Curvature in the Balance:

The Weyl Functional &

Scalar Curvature of 4-Manifolds

Claude LeBrun
Stony Brook University

Séminaire d'Analyse Géométrique
Université Paul Sabatier, 26 mai, 2025

On Riemannian n-manifold (M, g),

On Riemannian n-manifold (M, g), $n \geq 3$,

On Riemannian n-manifold (M, g), $n \geq 3$,

$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2} \mathring{r}^{[a}{}_{[c} \delta^{b]}{}_{d]} + \frac{2}{n(n-1)} \, s \, \delta^{a}_{[c} \delta^{b}_{d]}$$

On Riemannian n-manifold (M, g), $n \geq 3$,

$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2}\mathring{r}^{[a}_{[c}\delta^{b]}_{d]} + \frac{2}{n(n-1)}\, s\, \delta^{a}_{[c}\delta^{b}_{d]}$$

where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W = Weyl curvature

On Riemannian n-manifold (M, g), $n \geq 3$,

$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2} \mathring{r}^{[a}_{[c} \delta^{b]}_{d]} + \frac{2}{n(n-1)} \, s \, \delta^{a}_{[c} \delta^{b}_{d]}$$

where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W = Weyl curvature *(conformally invariant)*

On Riemannian n-manifold (M, g), $n \geq 3$,

$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2} \mathring{r}^{[a}{}_{[c} \delta^{b]}{}_{d]} + \frac{2}{n(n-1)} \, s \, \delta^{a}_{[c} \delta^{b}_{d]}$$

where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W = Weyl curvature *(conformally invariant)*

$W^{a}{}_{bcd}$ unchanged if $g \rightsquigarrow \hat{g} = u^2 g$.

On Riemannian n-manifold (M, g),

$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2}\mathring{r}^{[a}_{[c}\delta^{b]}_{d]} + \frac{2}{n(n-1)}\ s\ \delta^{a}_{[c}\delta^{b}_{d]}$$

where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W = Weyl curvature *(conformally invariant)*

Proposition. *Assume* $n \geq 4$. *Then* (M^n, g) *locally conformally flat* $\Longleftrightarrow W \equiv 0$.

On Riemannian n-manifold (M, g),

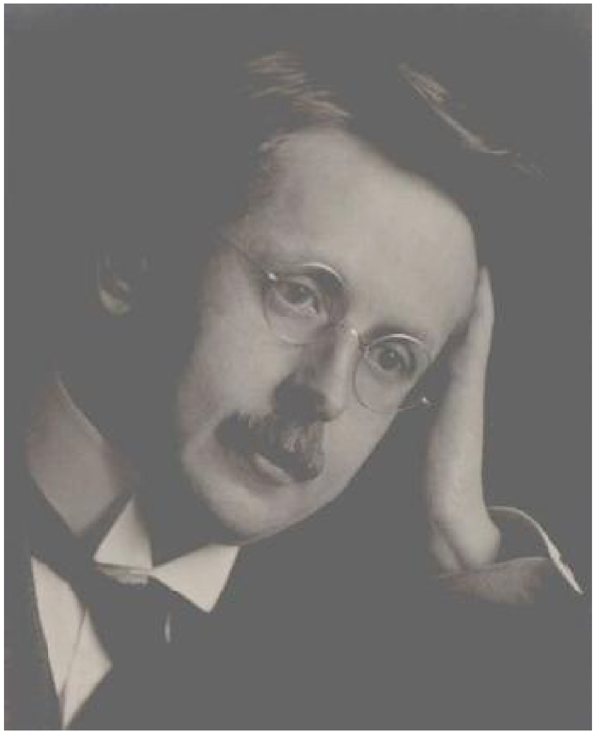
$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2}\mathring{r}^{[a}_{[c}\delta^{b]}_{d]} + \frac{2}{n(n-1)}\, s\, \delta^{a}_{[c}\delta^{b}_{d]}$$

where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W = Weyl curvature *(conformally invariant)*



For metrics on fixed M^n,

For metrics on fixed M^n,

now assumed to be compact, $n \geq 4$,

For metrics on fixed M^n,

For metrics on fixed M^n, Weyl functional

For metrics on fixed M^n, Weyl functional

$$\mathscr{W} : \mathcal{G}_M \longrightarrow \mathbb{R}$$

For metrics on fixed M^n, Weyl functional

$$\mathscr{W}(g) = \int_M |W_g|^{n/2} d\mu_g$$

For metrics on fixed M^n, Weyl functional

$$\mathscr{W}(g) = \int_M |W_g|^{n/2} d\mu_g$$

only depends on the conformal class

For metrics on fixed M^n, Weyl functional

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

only depends on the conformal class

$$[g] = \{u^2 g \mid u : M \xrightarrow{C^\infty} \mathbb{R}^+\}.$$

For metrics on fixed M^n, Weyl functional

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

only depends on the conformal class

$$[g] = \{u^2 g \mid u : M \xrightarrow{C^\infty} \mathbb{R}^+\}.$$

$$\mathscr{W} : \mathcal{G}_M/(C^\infty)^+ \longrightarrow \mathbb{R}$$

For metrics on fixed M^n, Weyl functional

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

only depends on the conformal class

$$[g] = \{u^2 g \mid u : M \xrightarrow{C^\infty} \mathbb{R}^+\}.$$

Measures deviation $[g]$ from conformal flatness.

