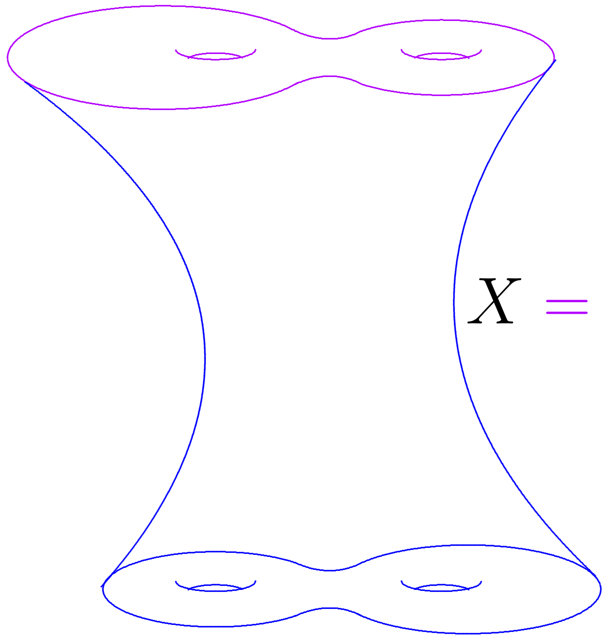

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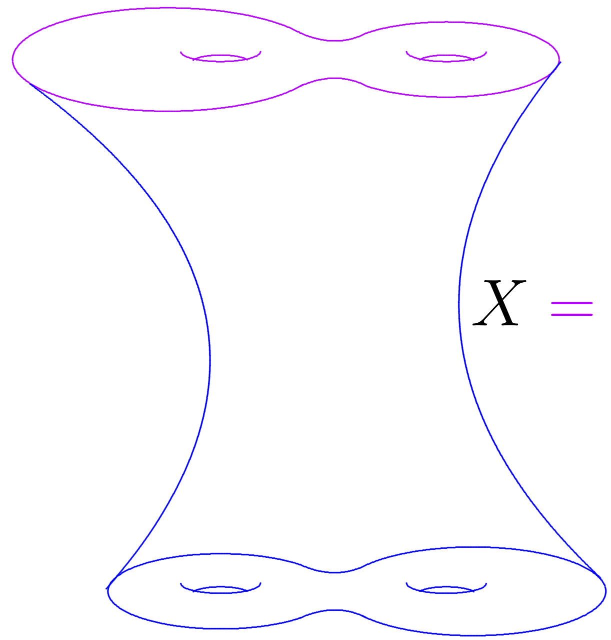

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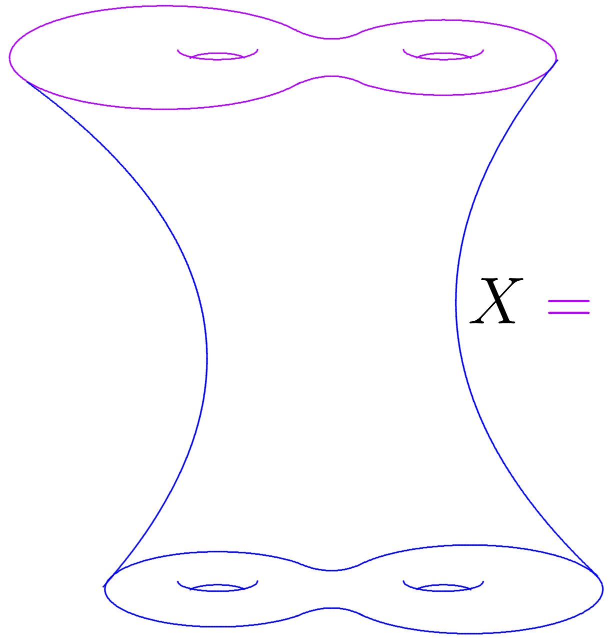

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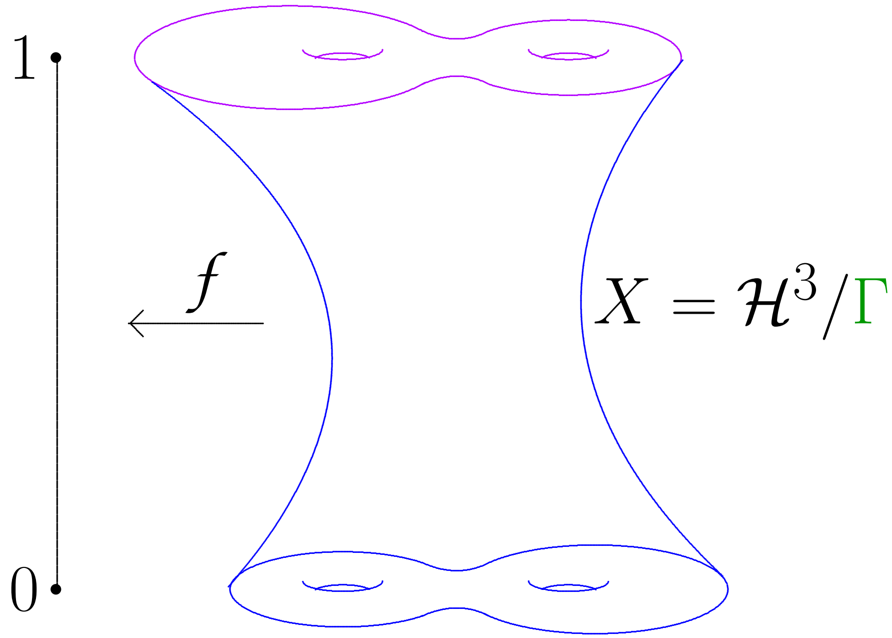

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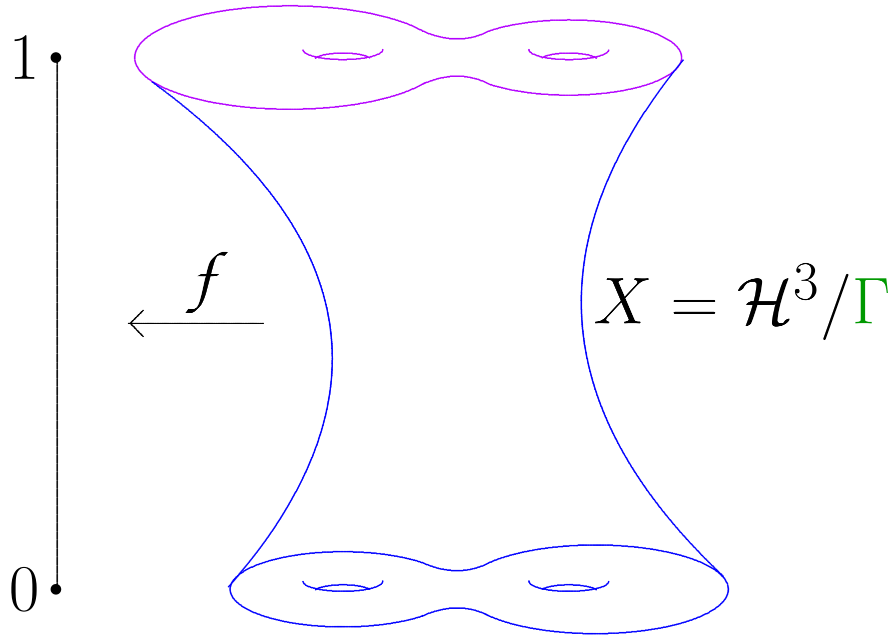

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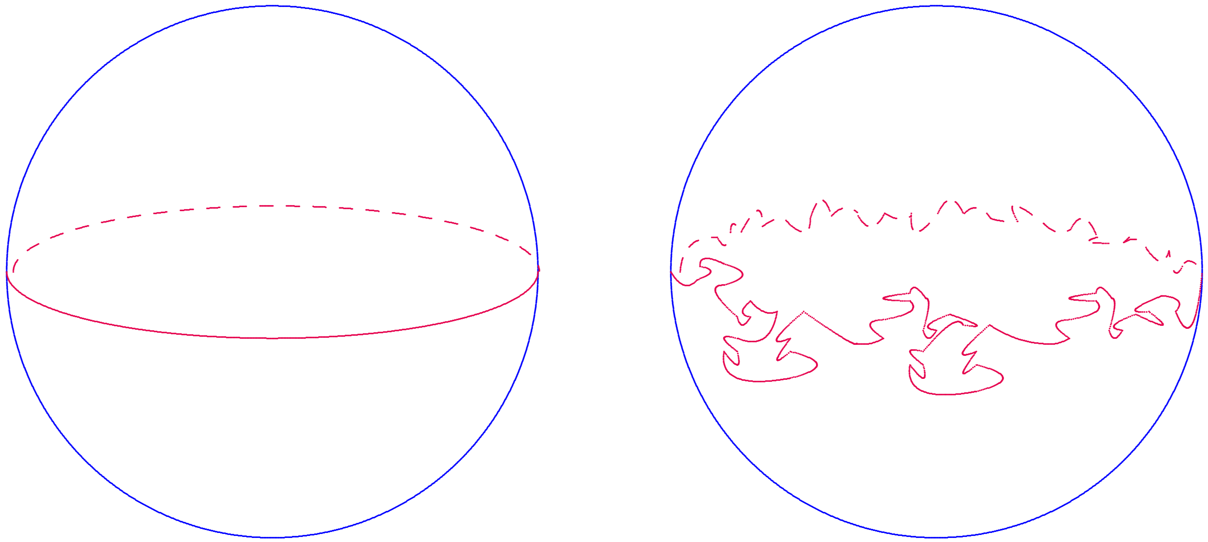

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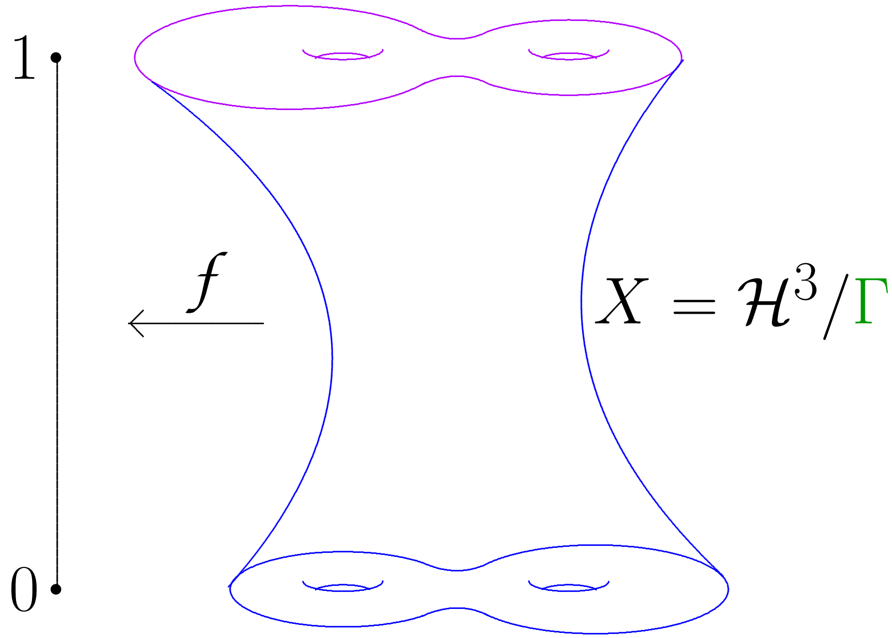

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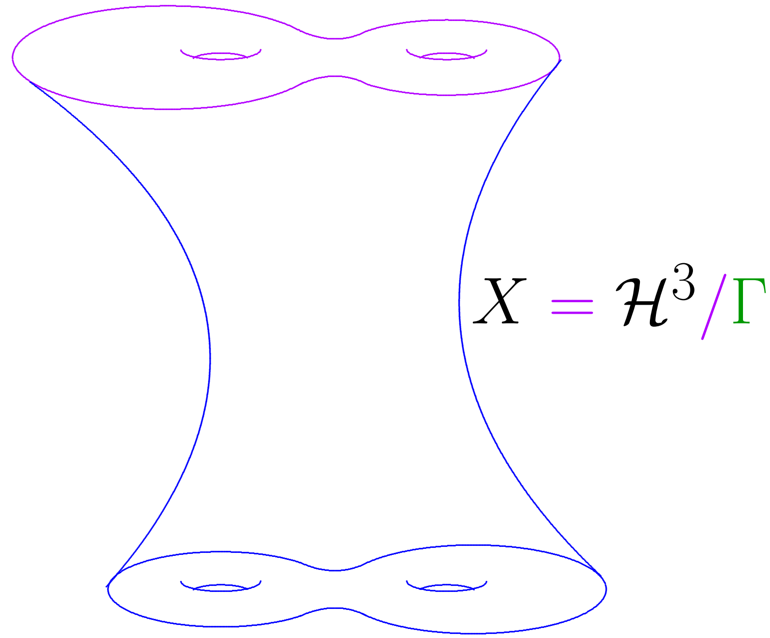
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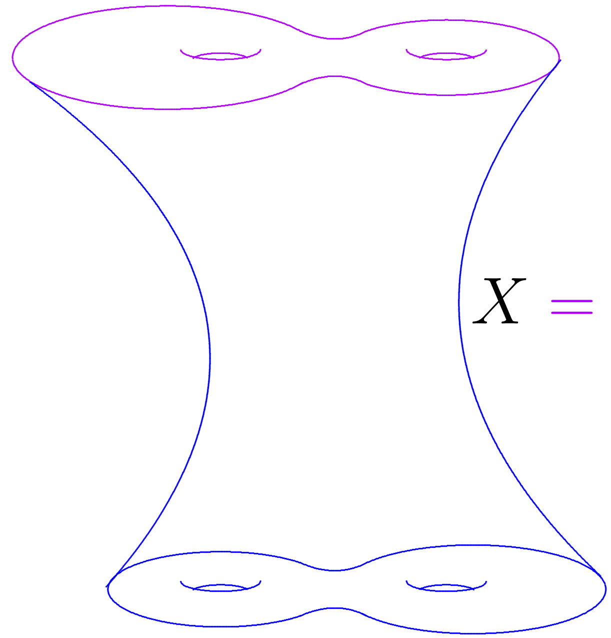

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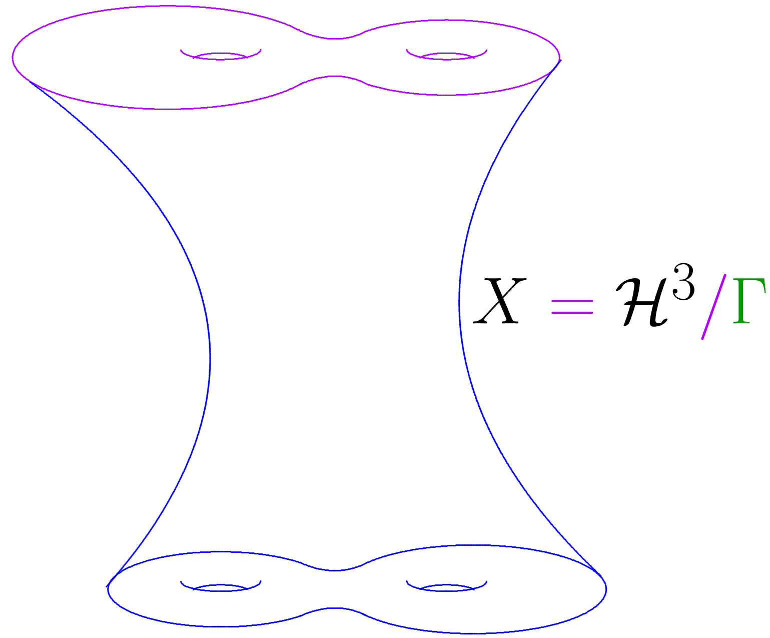
Construction of conformally flat 4-manifolds:



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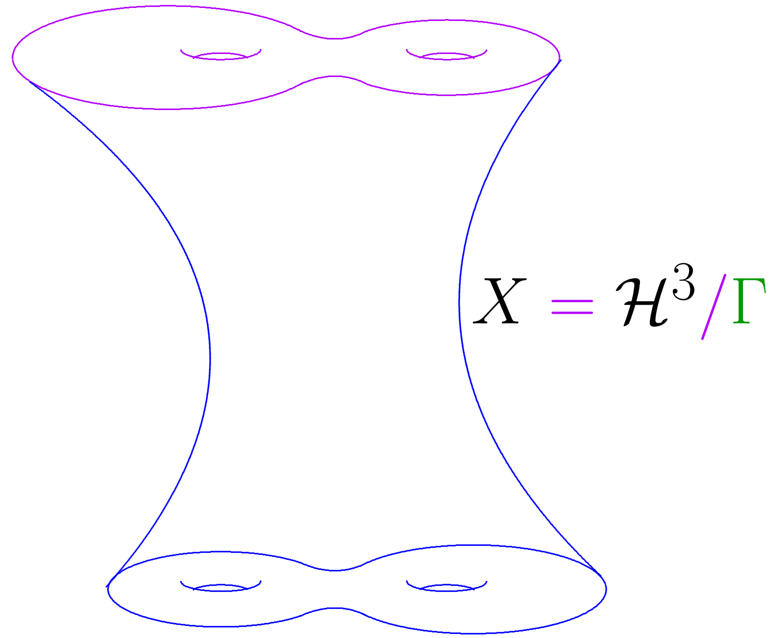

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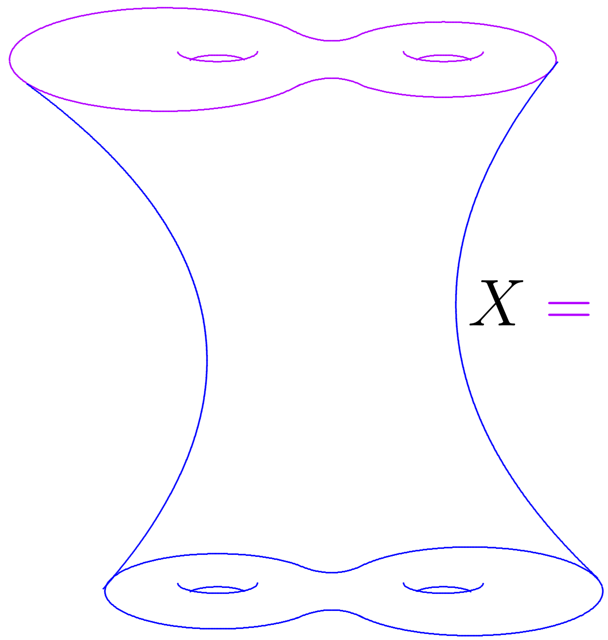
$\sim$: crush $\partial\overline{X} \times S^1$ to $\partial\overline{X}$.



