Einstein Metrics,

Four-Manifolds, &

Gravitational Instantons

Claude LeBrun
Stony Brook University

Geometry and Topology Seminar,
University College London,
October 8, 2025.

Some recent results joint with

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Olivier Biquard
Sorbonne Université

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Int. Math. Res. Not. IMRN

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Int. Math. Res. Not. IMRN
(2024) 13295-13311.

Will also briefly mention

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work in progress

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Tristan Ozuch
MIT

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— A. Einstein, to G. Gamow

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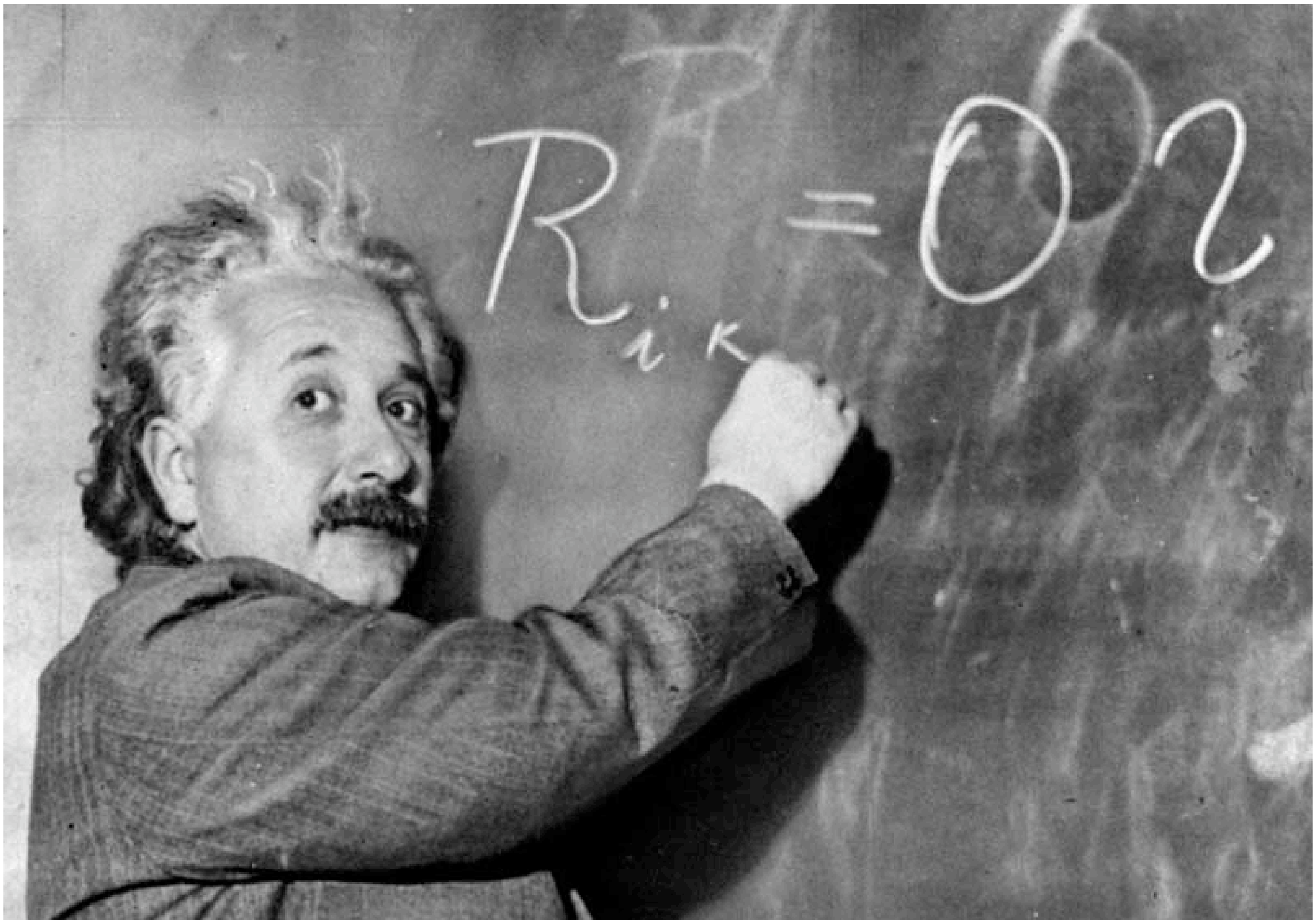

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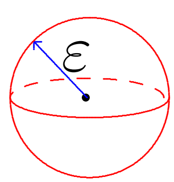
$$\frac{\mathrm{vol}_g(B_\varepsilon(p))}{c_n \varepsilon^n} = 1 - s\,\frac{\varepsilon^2}{6(n+2)} + O(\varepsilon^4)$$



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In high dimensions, the Einstein condition allows for a surprising degree of flexibility!