For metrics on fixed M^n, Weyl functional

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

only depends on the conformal class

$$[g] = \{u^2 g \mid u : M \xrightarrow{C^\infty} \mathbb{R}^+\}.$$

Measures deviation $[g]$ from conformal flatness.

Basic problems: For given smooth compact M,

For metrics on fixed M^n, Weyl functional

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

only depends on the conformal class

$$[g] = \{u^2 g \mid u : M \xrightarrow{C^\infty} \mathbb{R}^+\}.$$

Measures deviation $[g]$ from conformal flatness.

Basic problems: For given smooth compact M,

- What is $\inf \mathscr{W}$?

For metrics on fixed M^n, Weyl functional

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

only depends on the conformal class

$$[g] = \{u^2 g \mid u : M \xrightarrow{C^\infty} \mathbb{R}^+\}.$$

Measures deviation $[g]$ from conformal flatness.

Basic problems: For given smooth compact M,

- What is $\inf \mathscr{W}$?
- Do there exist minimizers?

Dimension Four is Exceptional

Dimension Four is Exceptional

For M^4,

Dimension Four is Exceptional

For M^4,

$$\mathscr{W}([g]) = \int_M |W_g|^2 d\mu_g$$

Dimension Four is Exceptional

For M^4,

$$\mathscr{W}([g]) = \int_M |W_g|^2 d\mu_g$$

Euler-Lagrange equations $B = 0$ elliptic mod gauge.

Dimension Four is Exceptional

For M^4,

$$\mathscr{W}([g]) = \int_M |W_g|^2 d\mu_g$$

Euler-Lagrange equations $B = 0$ elliptic mod gauge.

Here

$$B_{ab} := (\nabla^c \nabla^d + \frac{1}{2} \mathring{r}^{cd}) W_{acbd}$$

Dimension Four is Exceptional

For M^4,

$$\mathscr{W}([g]) = \int_M |W_g|^2 d\mu_g$$

Euler-Lagrange equations $B = 0$ elliptic mod gauge.

Here

$$B_{ab} := (\nabla^c \nabla^d + \frac{1}{2} \mathring{r}^{cd}) W_{acbd}$$

called Bach tensor.

Dimension Four is Exceptional

For M^4,

$$\mathscr{W}([g]) = \int_M |W_g|^2 d\mu_g$$

Euler-Lagrange equations $B = 0$ elliptic mod gauge.

Here

$$B_{ab} := (\nabla^c \nabla^d + \frac{1}{2} \mathring{r}^{cd}) W_{acbd}$$

called Bach tensor.

Solutions called Bach-flat metrics.

Dimension Four is Exceptional

For M^4,

$$\mathscr{W}([g]) = \int_M |W_g|^2 d\mu_g$$

Euler-Lagrange equations $B = 0$ elliptic mod gauge.

Here

$$B_{ab} := (\nabla^c \nabla^d + \frac{1}{2} \mathring{r}^{cd}) W_{acbd}$$

called Bach tensor.

Solutions called Bach-flat metrics.

Bianchi $\Longrightarrow$ Any Einstein (M^4, g) is Bach-flat.

Dimension Four is Exceptional

For M^4,

$$\mathscr{W}([g]) = \int_M |W_g|^2 d\mu_g$$

Euler-Lagrange equations $B = 0$ elliptic mod gauge.

Here

$$B_{ab} := (\nabla^c\nabla^d + \frac{1}{2}\mathring{r}^{cd})W_{acbd}$$

called Bach tensor.

Solutions called Bach-flat metrics.

Bianchi $\Longrightarrow$ Any Einstein (M^4, g) is Bach-flat.

$$r = \lambda g \quad \Longleftrightarrow \quad \mathring{r} = 0$$

Dimension Four is Exceptional

For M^4,

$$\mathscr{W}([g]) = \int_M |W_g|^2 d\mu_g$$

Euler-Lagrange equations $B = 0$ elliptic mod gauge.

Here

$$B_{ab} := (\nabla^c \nabla^d + \frac{1}{2} \mathring{r}^{cd}) W_{acbd}$$

called Bach tensor.

Solutions called Bach-flat metrics.

Bianchi $\Longrightarrow$ Any Einstein (M^4, g) is Bach-flat.

Dimension Four is Exceptional

For M^4,

$$\mathscr{W}([g]) = \int_M |W_g|^2 d\mu_g$$

Euler-Lagrange equations $B = 0$ elliptic mod gauge.

Here

$$B_{ab} := (\nabla^c \nabla^d + \frac{1}{2} \mathring{r}^{cd}) W_{acbd}$$

called Bach tensor.

Solutions called Bach-flat metrics.

Bianchi $\Longrightarrow$ Any Einstein (M^4, g) is Bach-flat.

Of course, conformally Einstein good enough!

By contrast:

By contrast:

For M^n,

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

By contrast:

For M^n,

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

has degenerate Euler-Lagrange equation

$$|W_g|^{(n-4)/2}(\nabla\nabla\cdot W + \cdots) = 0$$

when $n > 4$.

By contrast:

For M^n,

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

has degenerate Euler-Lagrange equation

$$|W_g|^{(n-4)/2}(\nabla\nabla\cdot W + \cdots) = 0$$

when $n > 4$.