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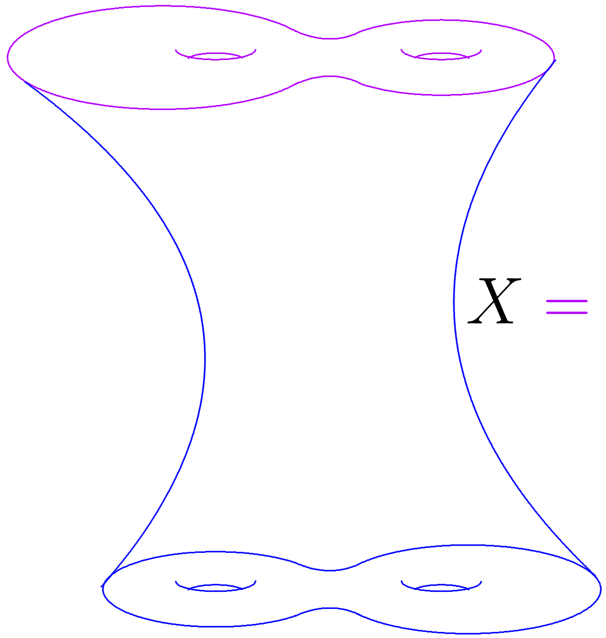

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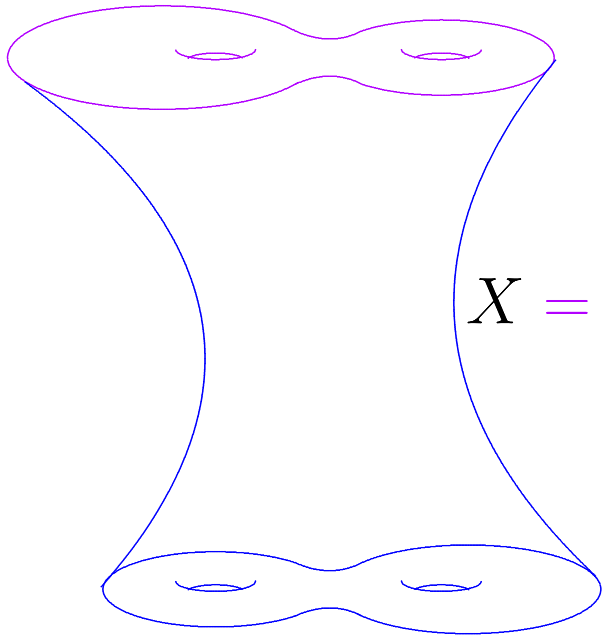

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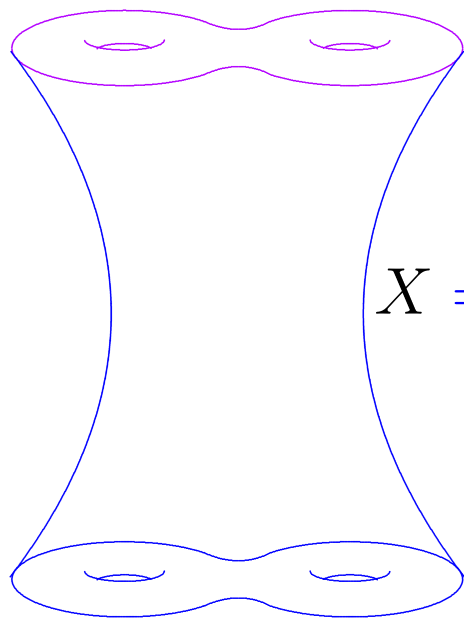

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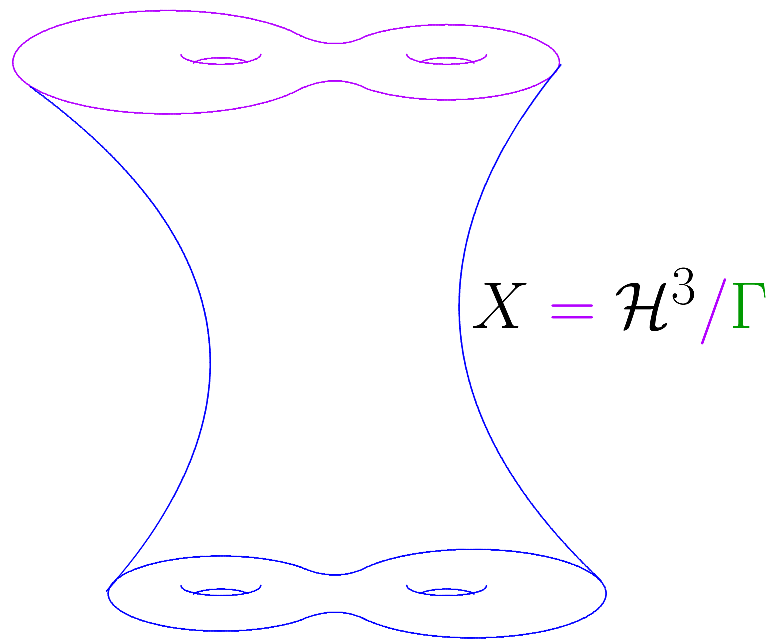
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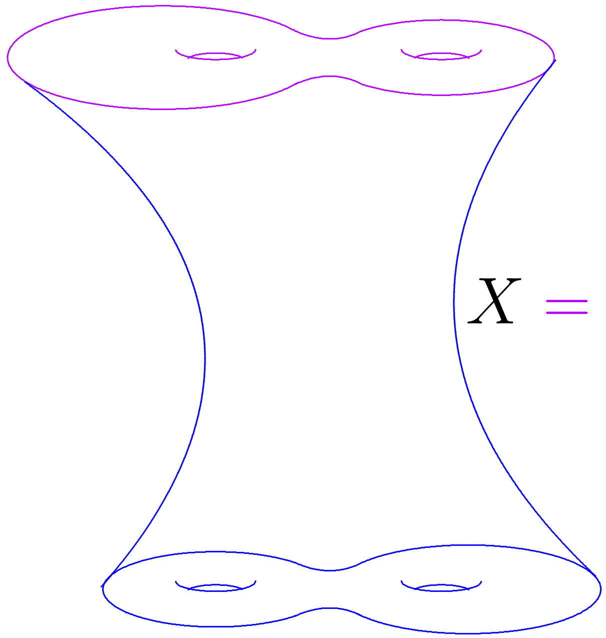

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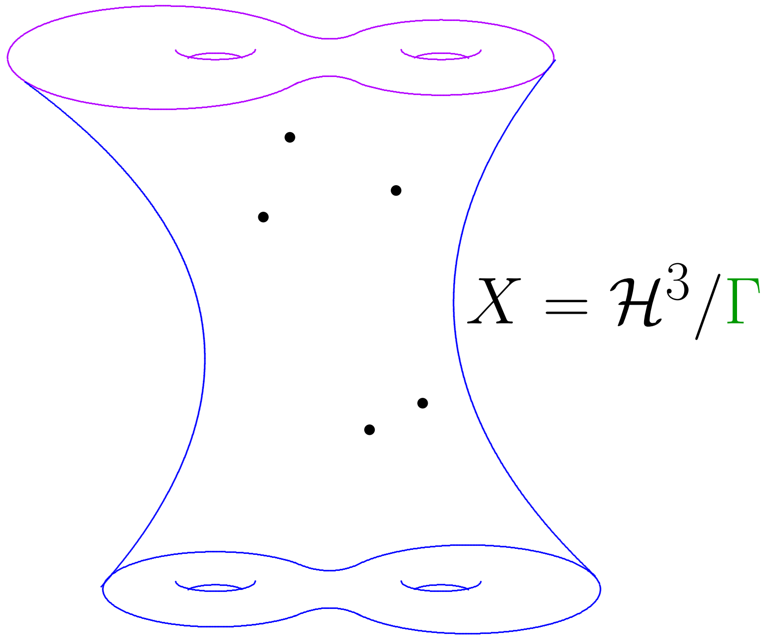

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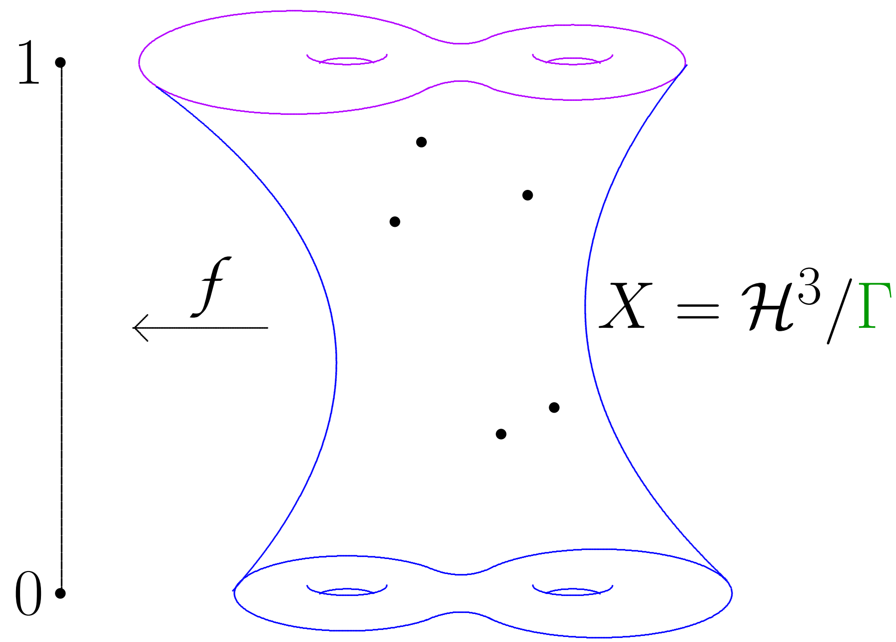

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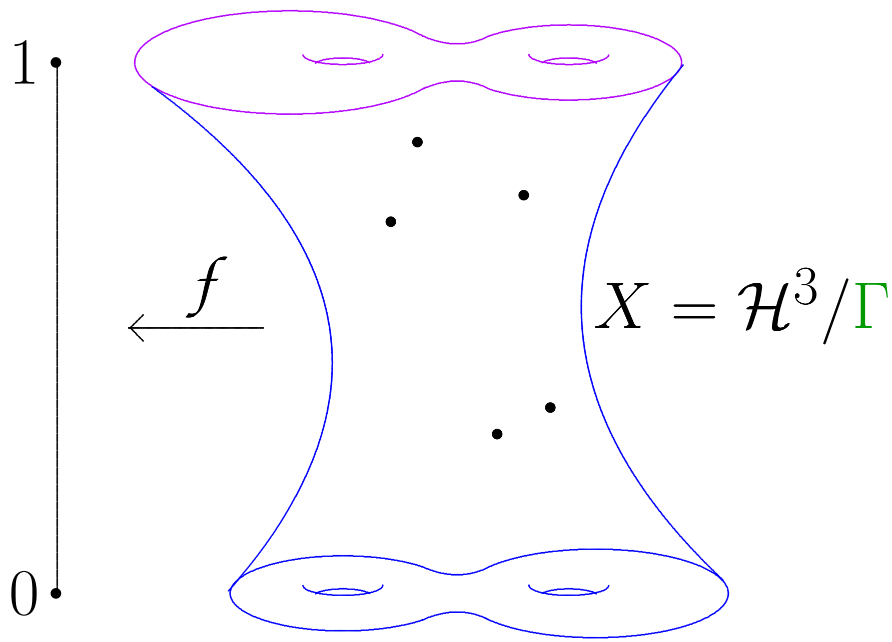

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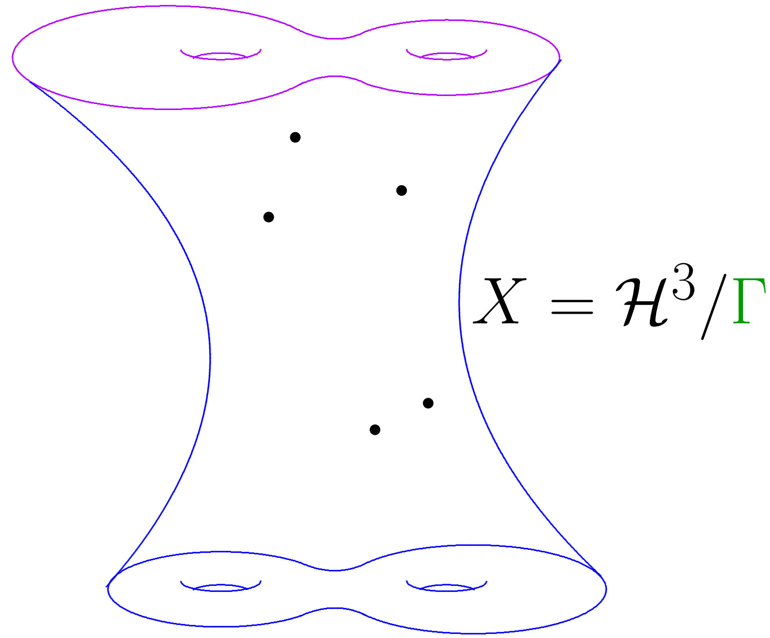

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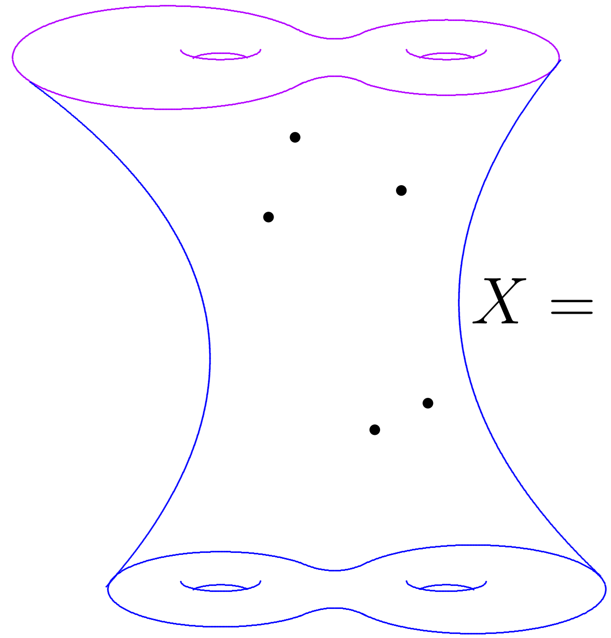