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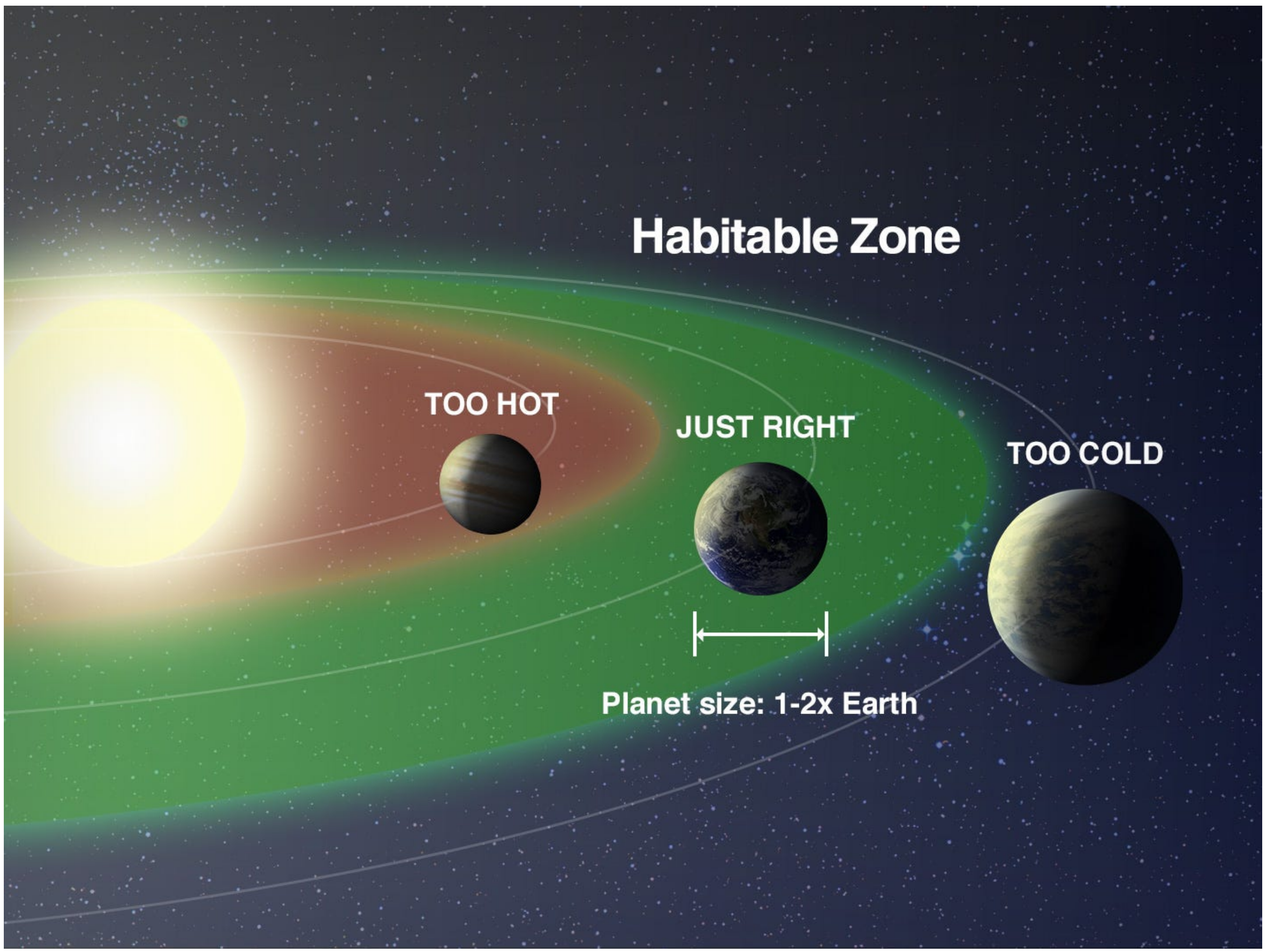

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Λ^+ self-dual 2-forms.
Λ^- anti-self-dual 2-forms.

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Which 4-manifolds admit Einstein metrics?

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$$d\omega = 0, \qquad \lrcorner\omega : TM \xrightarrow{\cong} T^*M.$$

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$$\omega = dx \wedge dy + dz \wedge dt$$

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Some Suggestive Questions. *If (M^4, ω) is a symplectic 4-manifold, when does M^4 admit an Einstein metric g (a priori unrelated to ω)? What if we also require $\lambda > 0$?*

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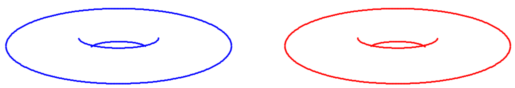

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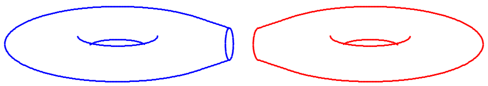

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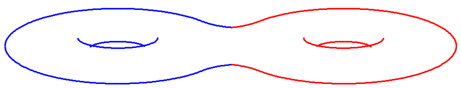

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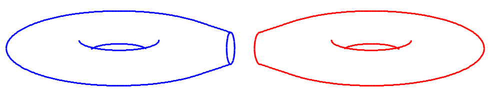

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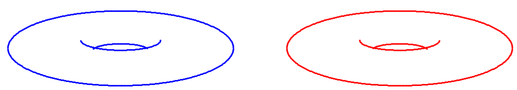

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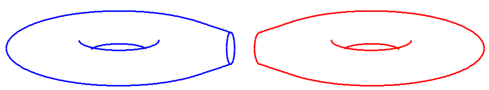

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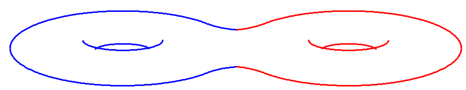

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Diffeotypes: exactly the Del Pezzo surfaces.

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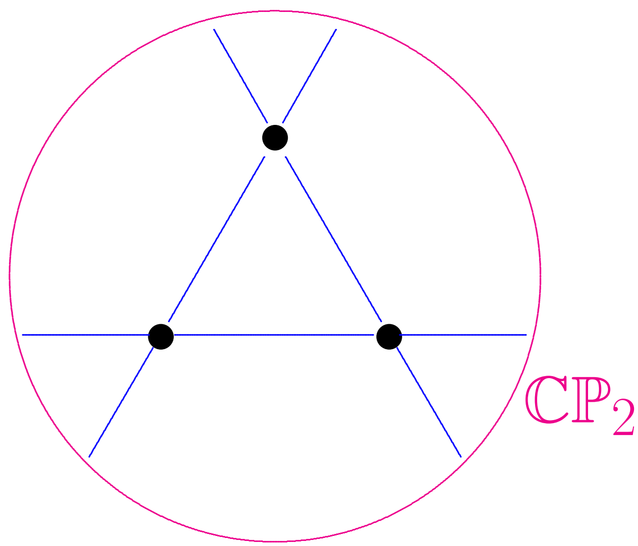

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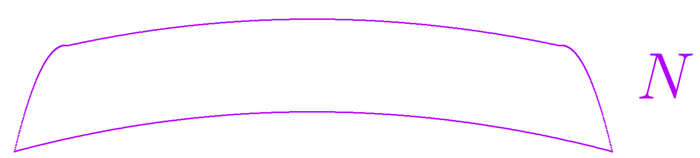

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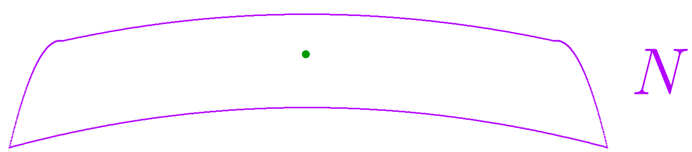

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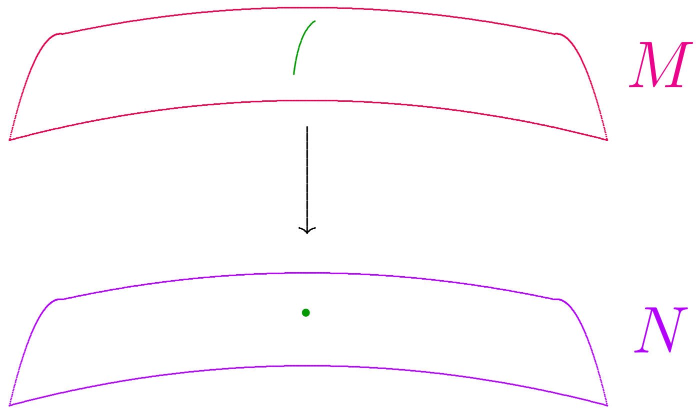