Einstein metrics are usually not critical points.

By contrast:

For M^n,

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

has degenerate Euler-Lagrange equation

$$|W_g|^{(n-4)/2}(\nabla\nabla\cdot W + \cdots) = 0$$

when $n > 4$.

Einstein metrics are usually not critical points.

For $n > 4$, product $K3 \times \mathbb{T}^{n-4}$ not critical,

By contrast:

For M^n,

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

has degenerate Euler-Lagrange equation

$$|W_g|^{(n-4)/2}(\nabla\nabla\cdot W + \cdots) = 0$$

when $n > 4$.

Einstein metrics are usually not critical points.

For $n > 4$, product $K3 \times \mathbb{T}^{n-4}$ not critical,

$$\text{CY} \times \text{flat}$$

By contrast:

For M^n,

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

has degenerate Euler-Lagrange equation

$$|W_g|^{(n-4)/2}(\nabla\nabla\cdot W + \cdots) = 0$$

when $n > 4$.

Einstein metrics are usually not critical points.

For $n > 4$, product $K3 \times \mathbb{T}^{n-4}$ not critical,

Ricci-flat $\Longrightarrow$ $W = \mathcal{R}$.

By contrast:

For M^n,

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

has degenerate Euler-Lagrange equation

$$|W_g|^{(n-4)/2}(\nabla\nabla\cdot W + \cdots) = 0$$

when $n > 4$.

Einstein metrics are usually not critical points.

For $n > 4$, product $K3 \times \mathbb{T}^{n-4}$ not critical,

since, for fixed CY on $K3$, $\quad \mathscr{W}(g) \propto \text{Vol}(\mathbb{T}^{n-4})$.

What's So Special About Dimension Four?

What's So Special About Dimension Four?

The Lie group $SO(4)$ is *not simple*:

What's So Special About Dimension Four?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

What's So Special About Dimension Four?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g),

What's So Special About Dimension Four?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g), $\Longrightarrow$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

What's So Special About Dimension Four?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g), $\Longrightarrow$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^\pm$ are (± 1)-eigenspaces of

$$\star : \Lambda^2 \to \Lambda^2,$$

$$\star^2 = 1.$$

What's So Special About Dimension Four?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g), $\Longrightarrow$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^\pm$ are (± 1)-eigenspaces of

$$\star : \Lambda^2 \to \Lambda^2,$$

$$\star^2 = 1.$$

Λ^+ self-dual 2-forms.

What's So Special About Dimension Four?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g), $\Longrightarrow$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^\pm$ are (± 1)-eigenspaces of

$$\star : \Lambda^2 \to \Lambda^2,$$

$$\star^2 = 1.$$

Λ^+ self-dual 2-forms.

Λ^- anti-self-dual 2-forms.

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W_- + \frac{s}{12} \end{array} \right)$$

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^+	$W_+ + \frac{s}{12}$	$\mathring{r}$
Λ^-	$\mathring{r}$	$W_- + \frac{s}{12}$

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^{+}	$W_{+} + \frac{s}{12}$	$\mathring{r}$
Λ^{-}	$\mathring{r}$	$W_{-} + \frac{s}{12}$

where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W_{+} = self-dual Weyl curvature

W_{-} = anti-self-dual Weyl curvature

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^+	$W_+ + \frac{s}{12}$	$\mathring{r}$
Λ^-	$\mathring{r}$	$W_- + \frac{s}{12}$

where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W_+ = self-dual Weyl curvature *(conformally invariant)*

W_- = anti-self-dual Weyl curvature ″

Polynomial Curvature Functionals

Polynomial Curvature Functionals

$$\begin{aligned} \mathcal{G}_M &\longrightarrow \mathbb{R} \\ g &\longmapsto \int_M P(\mathcal{R}_g)\, d\mu_g \end{aligned}$$

Polynomial Curvature Functionals

$$\begin{aligned} \mathcal{G}_M &\longrightarrow \mathbb{R} \\ g &\longmapsto \int_M P(\mathcal{R}_g)\, d\mu_g \end{aligned}$$

where $P(\mathcal{R})$ is $SO(4)$-invariant polynomial function of curvature.

Polynomial Curvature Functionals

$$\mathcal{G}_M \longrightarrow \mathbb{R}$$
$$g \longmapsto \int_M P(\mathcal{R}_g)\, d\mu_g$$

where $P(\mathcal{R})$ is $SO(4)$-invariant polynomial function of curvature.

Scale invariance $\Longrightarrow$ P quadratic.

Polynomial Curvature Functionals

$$\begin{aligned} \mathcal{G}_M &\longrightarrow \mathbb{R} \\ g &\longmapsto \int_M P(\mathcal{R}_g)\, d\mu_g \end{aligned}$$

where $P(\mathcal{R})$ is $SO(4)$-invariant polynomial function of curvature.

Scale invariance $\Longrightarrow$ P quadratic.

Any such $P(\mathcal{R})$ is linear combinations of

$$s^2, \quad |\mathring{r}|^2, \quad |W_+|^2, \quad |W_-|^2 \ .$$

Polynomial Curvature Functionals

$$\begin{aligned} \mathcal{G}_M &\longrightarrow \mathbb{R} \\ g &\longmapsto \int_M P(\mathcal{R}_g)\, d\mu_g \end{aligned}$$

where $P(\mathcal{R})$ is $SO(4)$-invariant polynomial function of curvature.