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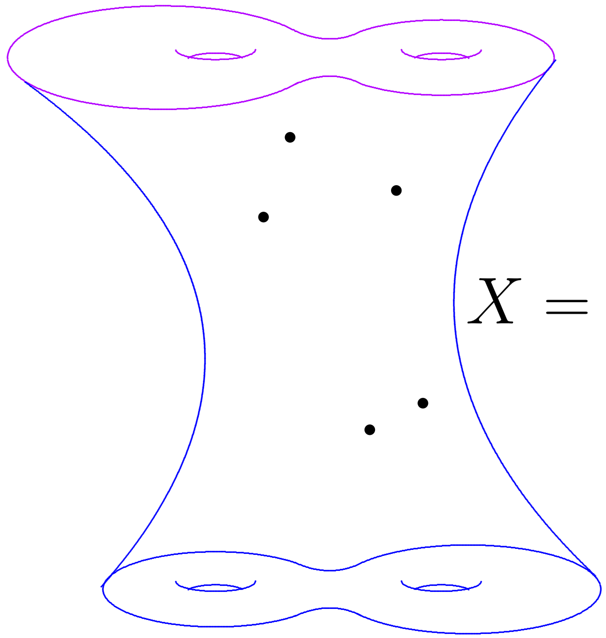
$$\Delta G_j = 2\pi\delta_{p_j}, \qquad G_j \to 0 \text{ at } \partial\overline{X}$$



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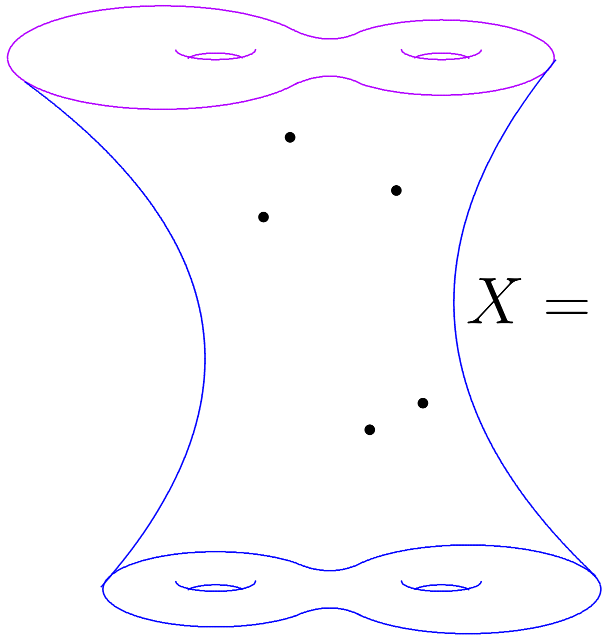
Let G_j be the Green's function of p_j, and set

$$V = 1 + \sum_{j=1}^{k} G_j.$$



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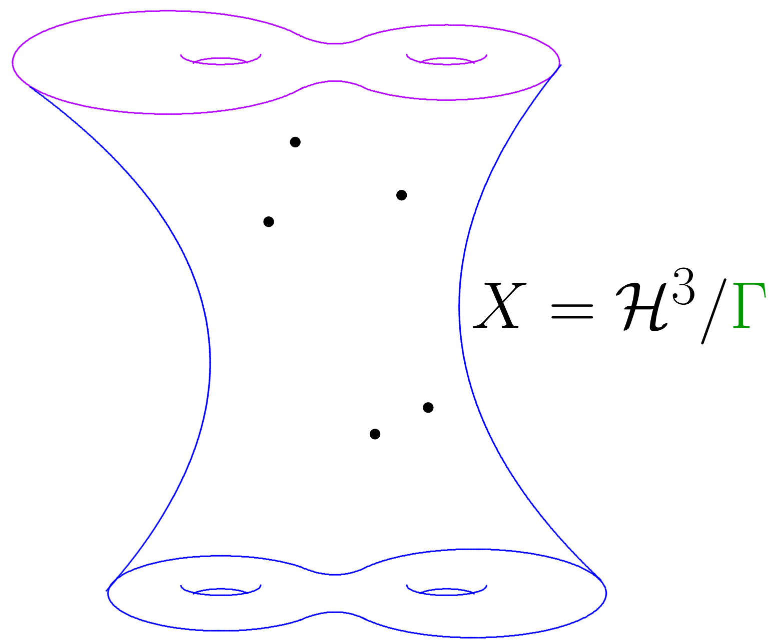

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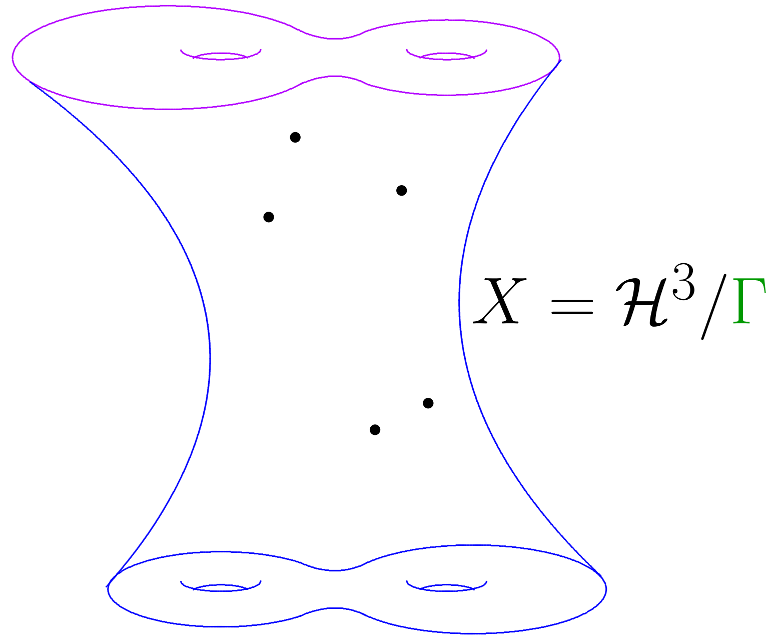
Choose $P \to (X - \{p_1, \ldots, p_k\})$ circle bundle with connection form θ such that

$$d\theta = \star dV.$$



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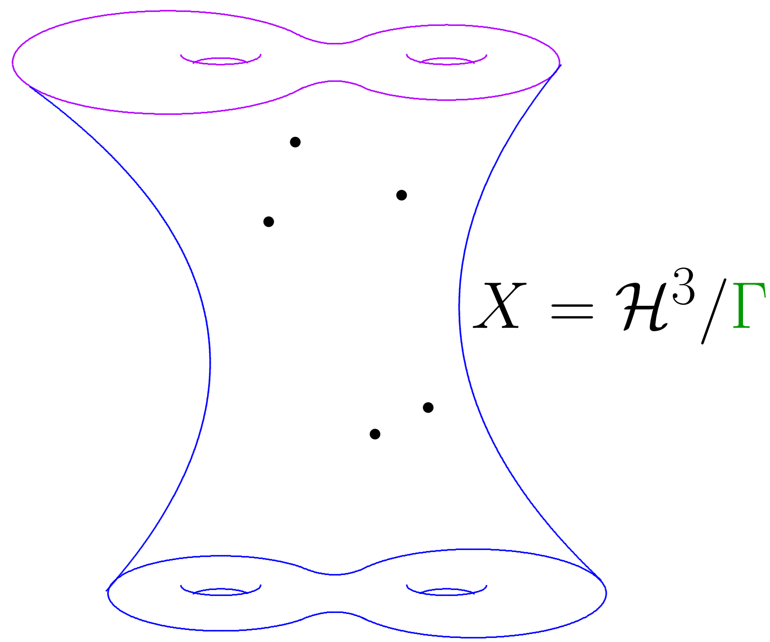

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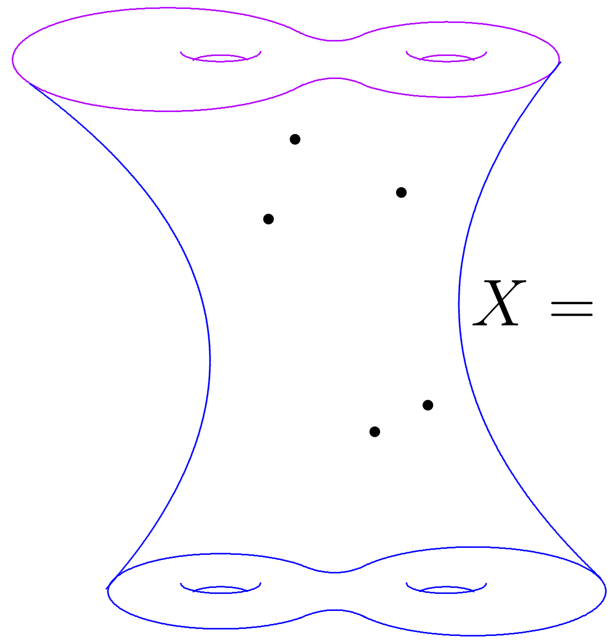

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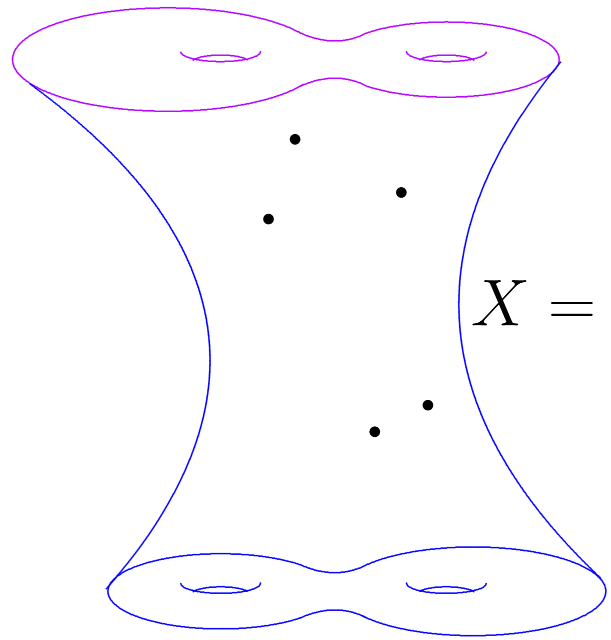

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\downarrow & & & \downarrow & & \downarrow & & \downarrow \\
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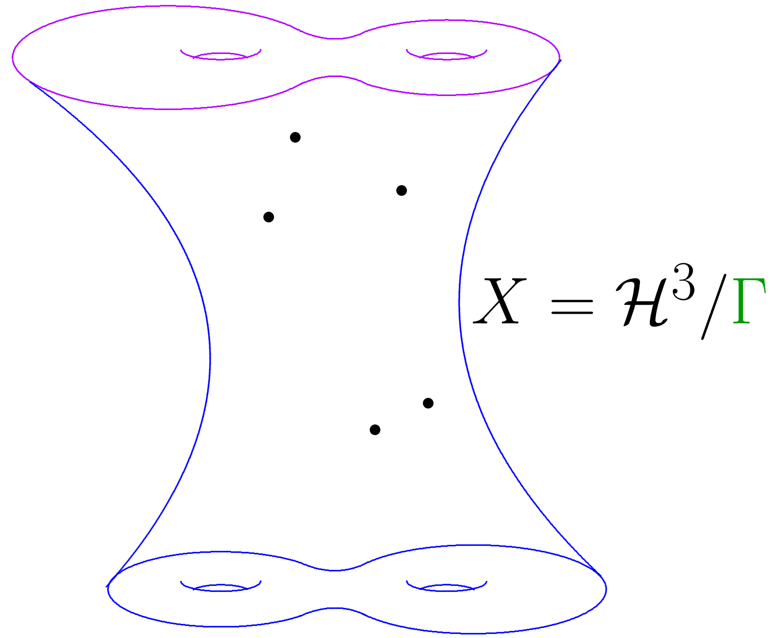
\end{array}$$



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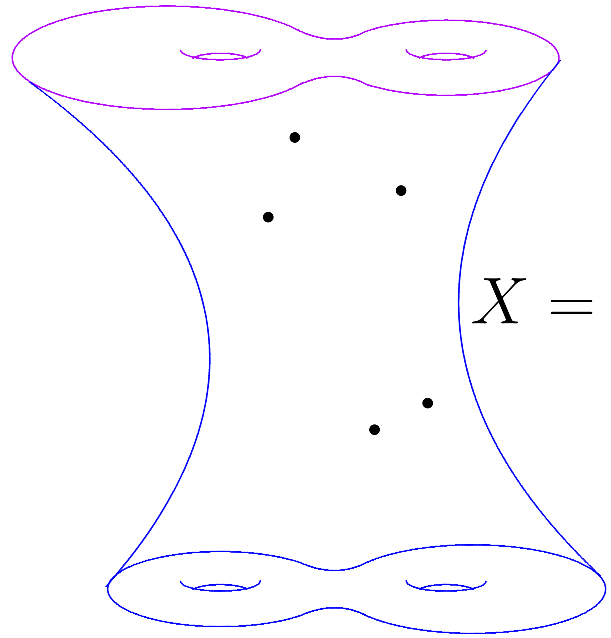

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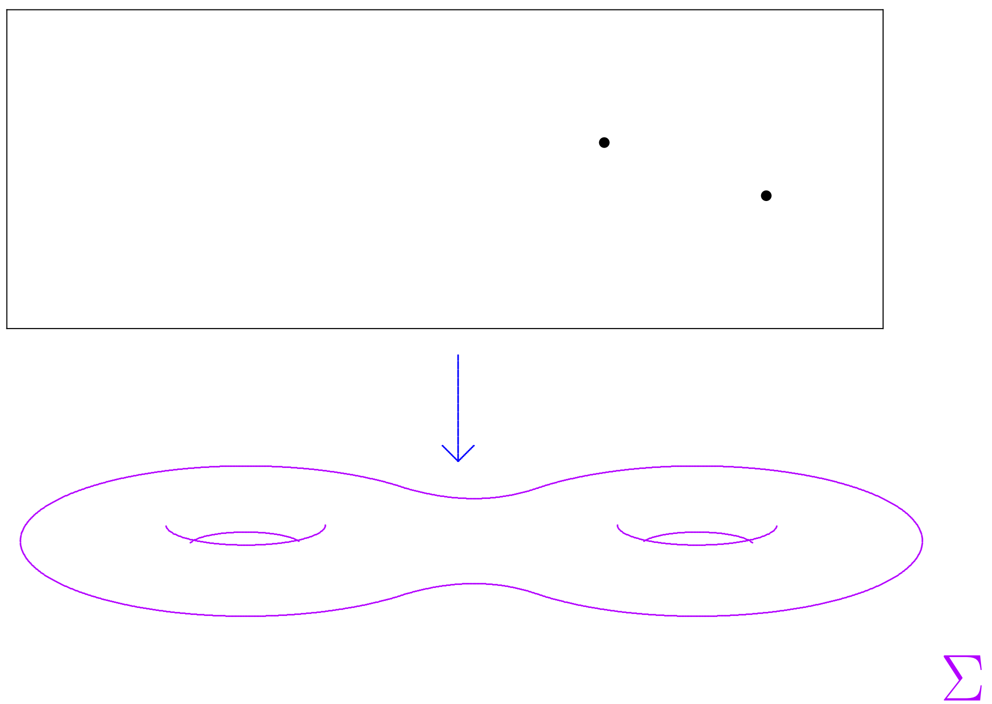

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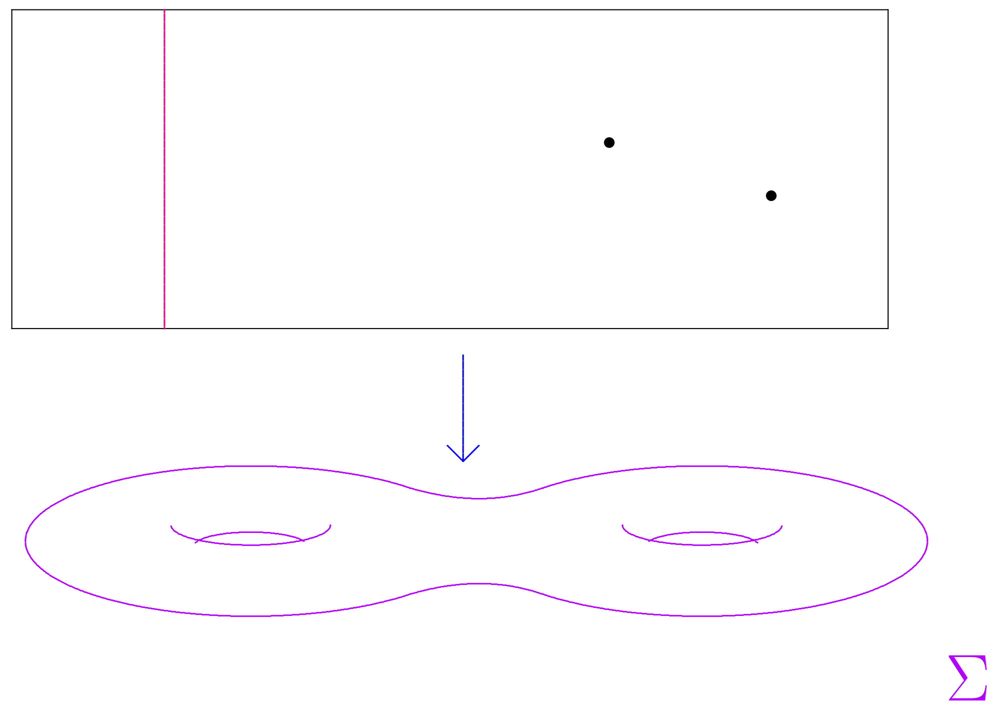