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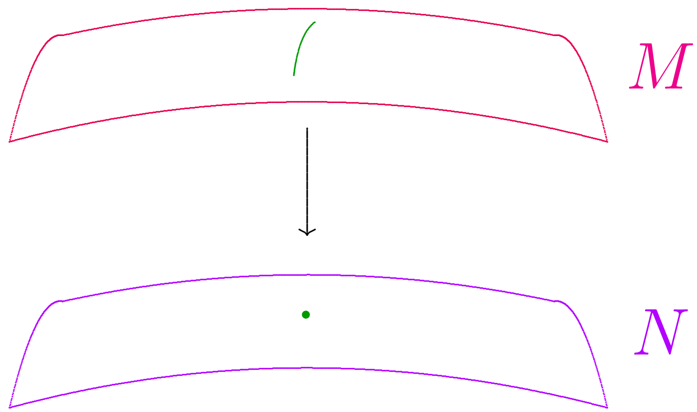

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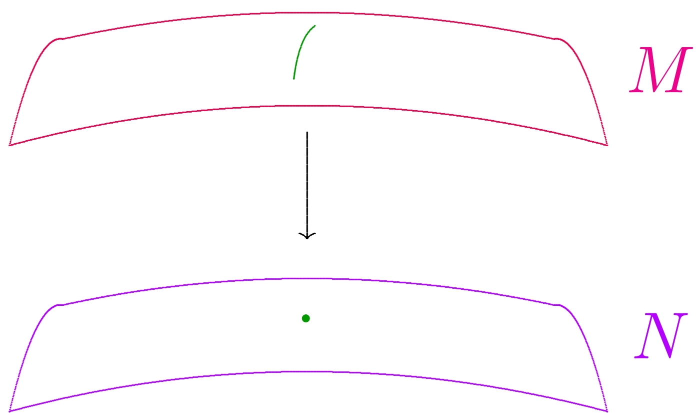

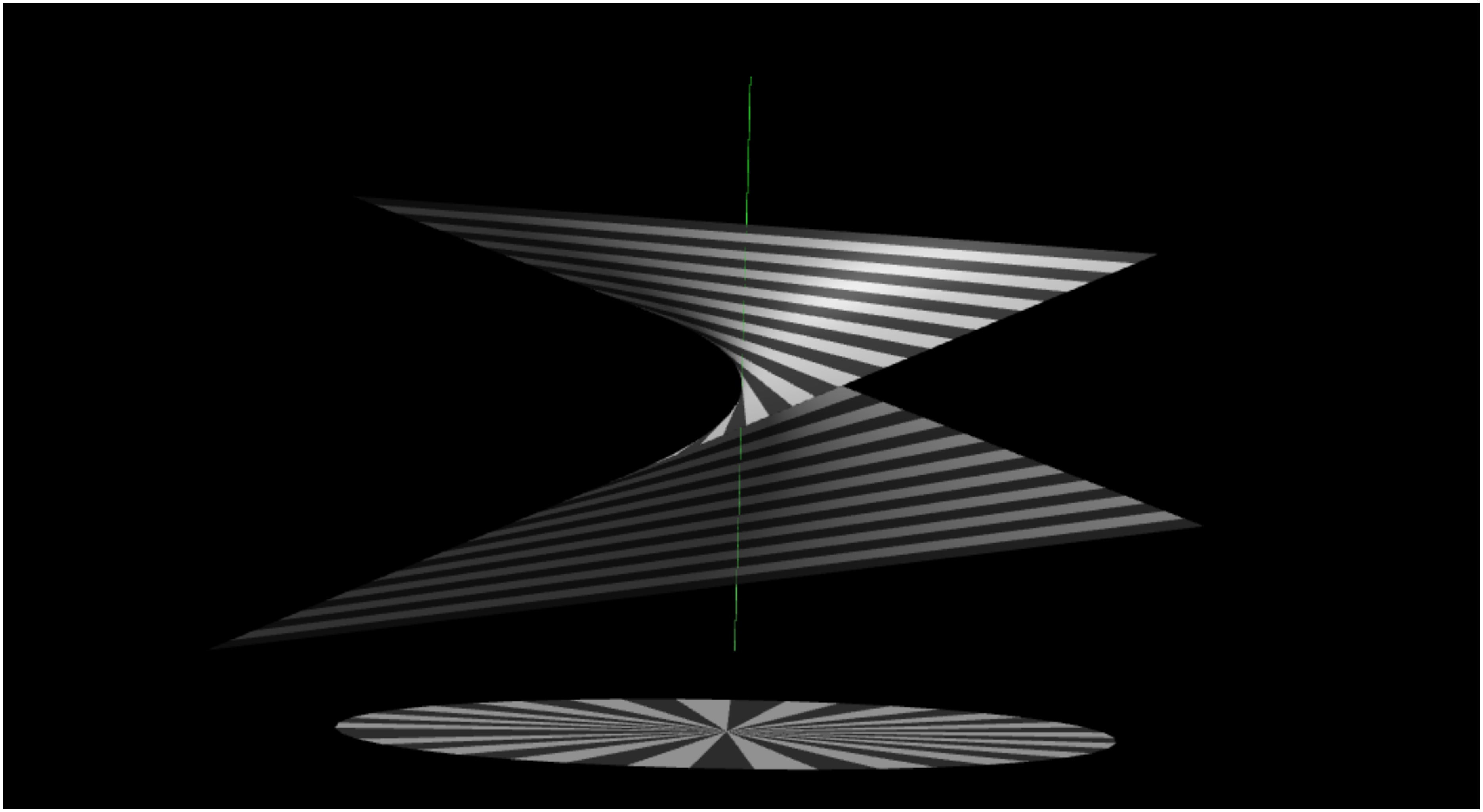

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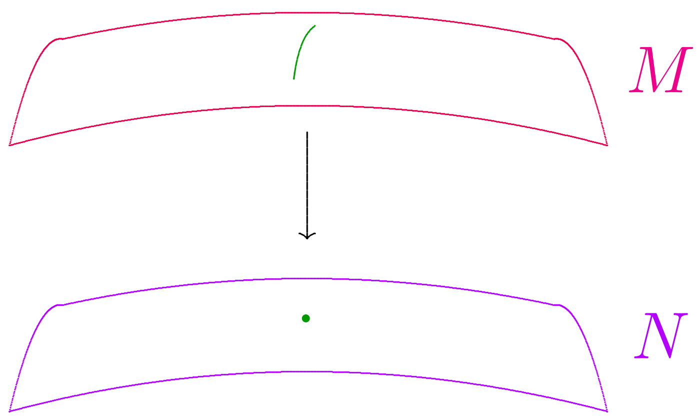