Scale invariance $\Longrightarrow$ P quadratic.

Any such $P(\mathcal{R})$ is linear combinations of

$$s^2, \quad |\mathring{r}|^2, \quad |W_+|^2, \quad |W_-|^2 \, .$$

Integrals give four scale-invariant functionals.

Four Basic Quadratic Curvature Functionals

Four Basic Quadratic Curvature Functionals

$$\mathcal{G}_M \longrightarrow \mathbb{R}$$

$$g \longmapsto \begin{cases} \int_M s^2 d\mu_g \\ \int_M |\mathring{r}|^2 d\mu_g \\ \int_M |W_+|^2 d\mu_g \\ \int_M |W_-|^2 d\mu_g \end{cases}$$

Four Basic Quadratic Curvature Functionals

$$\mathcal{G}_M \longrightarrow \mathbb{R}$$

$$g \longmapsto \begin{cases} \int_M s^2 d\mu_g \\ \int_M |\mathring{r}|^2 d\mu_g \\ \int_M |W_+|^2 d\mu_g \\ \int_M |W_-|^2 d\mu_g \end{cases}$$

However, these are not independent!

For (M^4, g) compact oriented Riemannian,

For (M^4, g) compact oriented Riemannian,

Euler characteristic

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(\frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu$$

For (M^4, g) compact oriented Riemannian,

Euler characteristic

$$\chi(M) = \frac{1}{8\pi^2}\int_M \left(\frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu$$

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

Thus any quadratic curvature functional expressible in terms of

Thus any quadratic curvature functional expressible in terms of e.g.

$$\int_M s^2 d\mu_g \quad \text{and} \quad \int_M |\mathring{r}|^2 d\mu_g \ .$$

Thus any quadratic curvature functional expressible in terms of e.g.

$$\int_M s^2 d\mu_g \quad \text{and} \quad \int_M |\mathring{r}|^2 d\mu_g \ .$$

Einstein metrics are critical for both.

Thus any quadratic curvature functional expressible in terms of e.g.

$$\int_M s^2 d\mu_g \quad \text{and} \quad \int_M |\mathring{r}|^2 d\mu_g \; .$$

Einstein metrics are critical for both.

$\therefore$ Einstein metrics critical $\forall$ quadratic functionals!

Thus any quadratic curvature functional expressible in terms of e.g.

$$\int_M s^2 d\mu_g \quad \text{and} \quad \int_M |\mathring{r}|^2 d\mu_g \ .$$

Einstein metrics are critical for both.

$\therefore$ Einstein metrics critical $\forall$ quadratic functionals!

e.g. critical for Weyl functional

$$g \longmapsto \int_M |W|_g^2 d\mu_g$$

But any quadratic curvature functional expressible in terms of

But any quadratic curvature functional expressible in terms of

$$\int_M s^2 d\mu_g \quad \text{and} \quad \int_M |W_+|^2 d\mu_g \ .$$

But any quadratic curvature functional expressible in terms of

$$\int_M s^2 d\mu_g \quad \text{and} \quad \int_M |W_+|^2 d\mu_g \ .$$

For example,

But any quadratic curvature functional expressible in terms of

$$\int_M s^2 d\mu_g \quad \text{and} \quad \int_M |W_+|^2 d\mu_g \ .$$

For example,

$$\mathscr{W}([g]) = -12\pi^2\tau(M) + 2\int_M |W_+|^2 d\mu_g$$

But any quadratic curvature functional expressible in terms of

$$\int_M s^2 d\mu_g \quad \text{and} \quad \int_M |W_+|^2 d\mu_g \ .$$

For example,

$$\mathscr{W}([g]) = -12\pi^2\tau(M) + 2\int_M |W_+|^2 d\mu_g$$

So $\int |W_+|^2 d\mu$ equivalent to Weyl functional.

But any quadratic curvature functional expressible in terms of

$$\int_M s^2 d\mu_g \quad \text{and} \quad \int_M |W_+|^2 d\mu_g \ .$$

But any quadratic curvature functional expressible in terms of

$$\int_M \frac{s^2}{24} d\mu_g \quad \text{and} \quad \int_M |W_+|^2 d\mu_g \ .$$

But any quadratic curvature functional expressible in terms of

$$\int_M \frac{s^2}{24} d\mu_g \quad \text{and} \quad \int_M |W_+|^2 d\mu_g \ .$$

Today's theme: How do these compare in size,

But any quadratic curvature functional expressible in terms of

$$\int_M \frac{s^2}{24} d\mu_g \quad \text{and} \quad \int_M |W_+|^2 d\mu_g \ .$$

Today's theme: How do these compare in size,

for specific classes of metrics on interesting 4-manifolds?

One motivation: **Kähler case.**

One motivation: **Kähler case.**

Suppose g Kähler metric on (M, J).

One motivation: **Kähler case.**

Suppose g Kähler metric on (M, J).

Give M orientation determined by J.