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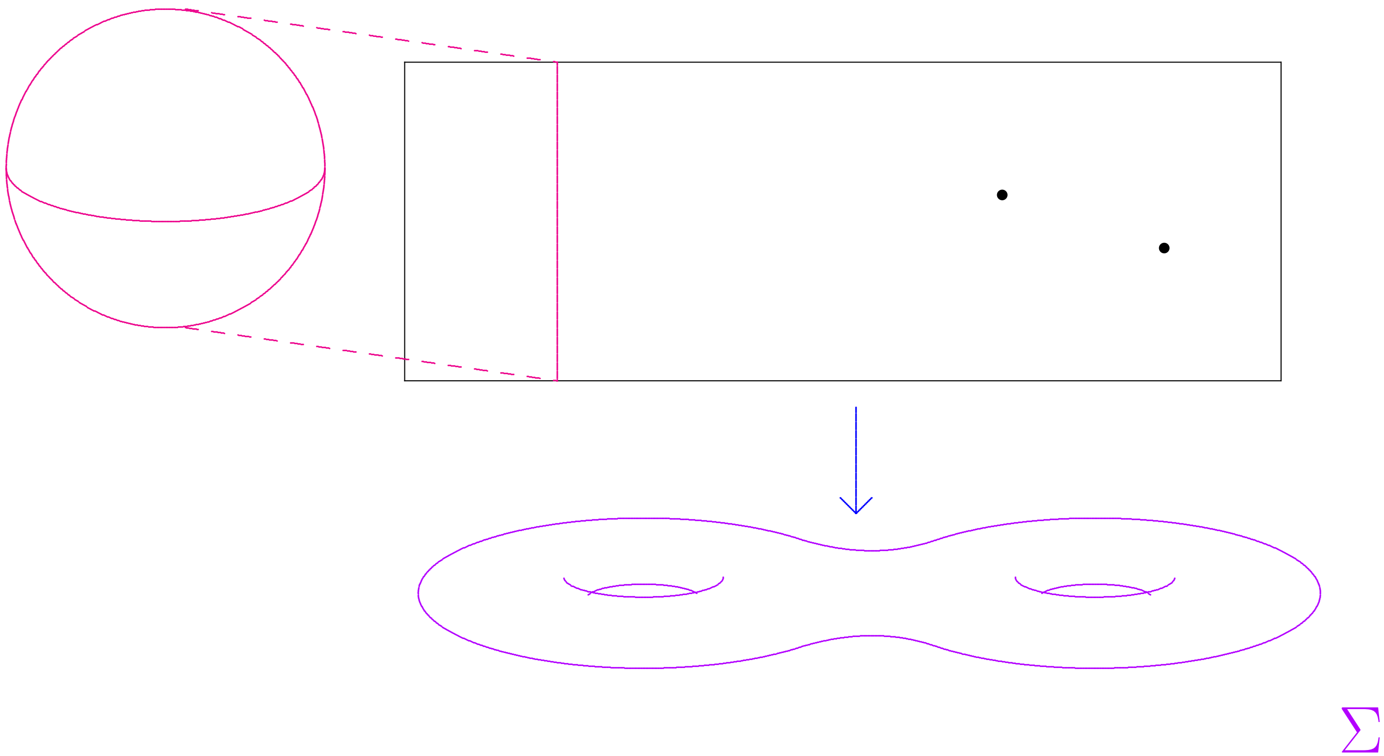

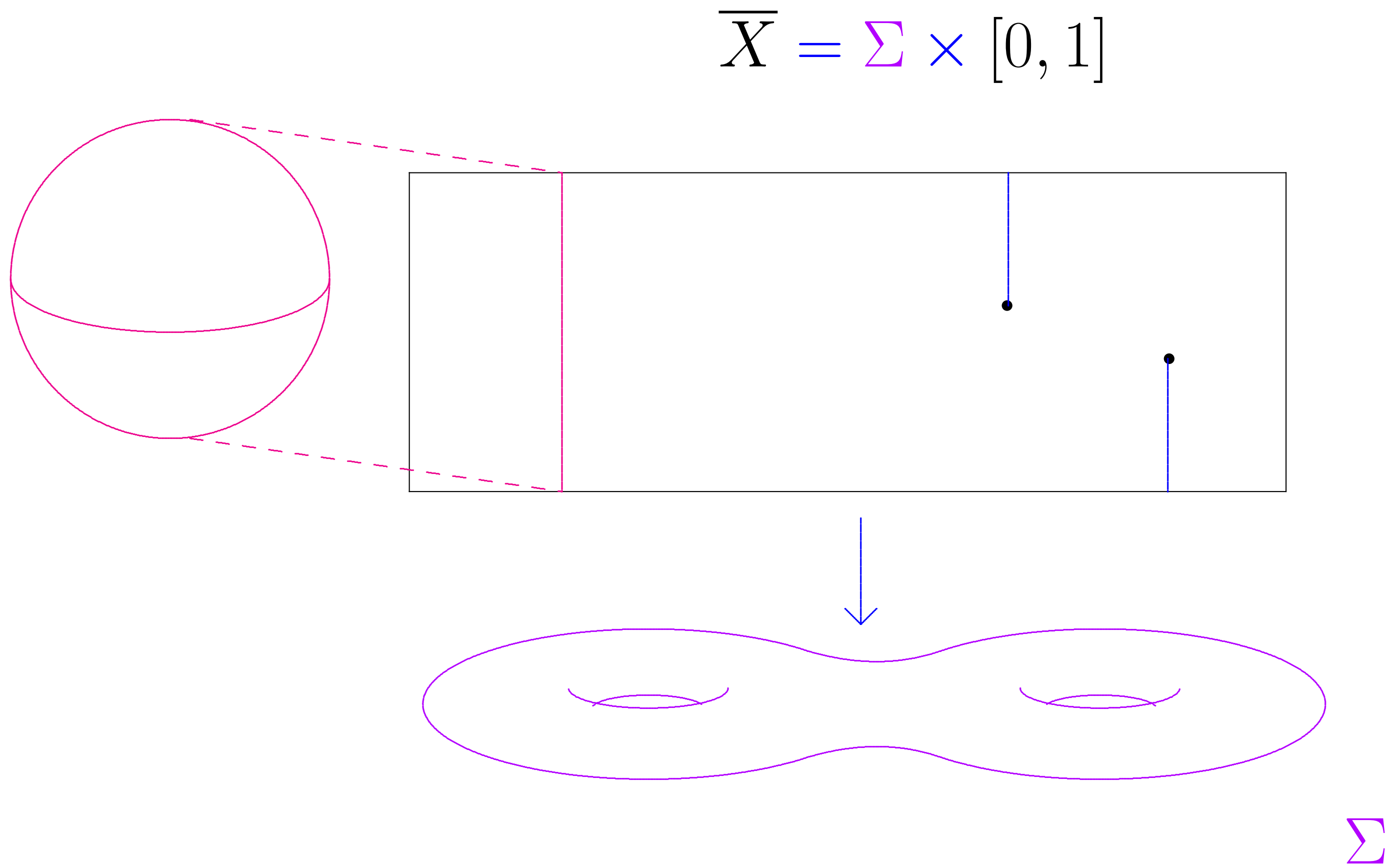
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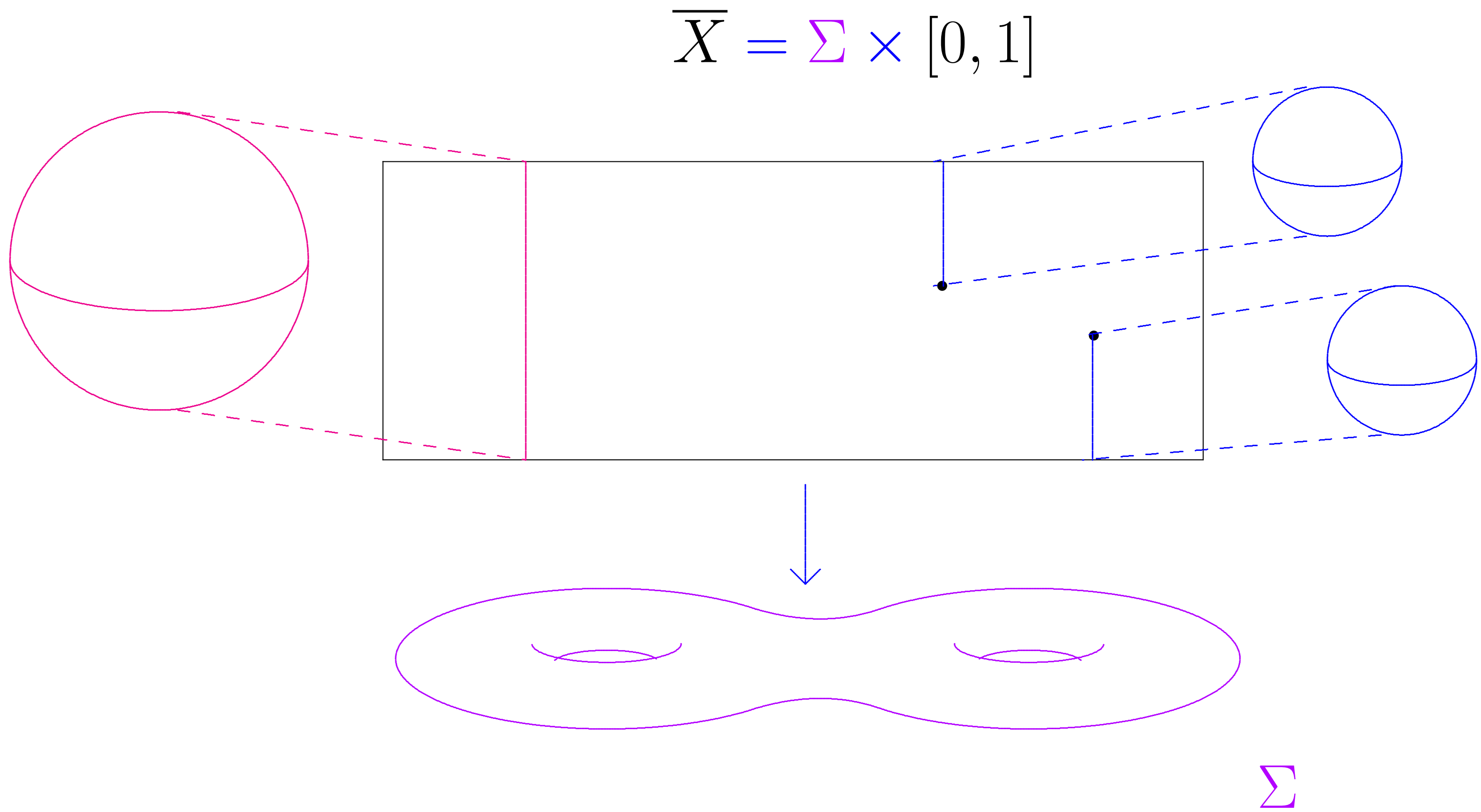
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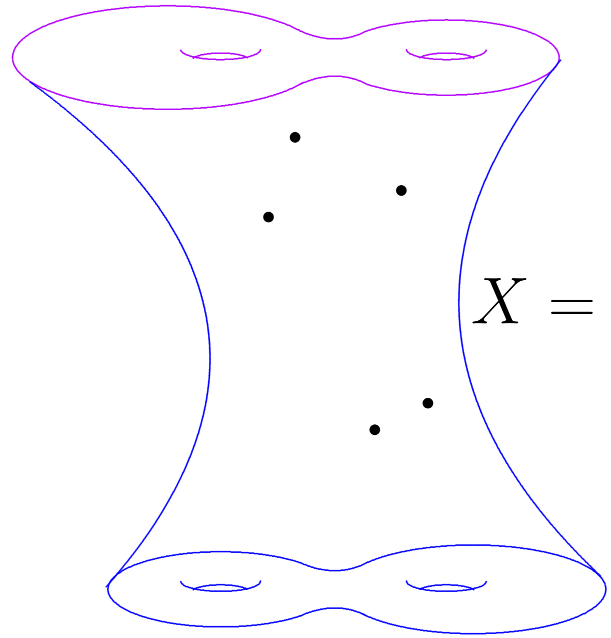

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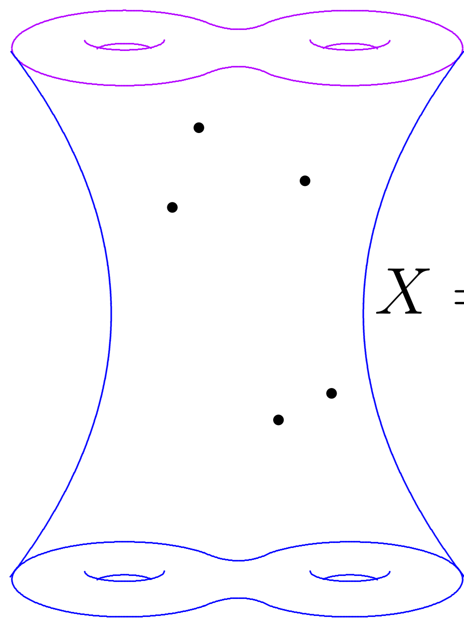

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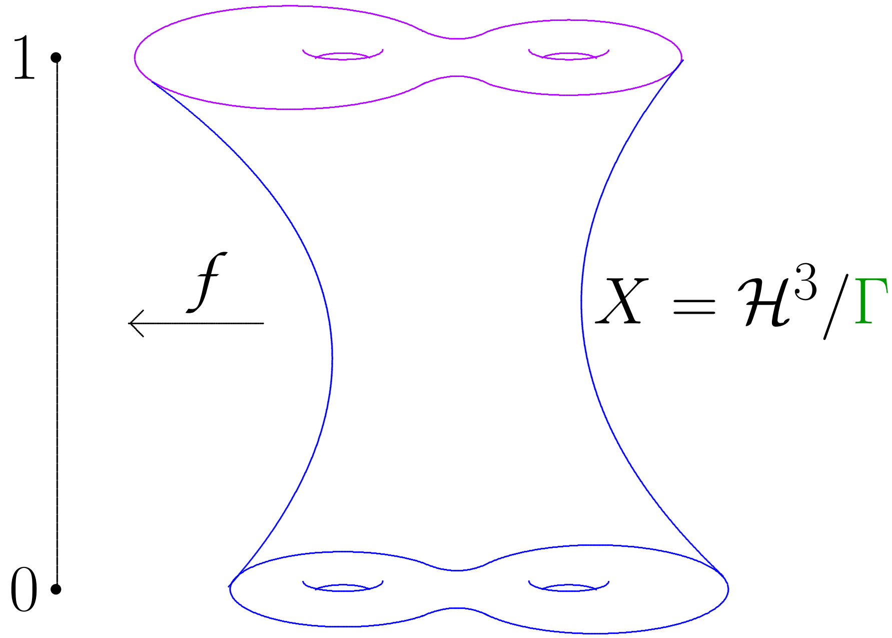