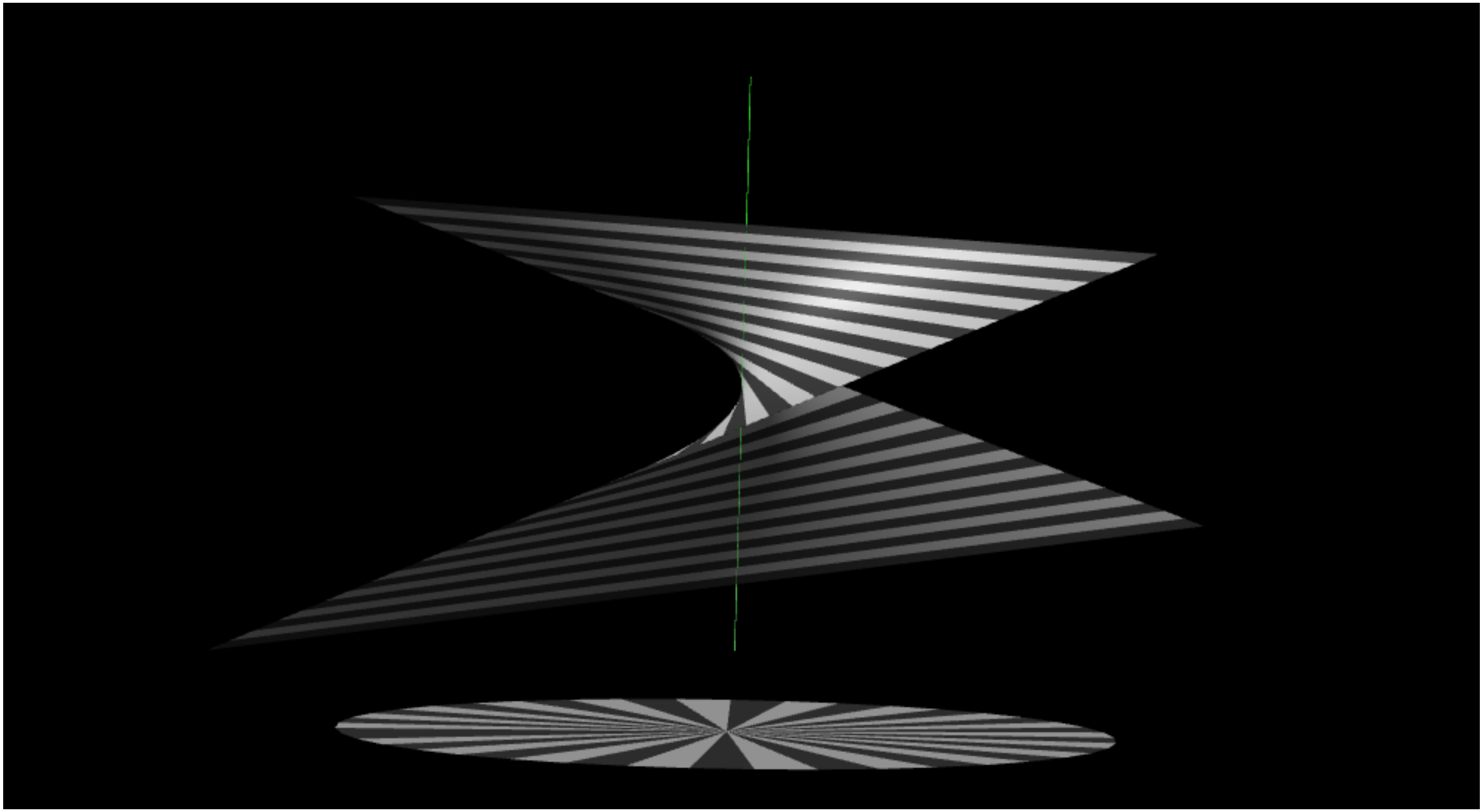

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If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

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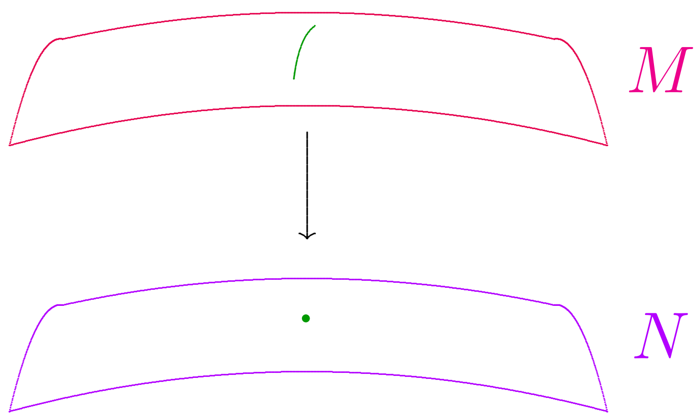