Kähler case:

$$\Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda^{-}$$

Kähler case:

$$\Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda^{-}$$

$$\Lambda^{+} = \mathbb{R}\omega \oplus \Re e(\Lambda^{2,0})$$

Kähler case:

$$\Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda^{-}$$

$$\Lambda^{+} = \mathbb{R}\omega \oplus \Re e(\Lambda^{2,0})$$

$$\nabla J = 0 \Longrightarrow \mathcal{R} \in \mathrm{End}(\Lambda^{1,1})$$

Kähler case:

$$\Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda^{-}$$

$$\Lambda^{+} = \mathbb{R}\omega \oplus \Re e(\Lambda^{2,0})$$

$$\nabla J = 0 \Longrightarrow \mathcal{R} \in \mathrm{End}(\Lambda^{1,1}) \Longrightarrow$$

$$W_{+} + \frac{s}{12} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & * \end{pmatrix}$$

Kähler case:

$$\Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda^{-}$$

$$\Lambda^{+} = \mathbb{R}\omega \oplus \Re e(\Lambda^{2,0})$$

$$\nabla J = 0 \Longrightarrow \mathcal{R} \in \mathrm{End}(\Lambda^{1,1}) \Longrightarrow$$

$$W_{+} + \frac{s}{12} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & \frac{s}{4} \end{pmatrix}$$

Kähler case:

$$\Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda^{-}$$

$$\Lambda^{+} = \mathbb{R}\omega \oplus \Re e(\Lambda^{2,0})$$

$$\nabla J = 0 \Longrightarrow \mathcal{R} \in \mathrm{End}(\Lambda^{1,1}) \Longrightarrow$$

$$W_{+} = \begin{pmatrix} -\frac{s}{12} & & \\ & -\frac{s}{12} & \\ & & \frac{s}{6} \end{pmatrix}$$

Kähler case:

$$\Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda^{-}$$

$$\Lambda^{+} = \mathbb{R}\omega \oplus \Re e(\Lambda^{2,0})$$

$$\nabla J = 0 \Longrightarrow \mathcal{R} \in \mathrm{End}(\Lambda^{1,1}) \Longrightarrow$$

$$|W_{+}|^{2} = \frac{s^{2}}{24}$$

One motivation: **Kähler case.**

Suppose g Kähler metric on (M, J).

Give M orientation determined by J.

One motivation: **Kähler case.**

Suppose g Kähler metric on (M, J).

Give M orientation determined by J.

Then

$$\frac{s^2}{24} = |W_+|^2$$

at every point.

One motivation: **Kähler case.**

Suppose g Kähler metric on (M, J).

Give M orientation determined by J.

Then

$$\frac{s^2}{24} = |W_+|^2$$

at every point.

$\therefore$ Two basic functionals agree on Kähler metrics!

One motivation: **Kähler case.**

Suppose g Kähler metric on (M, J).

Give M orientation determined by J.

Then

$$\frac{s^2}{24} = |W_+|^2$$

at every point.

$\therefore$ Two basic functionals agree on Kähler metrics!

$$\int_M \frac{s^2}{24} d\mu_g = \int_M |W_+|^2 d\mu_g \ .$$

One motivation: **Kähler case.**

Suppose g Kähler metric on (M, J).

Give M orientation determined by J.

Then

$$\frac{s^2}{24} = |W_+|^2$$

at every point.

$\therefore$ Two basic functionals agree on Kähler metrics!

$$\int_M \frac{s^2}{24} d\mu_g = \int_M |W_+|^2 d\mu_g \ .$$

More general Riemannian metrics?

Theorem (Gursky-L '99, Gursky '00). *Let* (M, g) *be a compact oriented Einstein 4-manifold*

Theorem (Gursky-L '99, Gursky '00). *Let* (M, g) *be a compact oriented Einstein* 4*-manifold with* $s > 0$

Theorem (Gursky-L '99, Gursky '00). *Let* (M, g) *be a compact oriented Einstein* 4*-manifold with* $s > 0$ *that is not an irreducible symmetric space.*

Theorem (Gursky-L '99, Gursky '00). *Let* (M, g) *be a compact oriented Einstein* 4*-manifold with* $s > 0$ *that is not an irreducible symmetric space.*

Excluded: Round S^4, Fubini-Study $\overline{\mathbb{CP}}_2$.

Theorem (Gursky-L '99, Gursky '00)**.** *Let* (M, g) *be a compact oriented Einstein* 4*-manifold with* $s > 0$ *that is not an irreducible symmetric space.*

Theorem (Gursky-L '99, Gursky '00). *Let* (M, g) *be a compact oriented Einstein* 4*-manifold with* $s > 0$ *that is not an irreducible symmetric space. Then*

$$\int_M |W_+|^2 d\mu_g \geq \int_M \frac{s^2}{24} d\mu_g$$

Theorem (Gursky-L '99, Gursky '00). *Let* (M, g) *be a compact oriented Einstein* 4*-manifold with* $s > 0$ *that is not an irreducible symmetric space. Then*

$$\int_M |W_+|^2 d\mu_g \geq \int_M \frac{s^2}{24} d\mu_g$$

with equality $\Leftrightarrow$ g *is locally Kähler-Einstein.*

Theorem (L '95, '09). *Let M be a smooth compact oriented 4-manifold that*

Theorem (L '95, '09). *Let M be a smooth compact oriented* 4*-manifold that*

- *admits a symplectic form ω, but*

Theorem (L '95, '09). *Let M be a smooth compact oriented* 4*-manifold that*

- *admits a symplectic form ω, but*
- *does not admit an Einstein metric with $s > 0$.*