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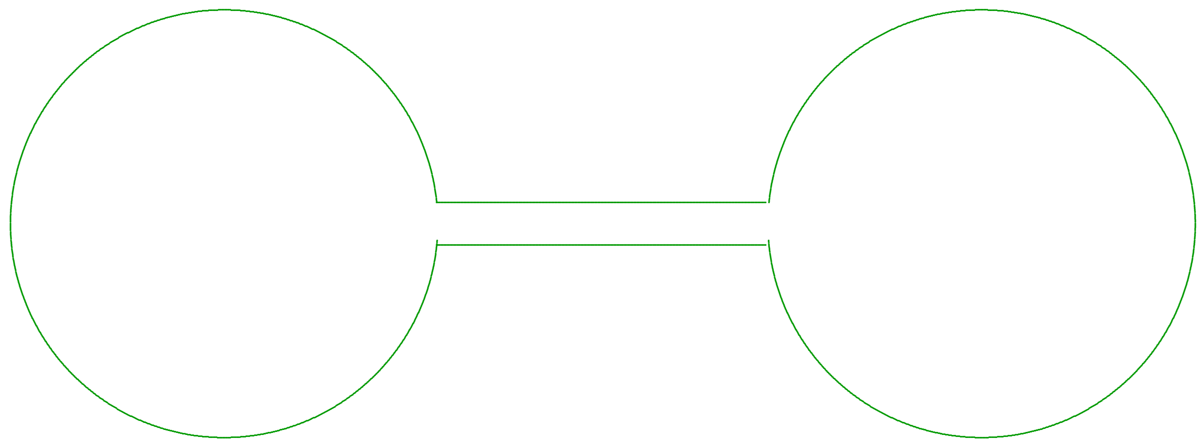
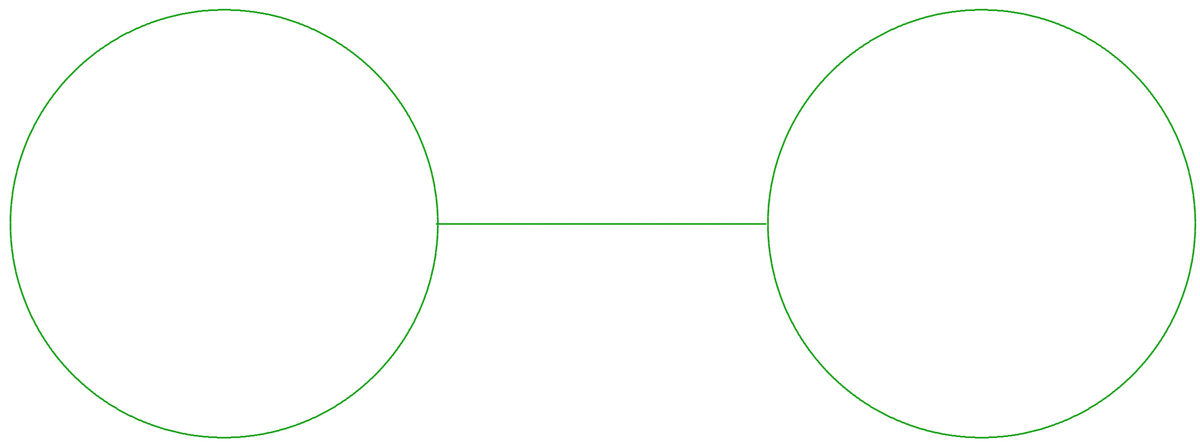
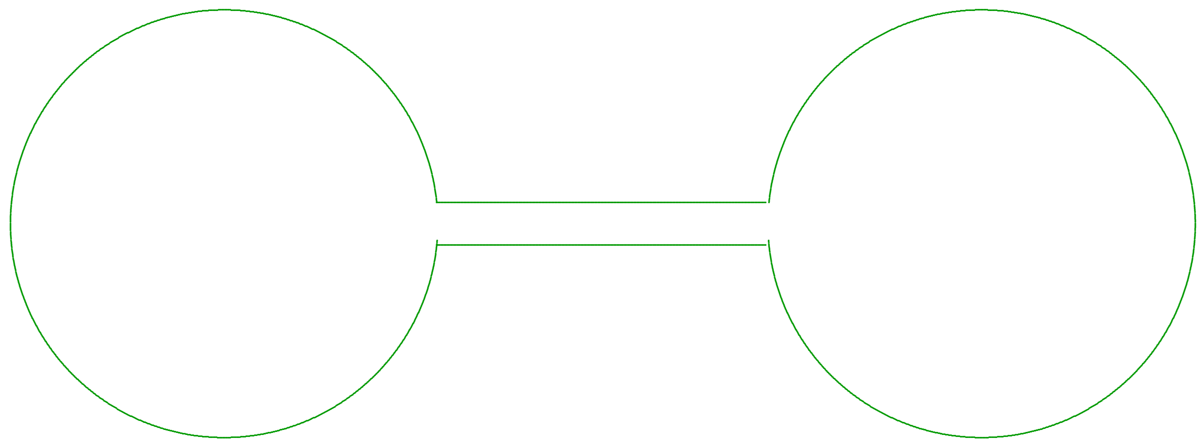
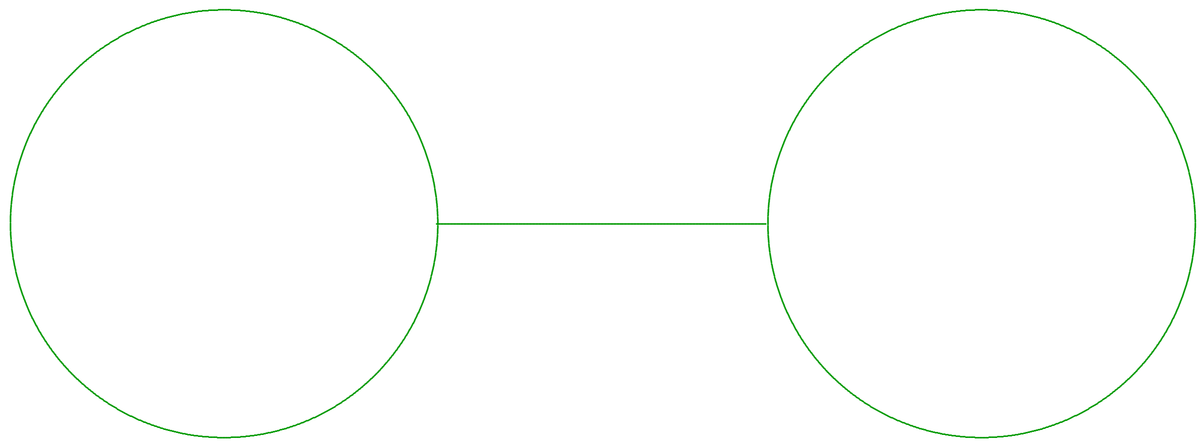
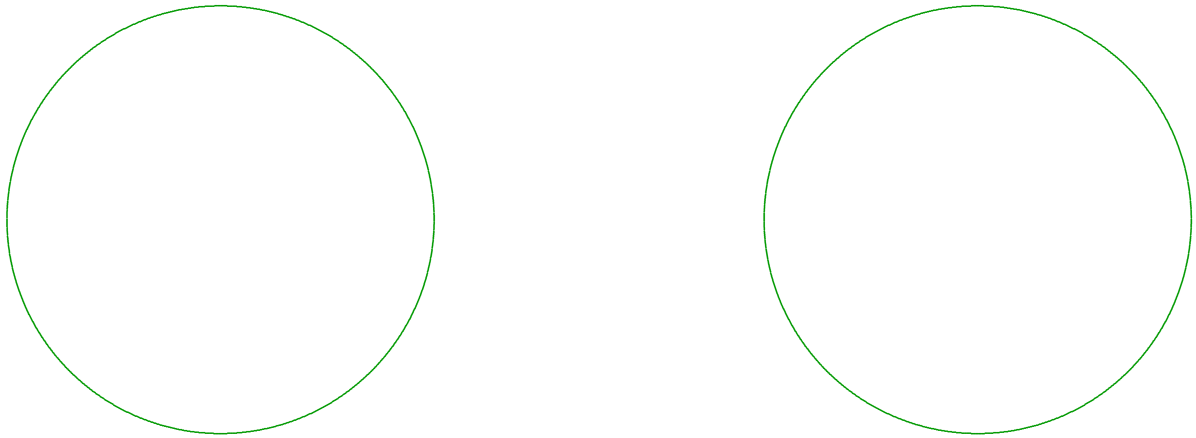
Ahlfors-Bers: Quasi-conformal mappings

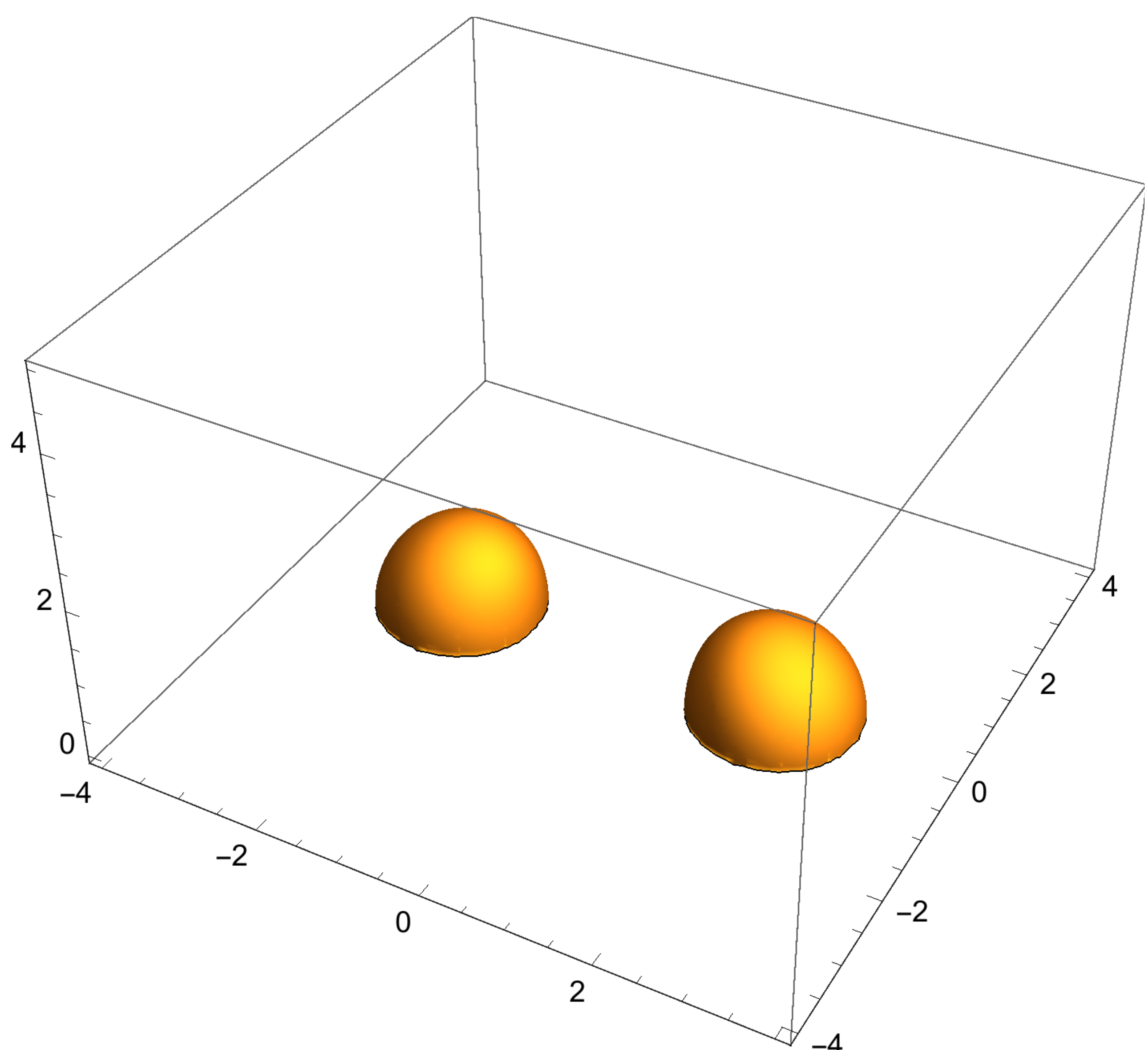








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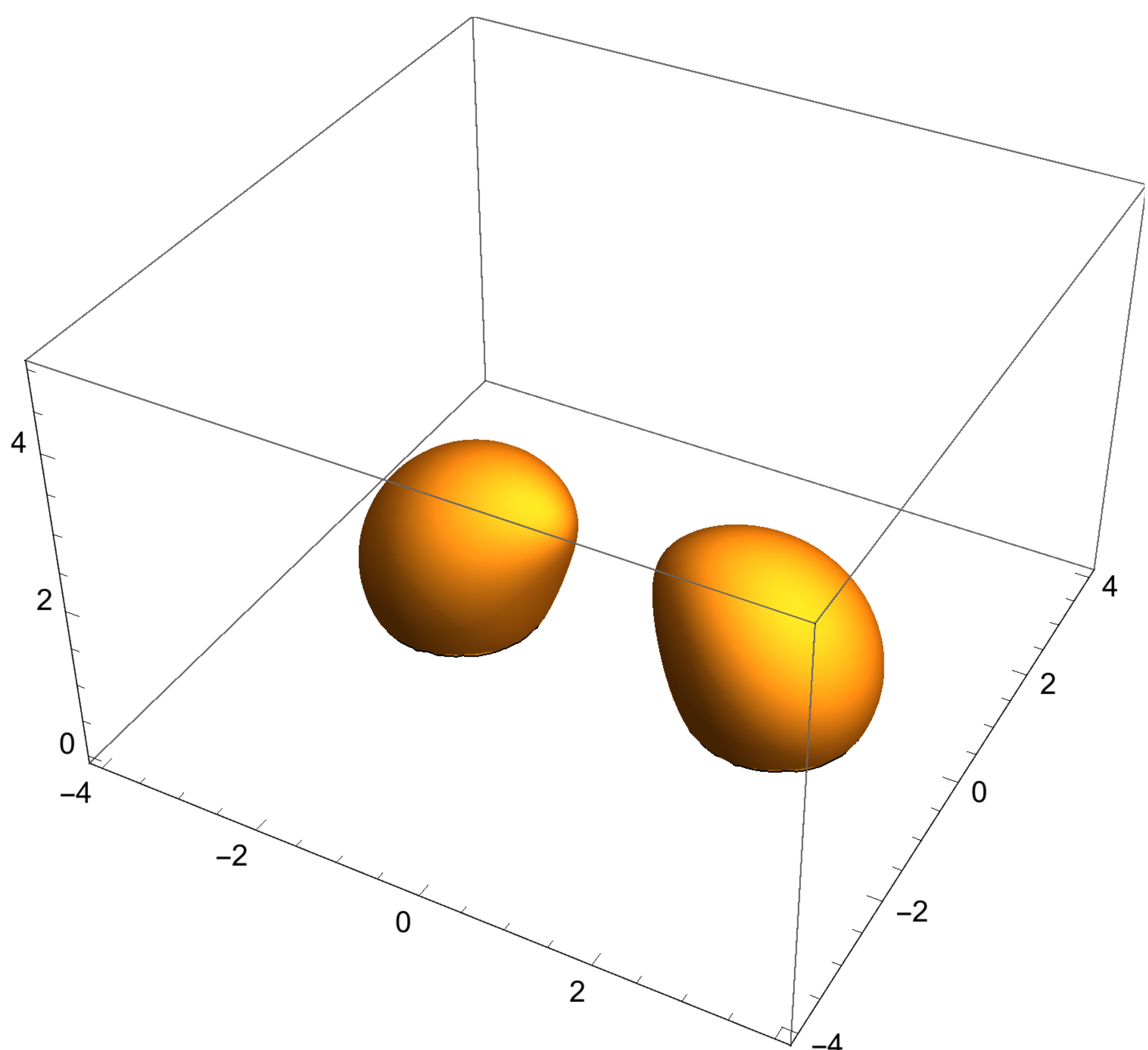
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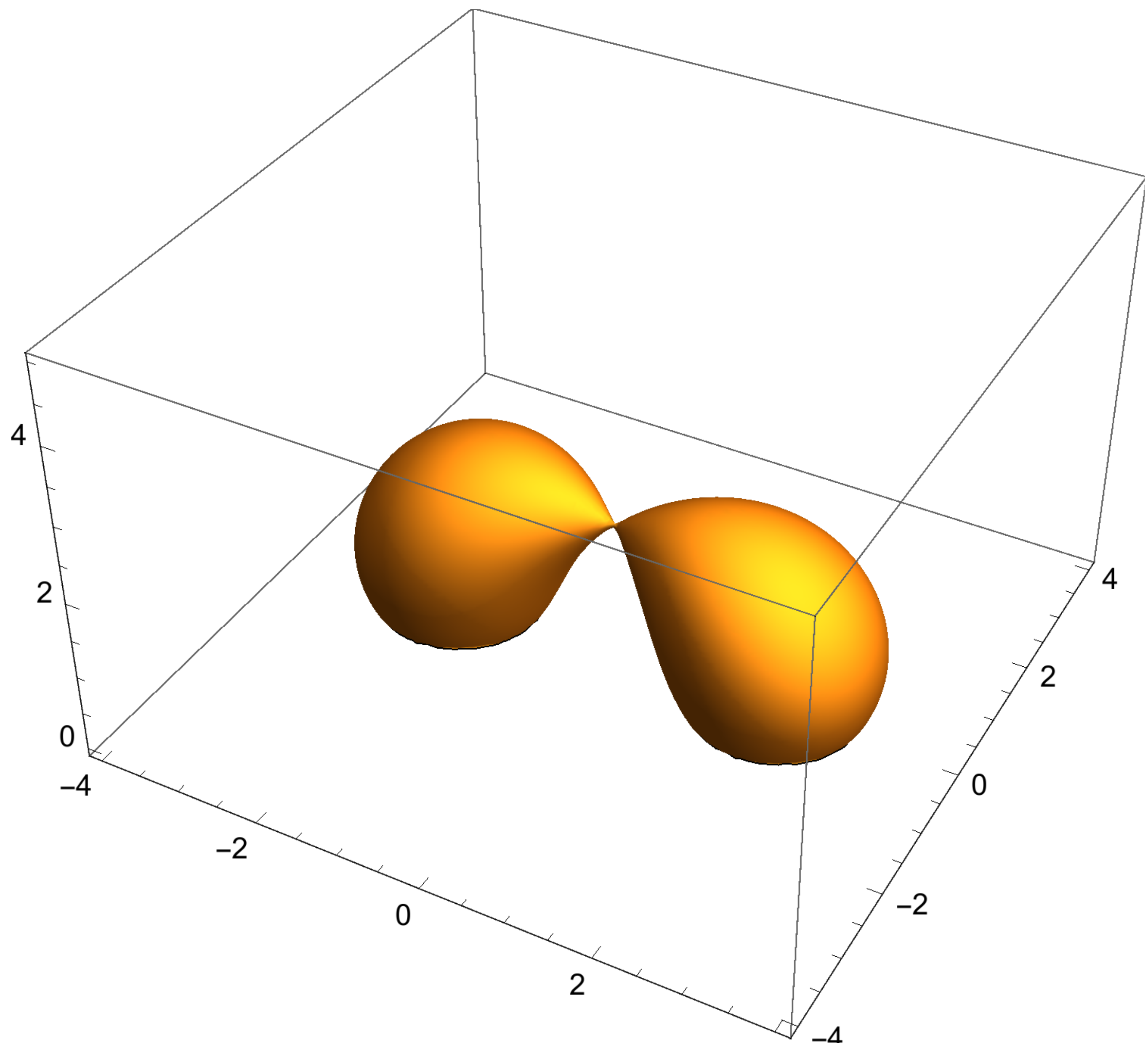