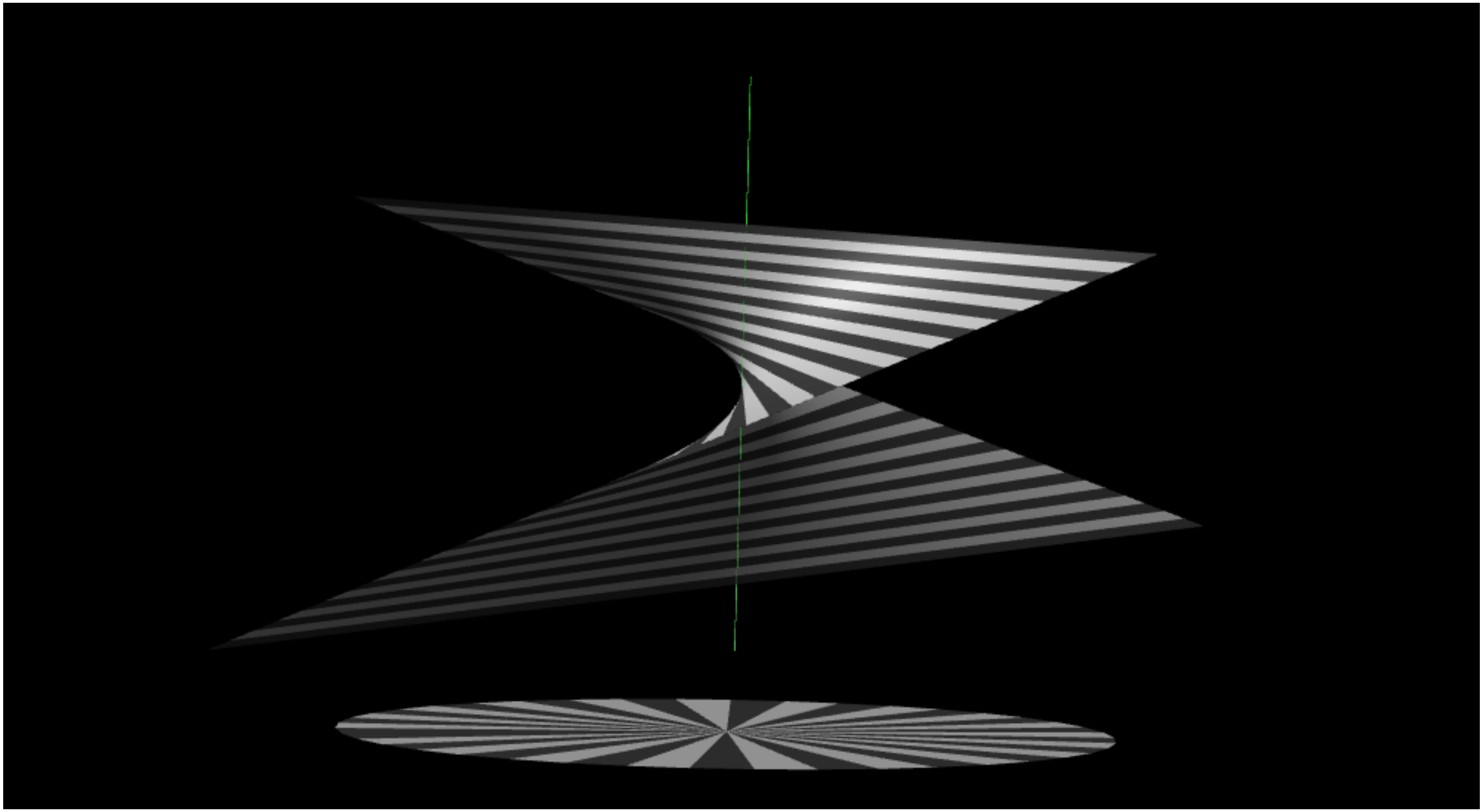

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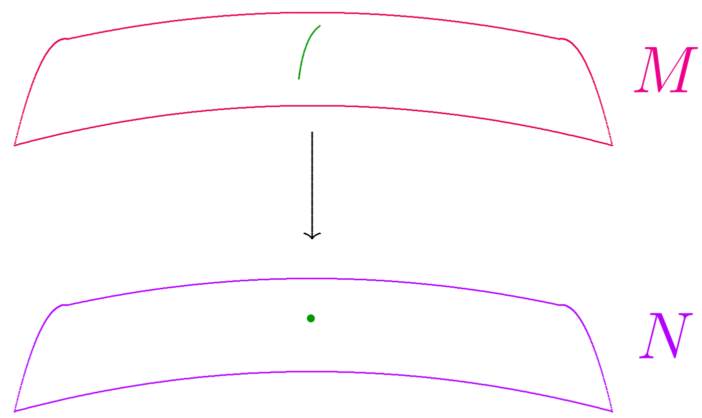
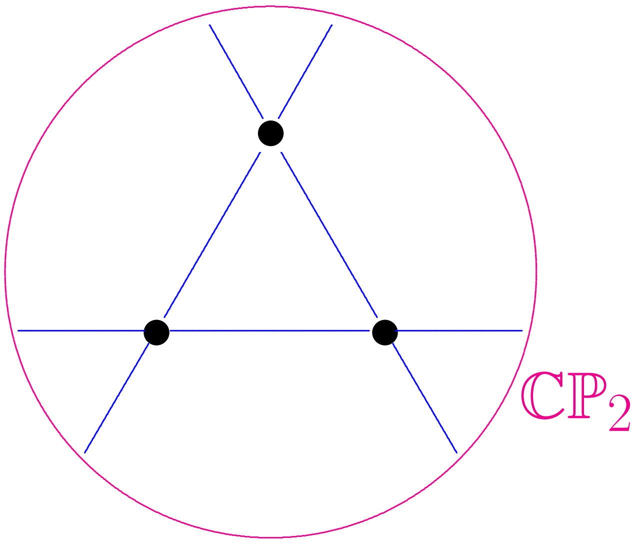


Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



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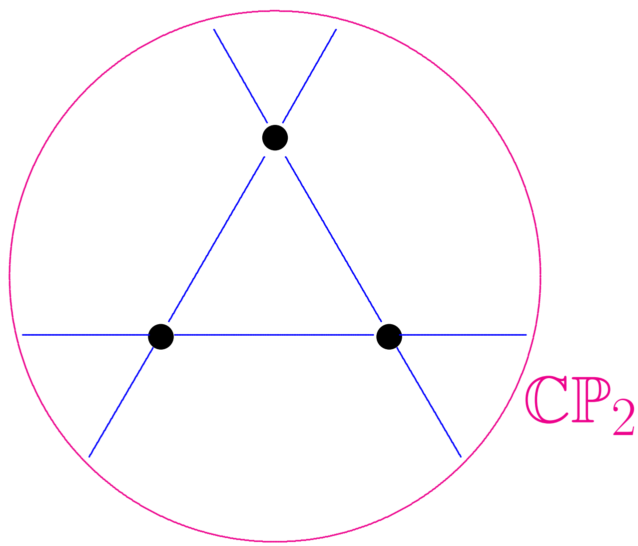

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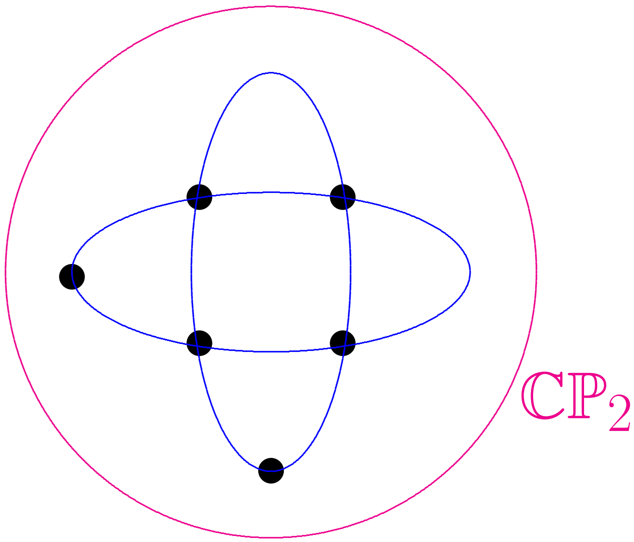