Theorem (L '95, '09). *Let M be a smooth compact oriented 4-manifold that*

- *admits a symplectic form ω, but*
- *does not admit an Einstein metric with $s > 0$.*

Excluded: Del Pezzo Surfaces (10 diffeotypes)

Theorem (L '95, '09). *Let M be a smooth compact oriented* 4*-manifold that*

- *admits a symplectic form ω, but*
- *does not admit an Einstein metric with $s > 0$.*

Theorem (L '95, '09). *Let M be a smooth compact oriented* 4*-manifold that*

- *admits a symplectic form ω, but*
- *does not admit an Einstein metric with $s > 0$.*

Then, with respect to the symplectic orientation,

Theorem (L '95, '09). *Let M be a smooth compact oriented* 4*-manifold that*

- *admits a symplectic form ω, but*
- *does* not *admit an Einstein metric with $s > 0$.*

Then, with respect to the symplectic orientation, any Einstein metric g on M satisfies

Theorem (L '95, '09). *Let M be a smooth compact oriented 4-manifold that*

- *admits a symplectic form ω, but*
- *does* not *admit an Einstein metric with $s > 0$.*

Then, with respect to the symplectic orientation, any Einstein metric g on M satisfies

$$\int_M \frac{s^2}{24} d\mu_g \geq \int_M |W_+|^2 d\mu_g \ ,$$

Theorem (L '95, '09). *Let M be a smooth compact oriented 4-manifold that*

- *admits a symplectic form ω, but*
- *does* not *admit an Einstein metric with $s > 0$.*

Then, with respect to the symplectic orientation, any Einstein metric g on M satisfies

$$\int_M \frac{s^2}{24} d\mu_g \geq \int_M |W_+|^2 d\mu_g \ ,$$

with equality $\Leftrightarrow$ g is a Kähler-Einstein metric.

How are these results proved?

Theorem (Gursky-L '99, Gursky '00). *Let* (M, g) *be a compact oriented Einstein* 4*-manifold with* $s > 0$ *that is not an irreducible symmetric space. Then*

$$\int_M |W_+|^2 d\mu_g \geq \int_M \frac{s^2}{24} d\mu_g$$

with equality $\Leftrightarrow$ g *is locally Kähler-Einstein.*

Theorem (Gursky-L '99, Gursky '00). *Let* (M, g) *be a compact oriented Einstein* 4*-manifold with* $s > 0$ *that is not an irreducible symmetric space. Then*

$$\int_M |W_+|^2 d\mu_g \ \geq \int_M \frac{s^2}{24} d\mu_g$$

with equality $\Leftrightarrow$ g *is locally Kähler-Einstein.*

Method: Weitzenböck formula for $\delta W_+ = 0$.

Theorem (Gursky-L '99, Gursky '00). *Let* (M, g) *be a compact oriented Einstein* 4*-manifold with* $s > 0$ *that is not an irreducible symmetric space. Then*

$$\int_M |W_+|^2 d\mu_g \ \geq \int_M \frac{s^2}{24} d\mu_g$$

with equality $\Leftrightarrow$ g *is locally Kähler-Einstein.*

Method: Weitzenböck formula for $\delta W_+ = 0$.

$$0 = \Delta |W_+|^2 + 2|\nabla W_+|^2 + s|W_+|^2 - 36 \det(W_+)$$

Theorem (Gursky-L '99, Gursky '00). *Let* (M, g) *be a compact oriented Einstein* 4*-manifold with* $s > 0$ *that is not an irreducible symmetric space. Then*

$$\int_M |W_+|^2 d\mu_g \geq \int_M \frac{s^2}{24} d\mu_g$$

with equality $\Leftrightarrow$ g *is locally Kähler-Einstein.*

Method: Weitzenböck formula for $\delta W_+ = 0$.

$$0 = \Delta|W_+|^2 + 2|\nabla W_+|^2 + s|W_+|^2 - 36\det(W_+)$$

$$\Longrightarrow \exists \widehat{g} = u^2 g \quad \text{s.t.} \quad \widehat{\mathfrak{s}} := \widehat{s} - 2\sqrt{6}\widehat{|W_+|} \leq 0.$$

Theorem (L '95, '09). *Let M be a smooth compact oriented 4-manifold that*

- *admits a symplectic form ω, but*
- *does* not *admit an Einstein metric with $s > 0$.*

Then, with respect to the symplectic orientation, any Einstein metric g on M satisfies

$$\int_M \frac{s^2}{24} d\mu_g \geq \int_M |W_+|^2 d\mu_g \ ,$$

with equality $\Leftrightarrow$ g is a Kähler-Einstein metric.

Theorem (L '95, '09). *Let M be a smooth compact oriented 4-manifold that*

- *admits a symplectic form ω, but*
- *does not admit an Einstein metric with $s > 0$.*

Then, with respect to the symplectic orientation, any Einstein metric g on M satisfies

$$\int_M \frac{s^2}{24} d\mu_g \geq \int_M |W_+|^2 d\mu_g \ ,$$

with equality $\Leftrightarrow$ g is a Kähler-Einstein metric.

Method: Seiberg-Witten theory.