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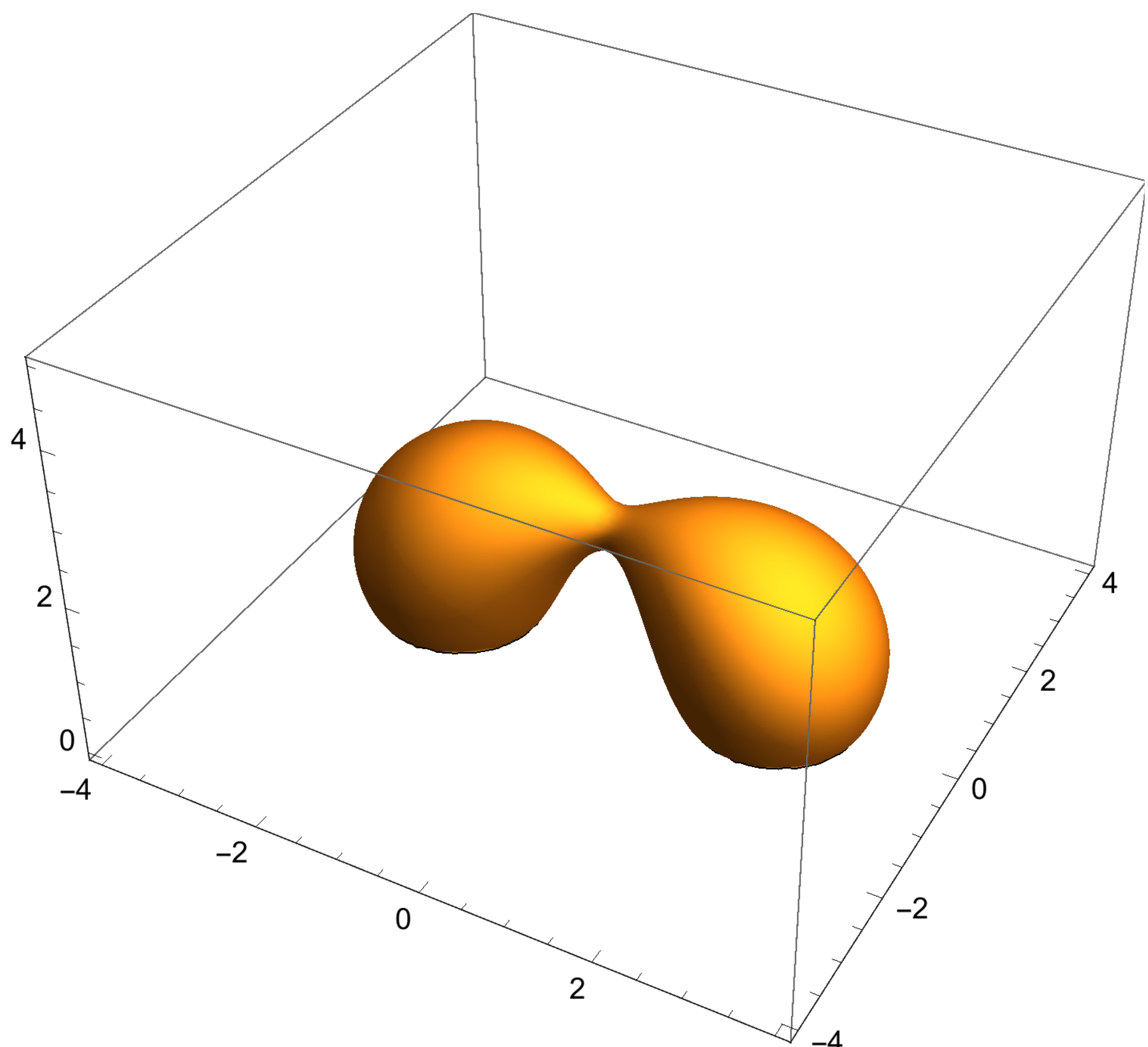
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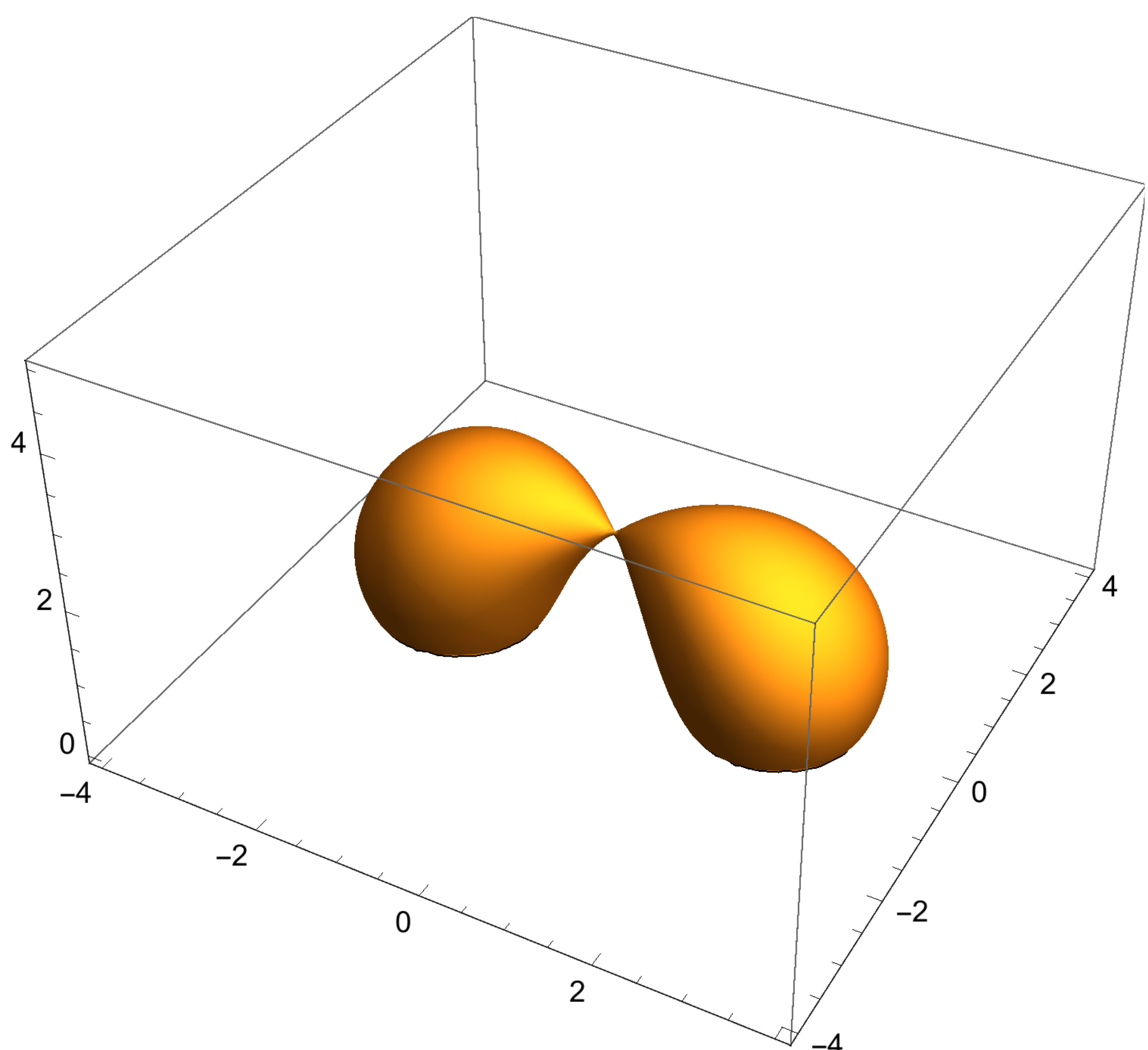
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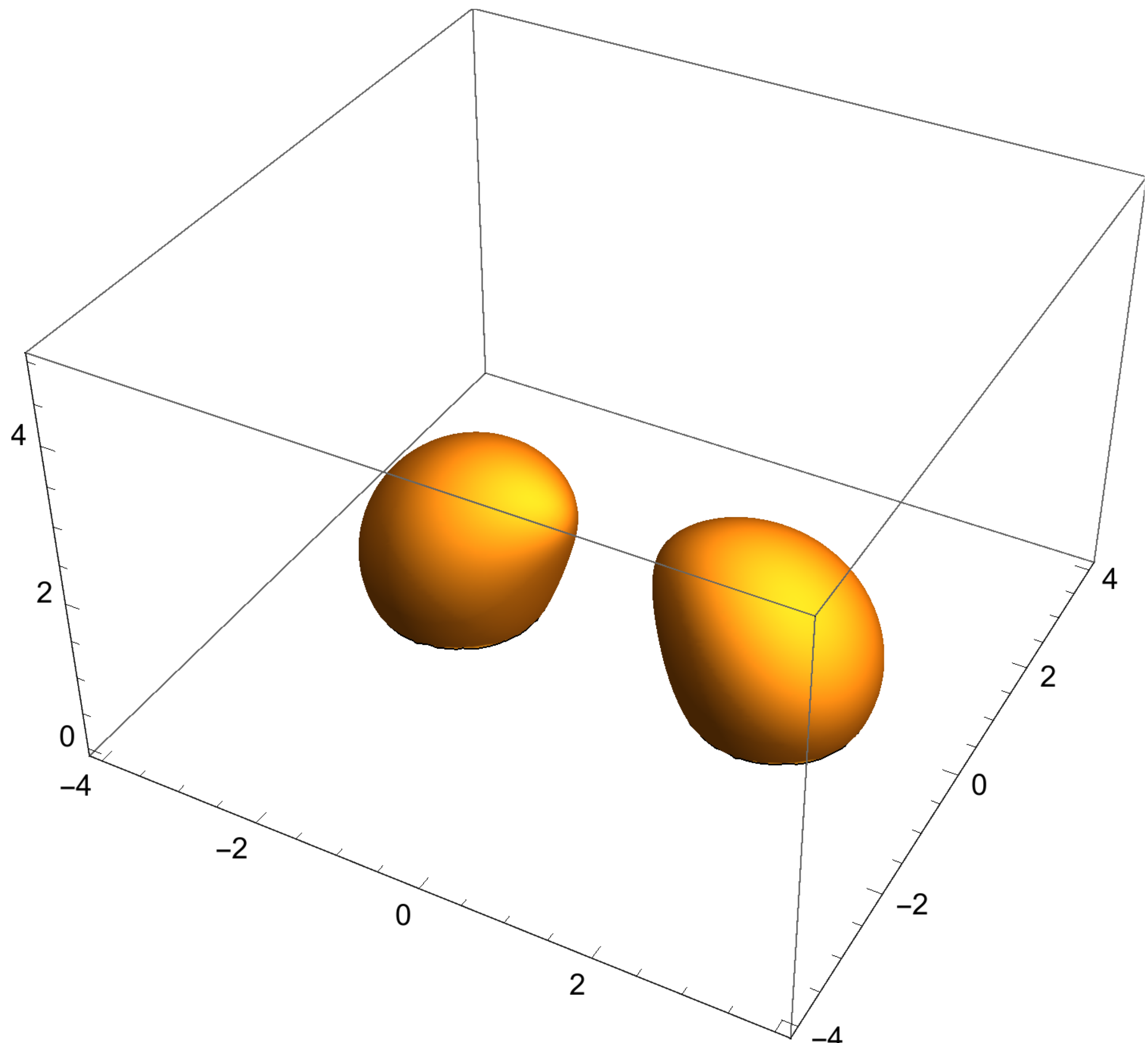