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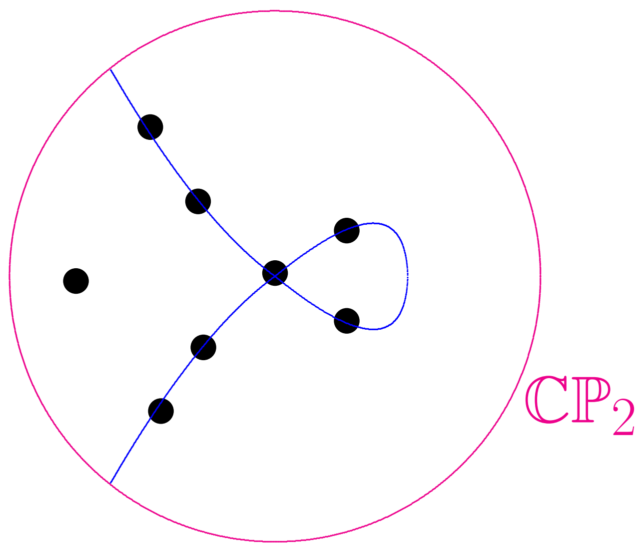

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Existence: Page

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Existence: Page-Derdziński,

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Existence: Page-Derdziński, Siu, Tian-Yau, Tian, Odaka-Spotti-Sun,

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Wu's criterion:

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Corollary. *Every simply-connected compact oriented Einstein* (M^4, h) *with* $\det(W_+) > 0$ *is diffeomorphic to a del Pezzo surface. Conversely, every del Pezzo* M^4 *carries Einstein* h *with* $\det(W_+) > 0$*, and these sweep out exactly one connected component of moduli space* $\mathscr{E}(M)$*.*

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This illustrates how gravitational instantons play a crucial role, even when studying compact case.

Gravitational Instantons?

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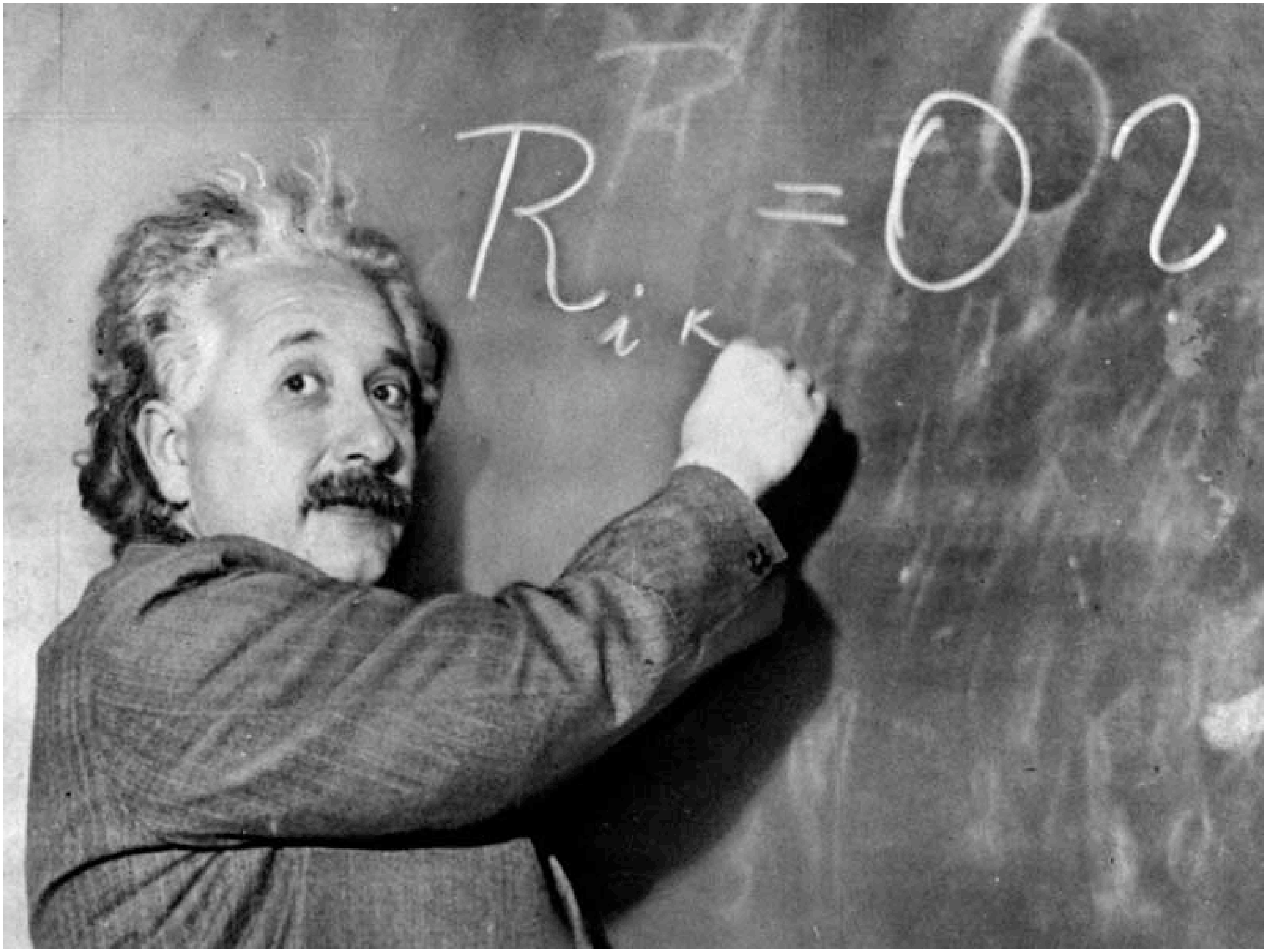

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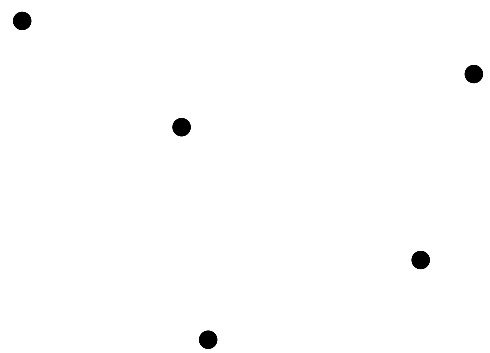