Theorem (L '95, '09). *Let M be a smooth compact oriented 4-manifold that*

- *admits a symplectic form ω, but*
- *does* not *admit an Einstein metric with $s > 0$.*

Then, with respect to the symplectic orientation, any Einstein metric g on M satisfies

$$\int_M \frac{s^2}{24} d\mu_g \geq \int_M |W_+|^2 d\mu_g \ ,$$

with equality $\Leftrightarrow$ g is a Kähler-Einstein metric.

Method: Seiberg-Witten theory.

Hypotheses $\Longrightarrow$ $\exists$ solution (Φ, θ) of SW equations for spin^c structure determined by ω.

Theorem (L '95, '09). *Let M be a smooth compact oriented 4-manifold that*

- *admits a symplectic form ω, but*
- *does* not *admit an Einstein metric with $s > 0$.*

Then, with respect to the symplectic orientation, any Einstein metric g on M satisfies

$$\int_M \frac{s^2}{24} d\mu_g \geq \int_M |W_+|^2 d\mu_g \ ,$$

with equality $\Leftrightarrow$ g is a Kähler-Einstein metric.

Method: Seiberg-Witten theory.

Hypotheses $\Longrightarrow$ $\exists$ solution (Φ, θ) of SW equations for spin^c structure determined by ω. $\Longrightarrow$

$$0 = 2\Delta|\Phi|^2 + 4|\nabla_\theta \Phi|^2 + s|\Phi|^2 + |\Phi|^4$$

Might therefore seem interesting to ask when

Might therefore seem interesting to ask when

$$\int_M |W_+|^2 d\mu_g \overset{?}{\geq} \int_M \frac{s^2}{24} d\mu_g$$

Might therefore seem interesting to ask when

$$\int_M |W_+|^2 d\mu_g \overset{?}{\geq} \int_M \frac{s^2}{24} d\mu_g$$

for all metrics g on M.

Might therefore seem interesting to ask when

$$\int_M |W_+|^2 d\mu_g \overset{?}{\geq} \int_M \frac{s^2}{24} d\mu_g$$

for all metrics g on M.

But this is actually a silly question!

Might therefore seem interesting to ask when

$$\int_M |W_+|^2 d\mu_g \overset{?}{\geq} \int_M \frac{s^2}{24} d\mu_g$$

for all metrics g on M.

But this is actually a silly question!

$\int_M |W_+|^2 d\mu_g$ conformally invariant.

Might therefore seem interesting to ask when

$$\int_M |W_+|^2 d\mu_g \overset{?}{\geq} \int_M \frac{s^2}{24} d\mu_g$$

for all metrics g on M.

But this is actually a silly question!

$\int_M |W_+|^2 d\mu_g$ conformally invariant.

$\int_M \frac{s^2}{24} d\mu_g$ is certainly not!

Might therefore seem interesting to ask when

$$\int_M |W_+|^2 d\mu_g \stackrel{?}{\geq} \int_M \frac{s^2}{24} d\mu_g$$

for all metrics g on M.

But this is actually a silly question!

$\int_M |W_+|^2 d\mu_g$ conformally invariant.

$\int_M \frac{s^2}{24} d\mu_g$ is certainly not!

Standard lore for Yamabe problem $\Longrightarrow$

Might therefore seem interesting to ask when

$$\int_M |W_+|^2 d\mu_g \stackrel{?}{\geq} \int_M \frac{s^2}{24} d\mu_g$$

for all metrics g on M.

But this is actually a silly question!

$\int_M |W_+|^2 d\mu_g$ conformally invariant.

$\int_M \frac{s^2}{24} d\mu_g$ is certainly not!

Standard lore for Yamabe problem $\Longrightarrow$

Any complex surface M with b_1 even carries both metrics with $>$ and with $<$.

Thus, our question only becomes sensible if we

Thus, our question only becomes sensible if we

- restrict our question to a class of metrics where general conformal rescaling is not possible:

Thus, our question only becomes sensible if we

- restrict our question to a class of metrics where general conformal rescaling is not possible:
  - Kähler metrics;

Thus, our question only becomes sensible if we

- restrict our question to a class of metrics where general conformal rescaling is not possible:
  - Kähler metrics;
  - Einstein metrics;

Thus, our question only becomes sensible if we

- restrict our question to a class of metrics where general conformal rescaling is not possible:
  - Kähler metrics;
  - Einstein metrics;
  - almost-Kähler metrics.

Thus, our question only becomes sensible if we

- restrict our question to a class of metrics where general conformal rescaling is not possible; or

Thus, our question only becomes sensible if we

- restrict our question to a class of metrics where general conformal rescaling is not possible; or
- modify problem to make it conformally invariant.

Thus, our question only becomes sensible if we

- restrict our question to a class of metrics where general conformal rescaling is not possible; or
- modify problem to make it conformally invariant.

One conformally-invariant version:

Thus, our question only becomes sensible if we

- restrict our question to a class of metrics where general conformal rescaling is not possible; or
- modify problem to make it conformally invariant.

One conformally-invariant version:

$$\int_M |W_+|^2 d\mu_g \overset{?}{\geq} \int_M \left(\frac{s^2}{24} - \frac{|\mathring{r}|^2}{2} \right) d\mu_g$$

Thus, our question only becomes sensible if we

- restrict our question to a class of metrics where general conformal rescaling is not possible; or
- modify problem to make it conformally invariant.