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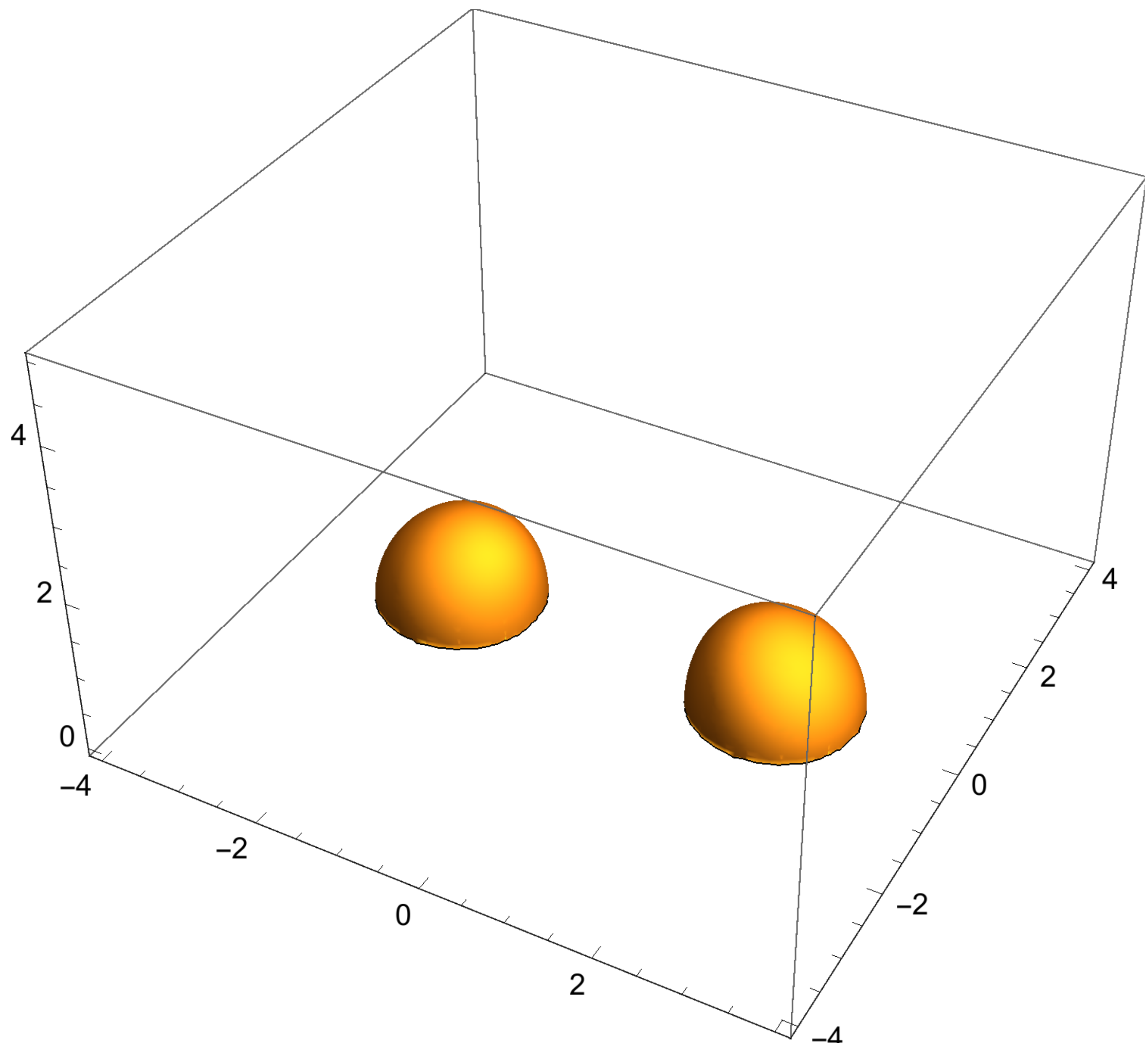
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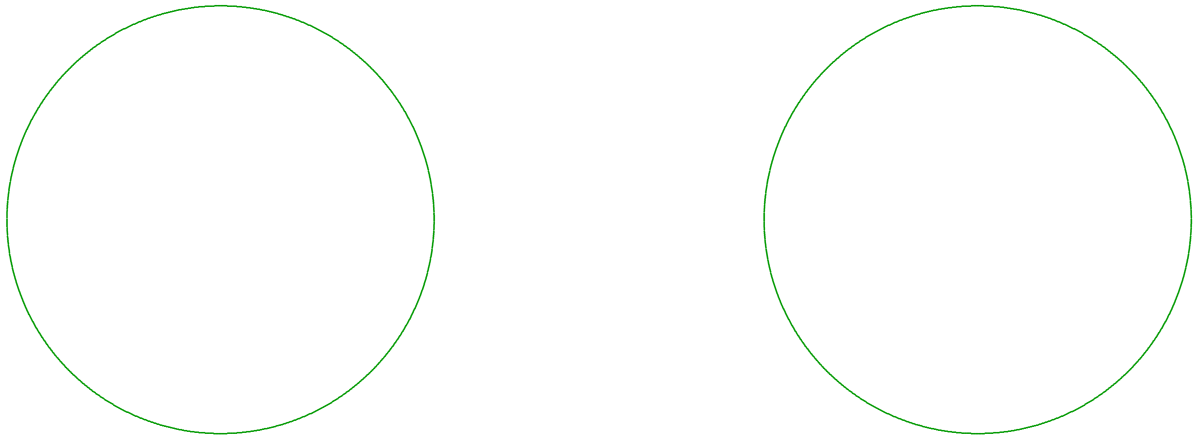
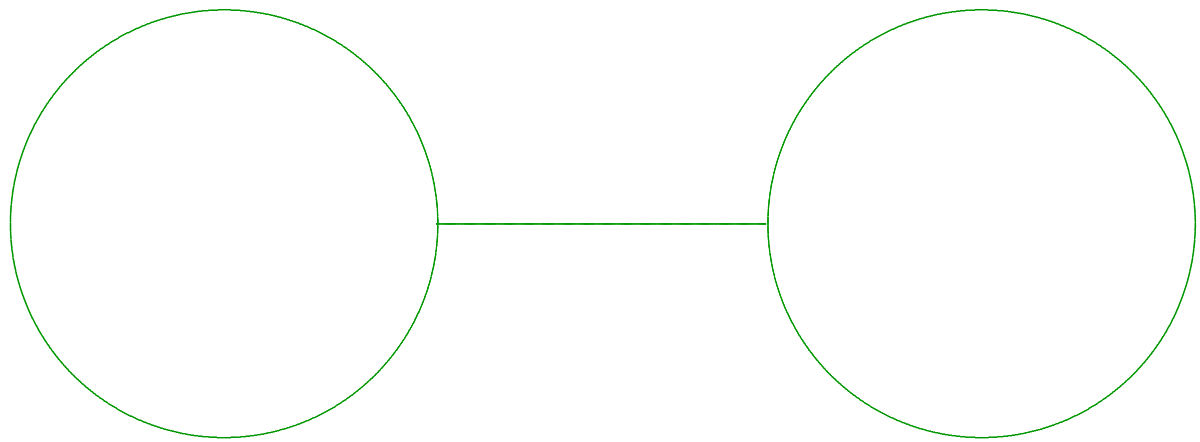
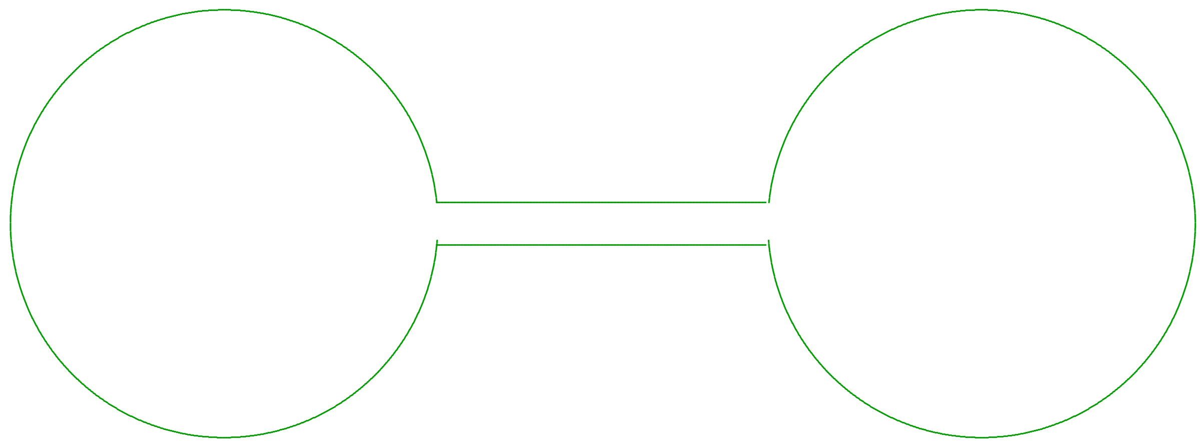
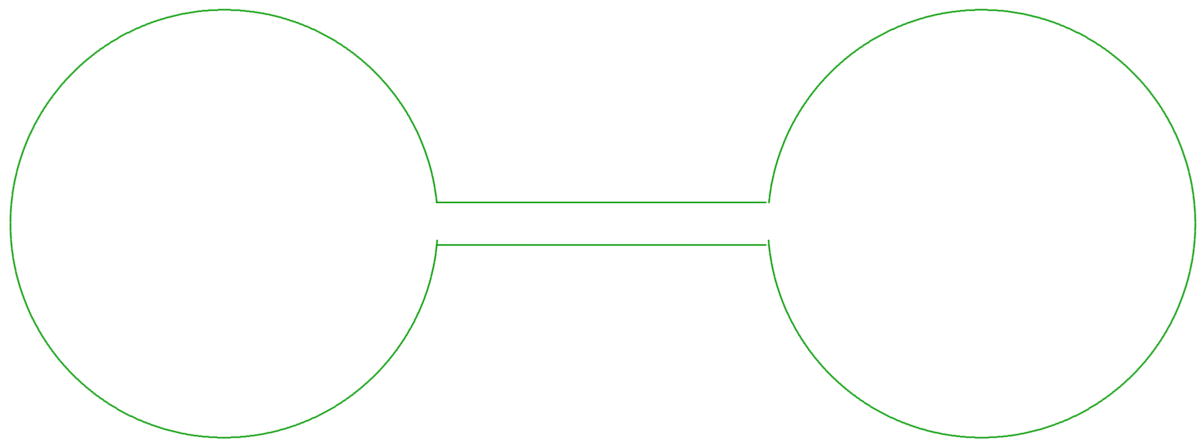
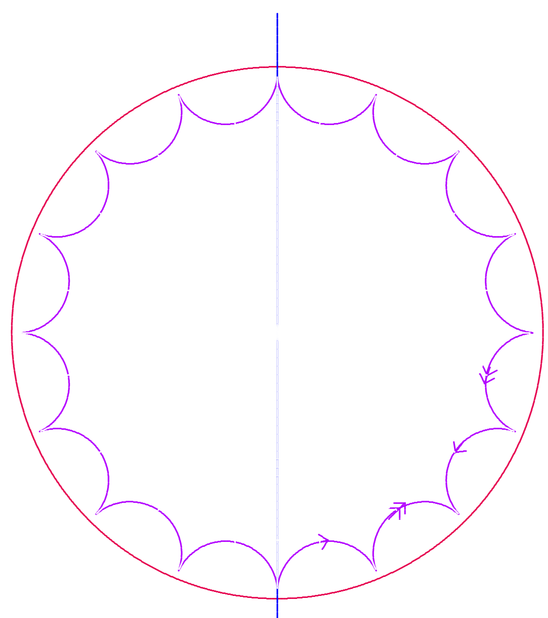
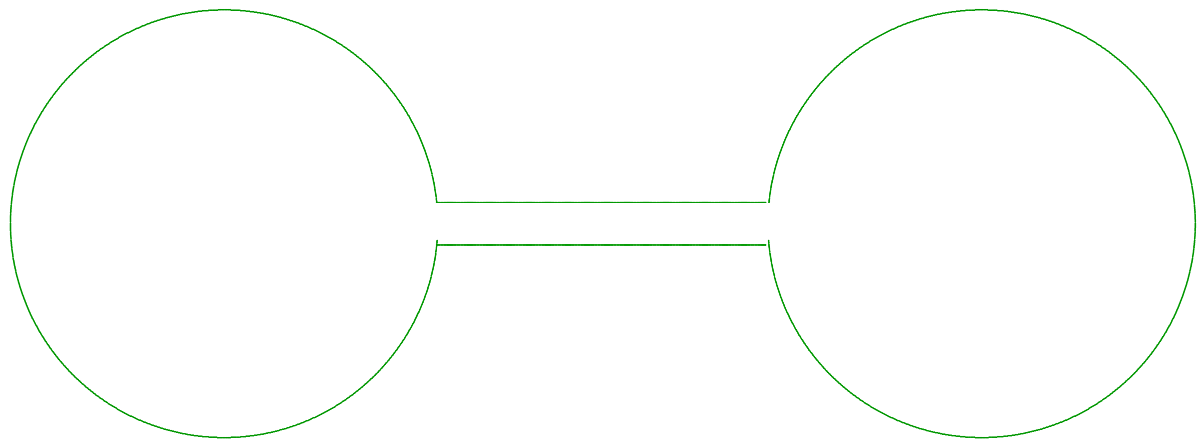