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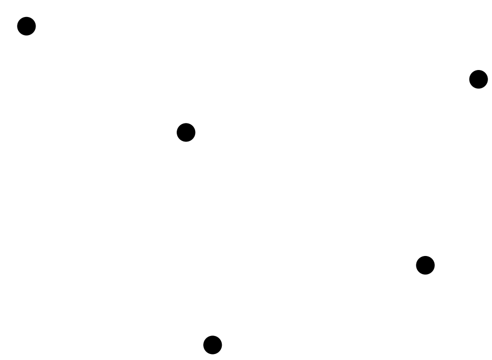
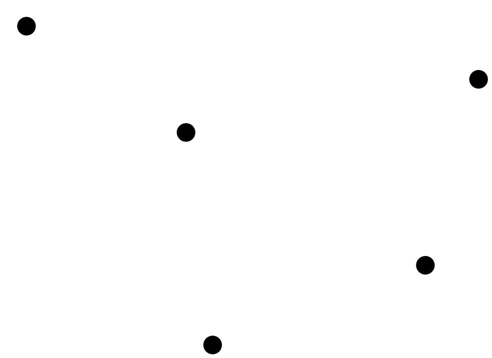


Data: ℓ points in $\mathbb{R}^3$ and κ^2



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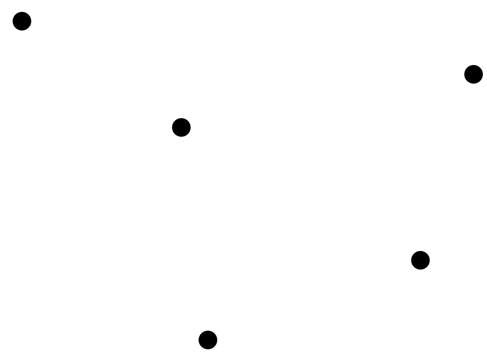

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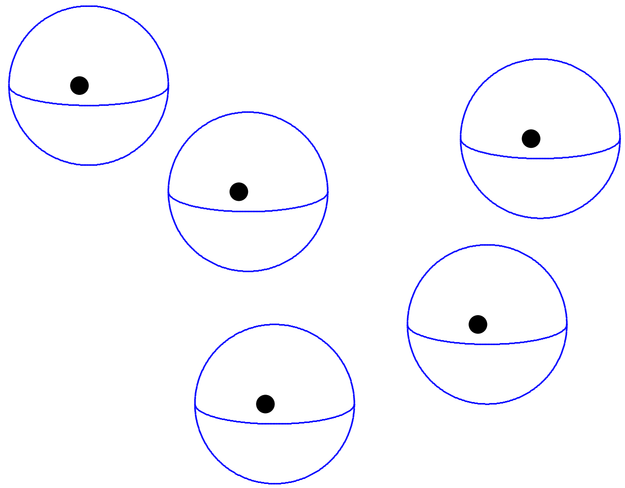

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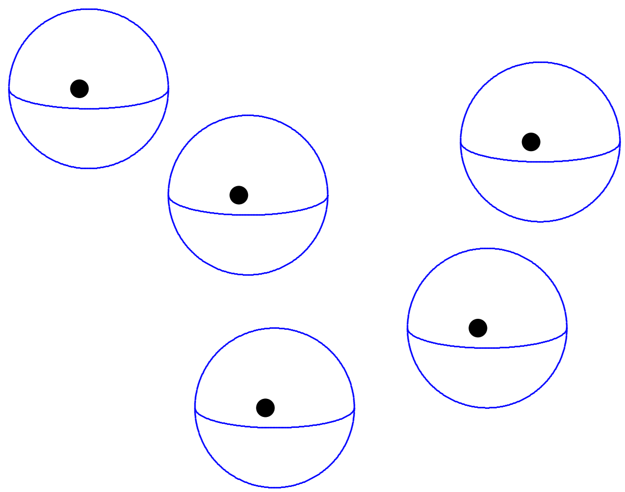

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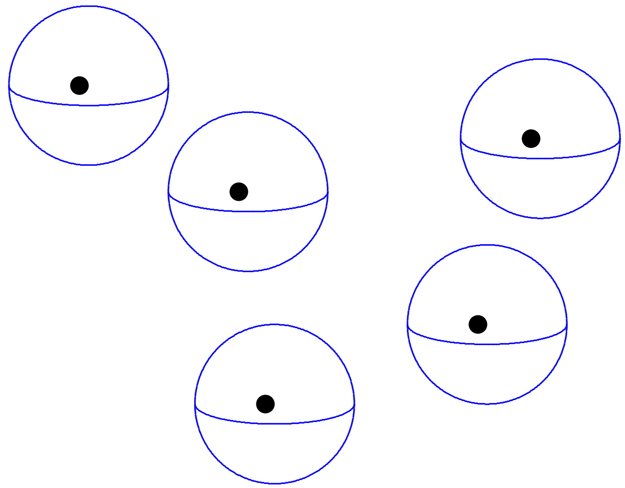

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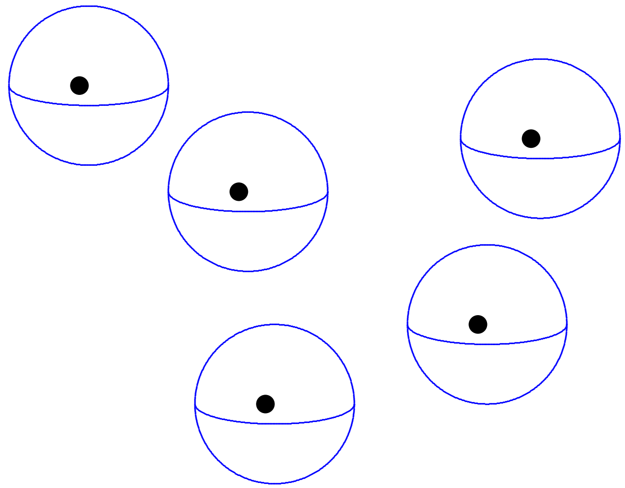

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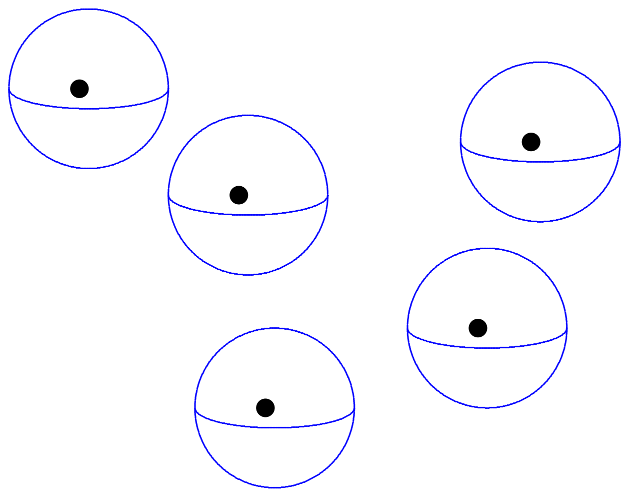

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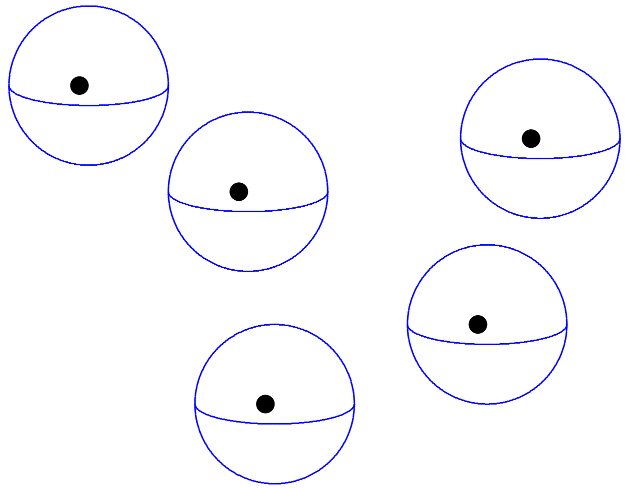

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This last property distinguishes the ALF spaces from other classes of gravitational instantons:

ALG, ALH, ALG*, ALH*, ...

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Non-Kähler, but conformally Kähler!

Hawking also explored non-hyper-Kähler examples. . .

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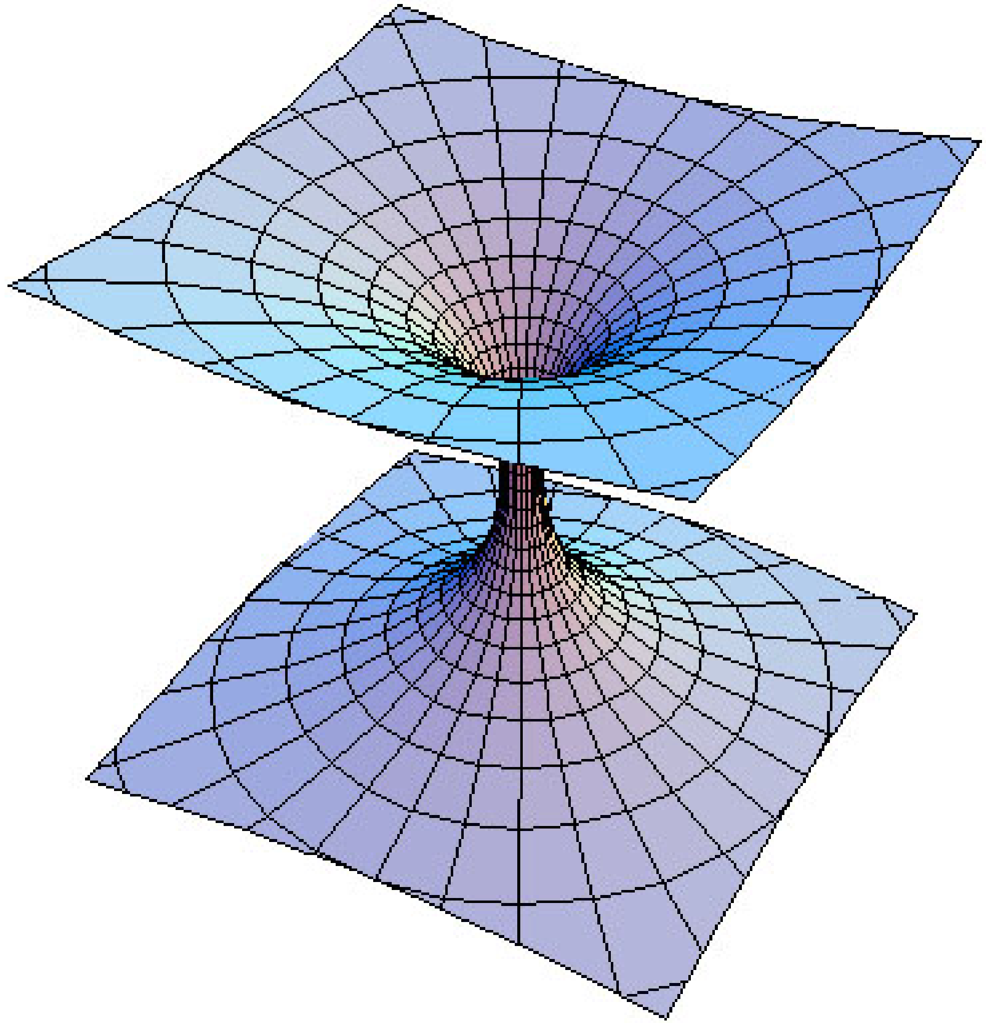

$\mathbb{R} \times S^2 \subset \mathbb{R}^2 \times S^2$

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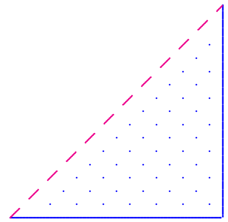

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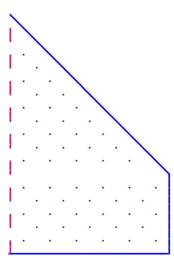

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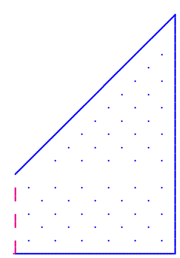

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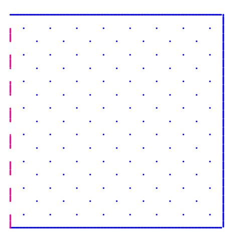

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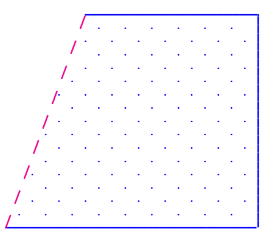

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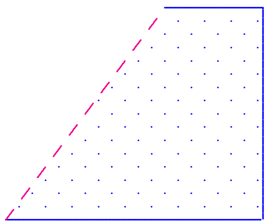

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Y. Chen & E. Teo, 2011

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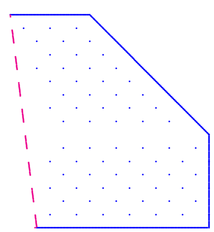

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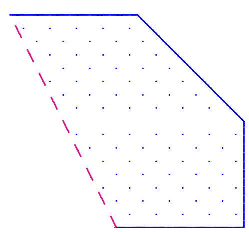

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allowing one to generalize methods first explored in the compact case.

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Set $h = \alpha^{2/3} g$, where α top eigenvalue of W_{+g}, and choose top eigenform $\omega \in \Lambda^+$ with $|\omega|_h \equiv 1$. Then

$$0 \geq |\nabla\omega|^2 + 3\langle \omega, (d + d^*)^2 \omega\rangle$$

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This optimal result combines **Theorem A** with a result of Mingyang Li, `arXiv:2310.13197`.

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