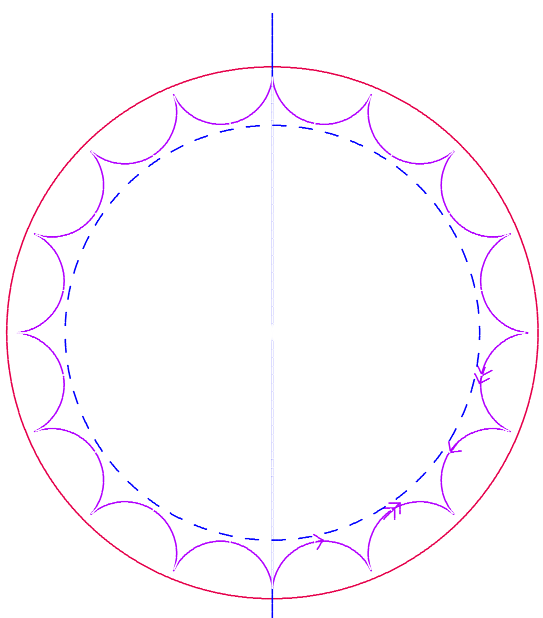


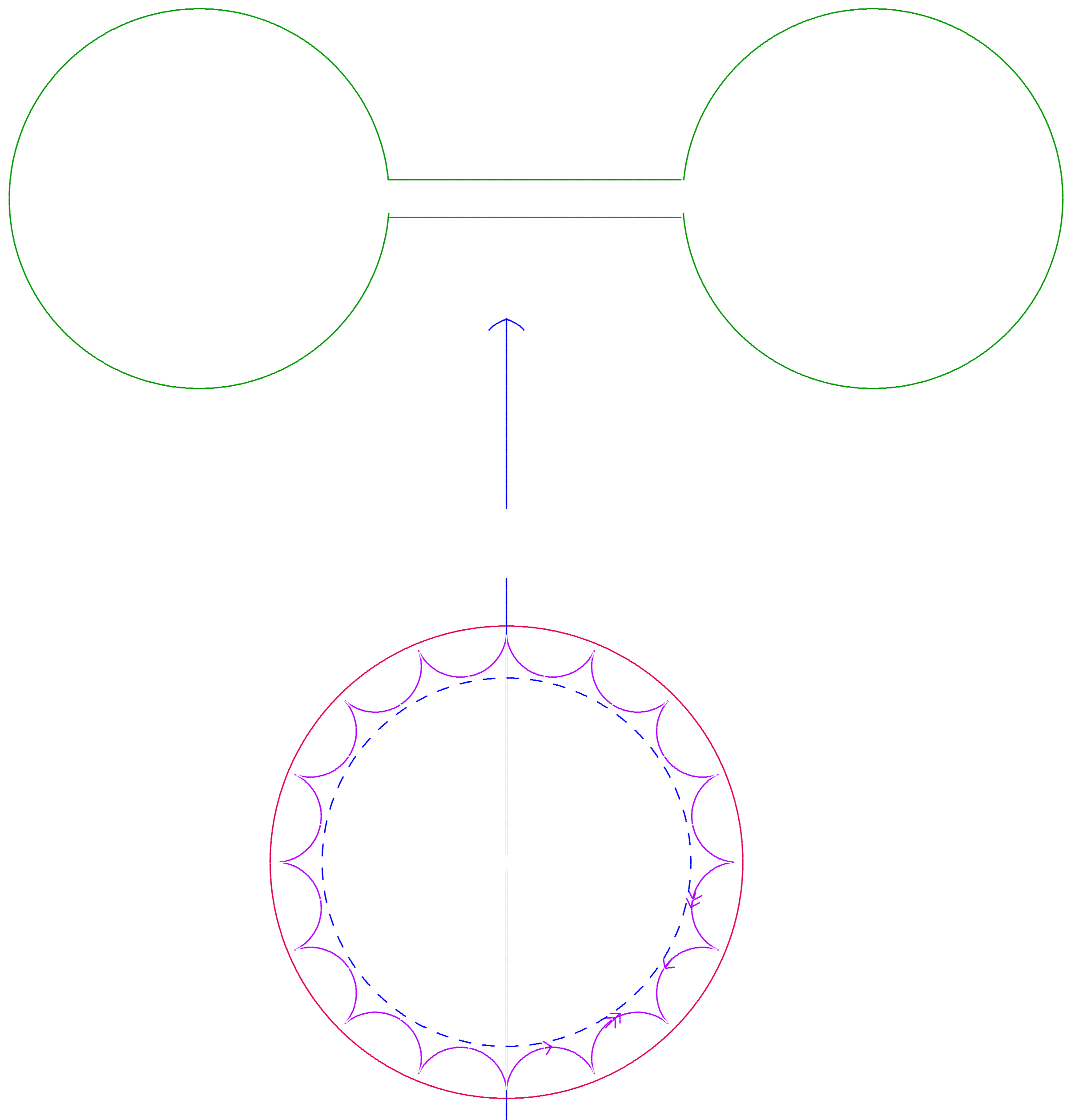


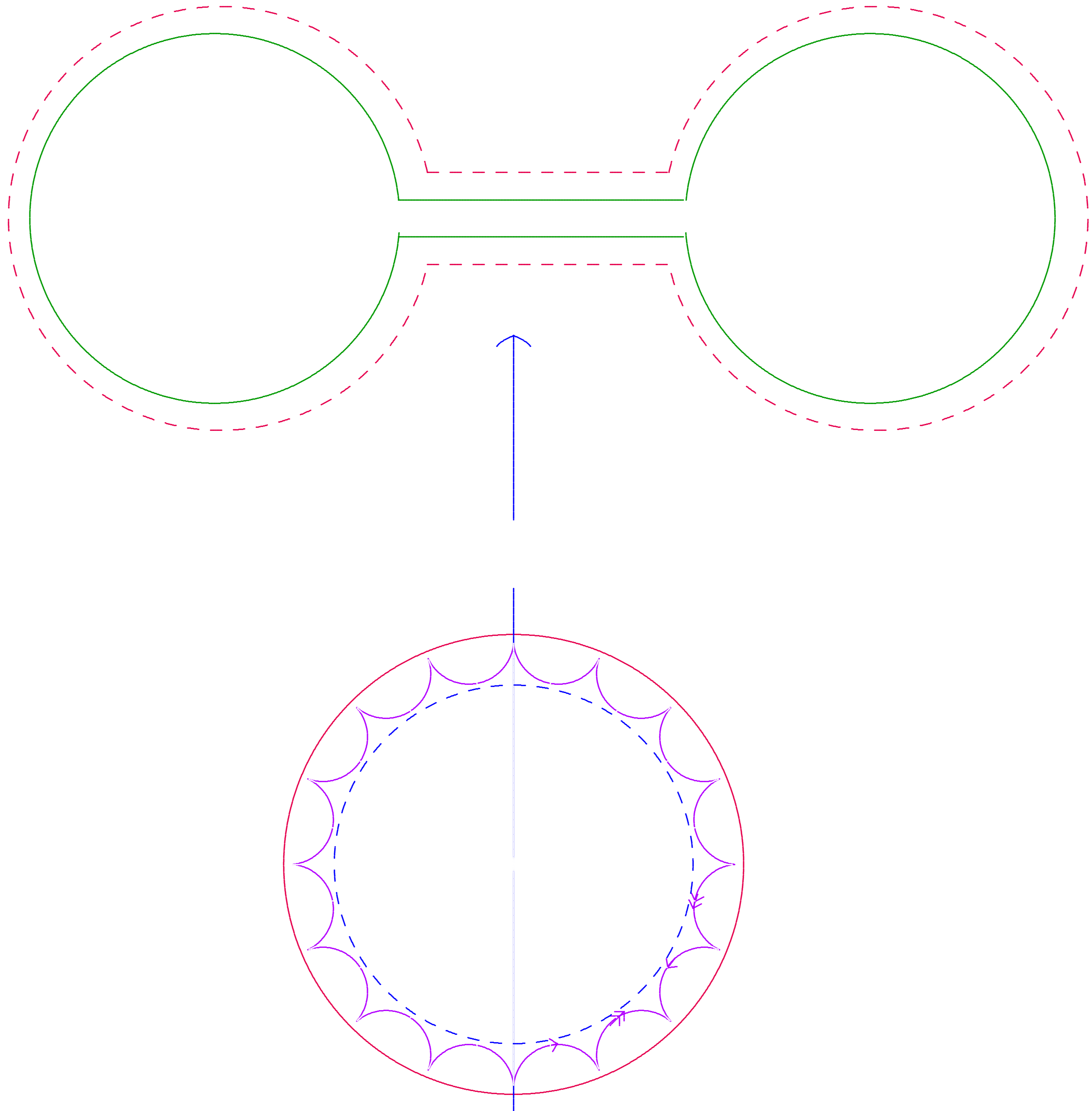


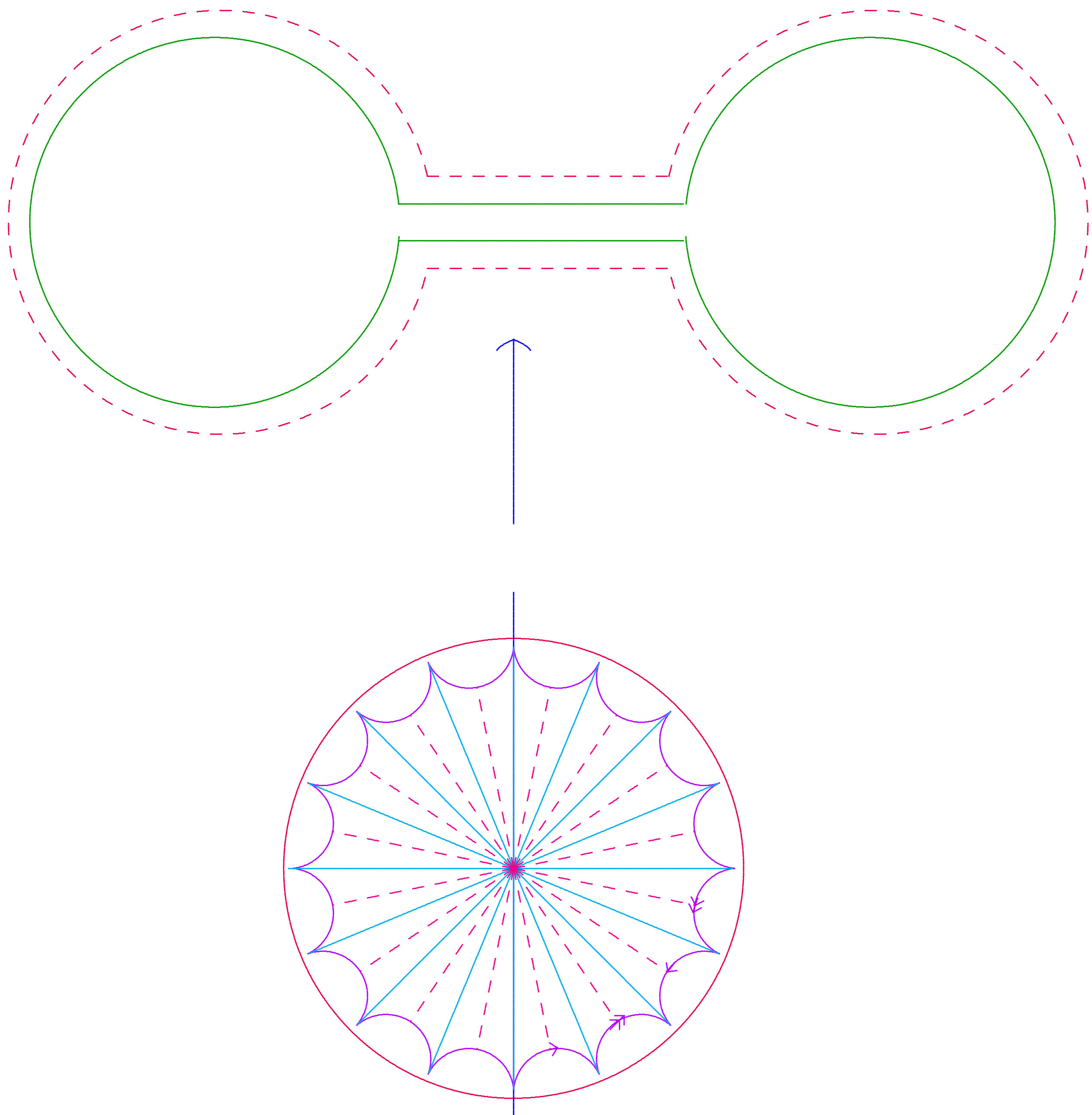


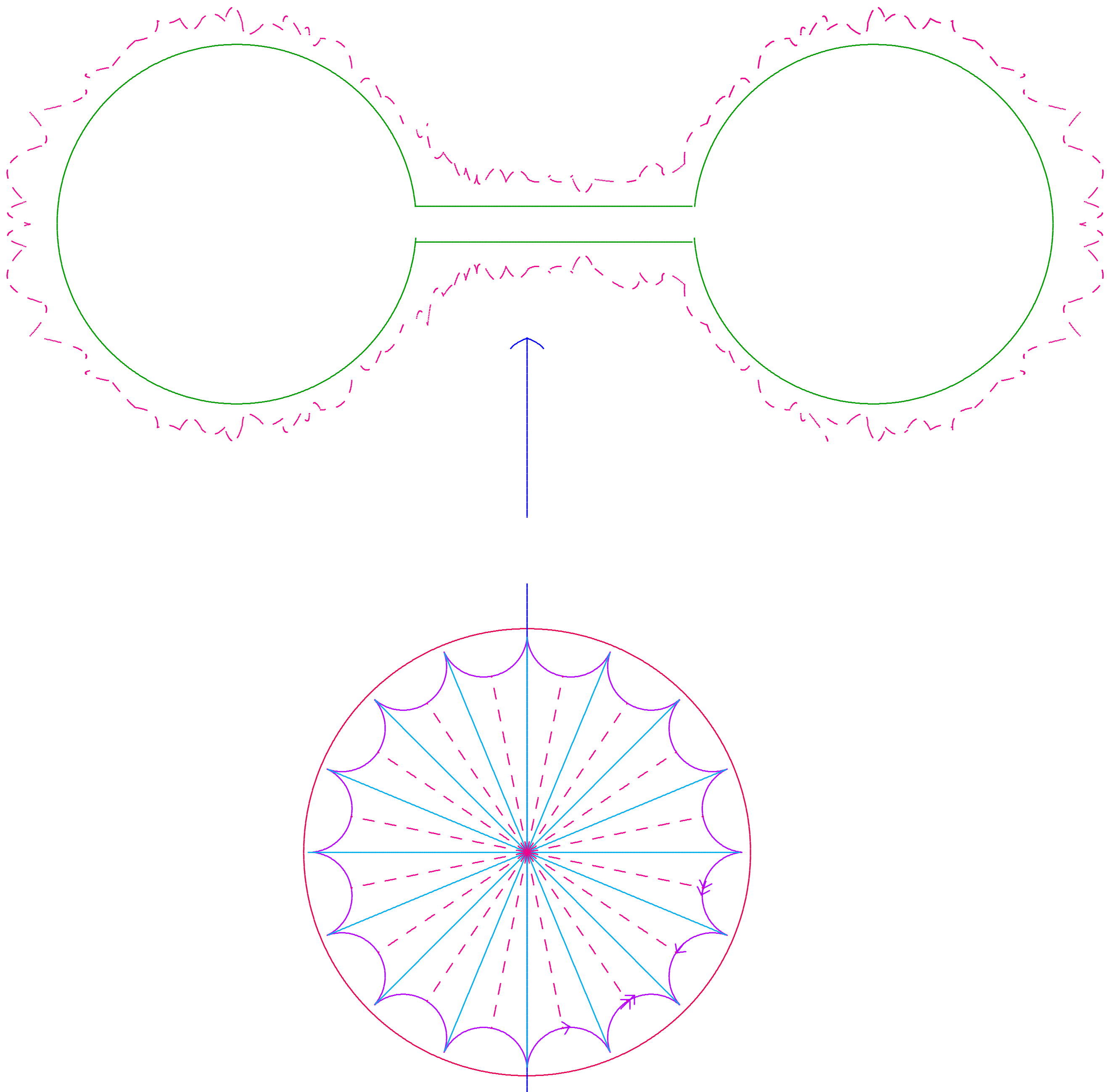








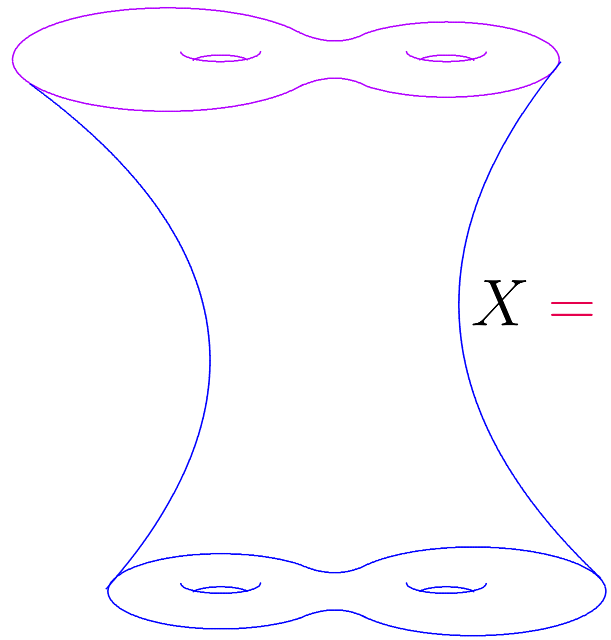




Theorem A. *Consider* 4-*manifolds* $M = \Sigma \times S^2$, *where* Σ *compact Riemann surface of genus* $\mathbf{g}$.

Then $\forall$ *even* $\mathbf{g} \gg 0$, $\exists$ *family* $[g_t]$, $t \in [0, 1]$, *of locally-conformally-flat classes on* M, *such that*

- $\exists$ *scalar-flat Kähler metric* $g_0 \in [g_0]$; *but*
- $\nexists$ *almost-Kähler metric* $g \in [g_1]$.



$$X = \mathcal{H}^3/\Gamma$$

Construction of conformally flat 4-manifolds:

$$M = [\overline{X} \times S^1]/\sim$$

$$g = f(1-f)[h + dt^2]$$

Theorem B. *Fix an integer* $k \geq 2$, *and then consider the 4-manifolds* $M = (\Sigma \times S^2)\#k\overline{\mathbb{CP}}_2$, *where* Σ *compact Riemann surface of genus* $\mathfrak{g}$.

Then $\forall$ *even* $\mathfrak{g} \gg 0$, $\exists$ *family* $[g_t]$, $t \in [0, 1]$, *of anti-self-dual conformal classes on* M, *such that*

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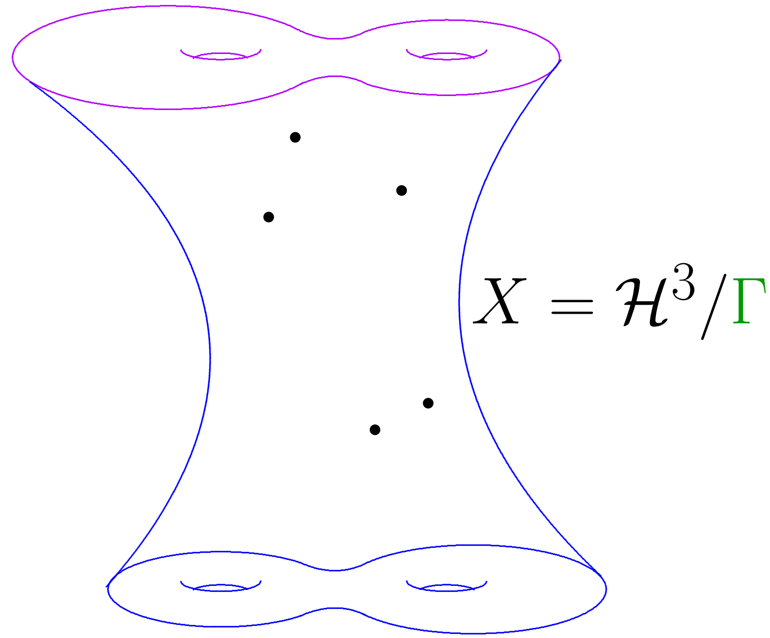

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$$g = f(1-f)[Vh + V^{-1}\theta^2]$$

$$V = 1 + \sum_{j=1}^{k} G_j$$

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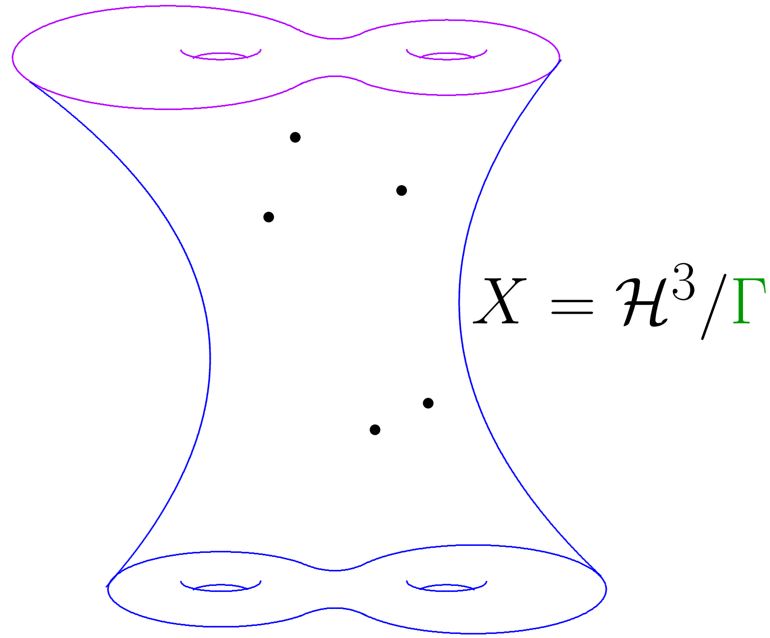

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Or are they actually logically independent?

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It's a real pleasure to be here!