One conformally-invariant version:

$$\int_M |W_+|^2 d\mu_g \overset{?}{\geq} \int_M \left(\frac{s^2}{24} - \frac{|\mathring{r}|^2}{2} \right) d\mu_g$$

agrees with previous question in the Einstein case.

Thus, our question only becomes sensible if we

- restrict our question to a class of metrics where general conformal rescaling is not possible; or
- modify problem to make it conformally invariant.

One conformally-invariant version:

$$\int_M |W_+|^2 d\mu_g \overset{?}{\geq} \int_M \left(\frac{s^2}{24} - \frac{|\mathring{r}|^2}{2} \right) d\mu_g$$

agrees with previous question in the Einstein case.

Equivalent to

Thus, our question only becomes sensible if we

- restrict our question to a class of metrics where general conformal rescaling is not possible; or
- modify problem to make it conformally invariant.

One conformally-invariant version:

$$\int_M |W_+|^2 d\mu_g \overset{?}{\geq} \int_M \left(\frac{s^2}{24} - \frac{|\mathring{r}|^2}{2} \right) d\mu_g$$

agrees with previous question in the Einstein case.

Equivalent to

$$\frac{1}{4\pi^2} \int_M |W_+|^2 d\mu_g \overset{?}{\geq} \frac{1}{3}(2\chi + 3\tau)(M).$$

Since

$$\mathscr{W}([g]) = -12\pi^2\tau(M) + 2\int_M |W_+|^2 d\mu_g$$

this is really a question about $\inf \mathscr{W}$.

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

Here $\tau(M) = b_+(M) - b_-(M)$ defined in terms of intersection pairing

Here $\tau(M) = b_+(M) - b_-(M)$ defined in terms of intersection pairing

$$\begin{aligned} H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) &\longrightarrow \mathbb{R} \\ (\,[\varphi]\,,\,[\psi]\,) &\longmapsto \int_M \varphi \wedge \psi \end{aligned}$$

Here $\tau(M) = b_+(M) - b_-(M)$ defined in terms of intersection pairing

$$\begin{array}{ccc} H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) & \longrightarrow & \mathbb{R} \\ (\ [\varphi]\ ,\ [\psi]\ ) & \longmapsto & \int_M \varphi \wedge \psi \end{array}$$

Diagonalize:

Here $\tau(M) = b_+(M) - b_-(M)$ defined in terms of intersection pairing

$$\begin{array}{ccc} H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) & \longrightarrow & \mathbb{R} \\ (\,[\varphi]\,,\,[\psi]\,) & \longmapsto & \displaystyle\int_M \varphi \wedge \psi \end{array}$$

Diagonalize:

$$\begin{bmatrix} +1 & & & & & \\ & \ddots & & & & \\ & & +1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{bmatrix}.$$

Here $\tau(M) = b_+(M) - b_-(M)$ defined in terms of intersection pairing

$$
\begin{array}{ccc}
H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) & \longrightarrow & \mathbb{R} \\
(\, [\varphi] \,,\, [\psi] \,) & \longmapsto & \int_M \varphi \wedge \psi
\end{array}
$$

Diagonalize:

$$
\left[\begin{array}{cccccc}
\underbrace{\begin{matrix} +1 & & \\ & \ddots & \\ & & +1 \end{matrix}}_{b_+(M)} & \\
& b_-(M)\left\{\begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix}\right.
\end{array}\right].
$$

Hodge theory:

$$H^2(M, \mathbb{R}) = \{\varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0,\ d \star \varphi = 0\}.$$

Hodge theory:

$$H^2(M, \mathbb{R}) = \{\varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0, \; d \star \varphi = 0\}.$$

Since $\star$ is involution of RHS, $\Longrightarrow$

$$H^2(M, \mathbb{R}) = \mathcal{H}^+_g \oplus \mathcal{H}^-_g,$$

Hodge theory:

$$H^2(M, \mathbb{R}) = \{\varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0,\ d \star \varphi = 0\}.$$

Since $\star$ is involution of RHS, $\Longrightarrow$

$$H^2(M, \mathbb{R}) = \mathcal{H}^+_g \oplus \mathcal{H}^-_g,$$

where

$$\mathcal{H}^\pm_g = \{\varphi \in \Gamma(\Lambda^\pm) \mid d\varphi = 0\}$$

self-dual & anti-self-dual harmonic forms.

Hodge theory:

$$H^2(M,\mathbb{R}) = \{\varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0,\ d\star\varphi = 0\}.$$

Since $\star$ is involution of RHS, $\Longrightarrow$

$$H^2(M,\mathbb{R}) = \mathcal{H}^+_g \oplus \mathcal{H}^-_g,$$

where

$$\mathcal{H}^\pm_g = \{\varphi \in \Gamma(\Lambda^\pm) \mid d\varphi = 0\}$$

self-dual & anti-self-dual harmonic forms. Then

$$b_\pm(M) = \dim \mathcal{H}^\pm_g.$$

Hodge theory:

$$H^2(M, \mathbb{R}) = \{\varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0,\ d \star \varphi = 0\}.$$

Since $\star$ is involution of RHS, $\Longrightarrow$

$$H^2(M, \mathbb{R}) = \mathcal{H}^+_g \oplus \mathcal{H}^-_g,$$

where

$$\mathcal{H}^\pm_g = \{\varphi \in \Gamma(\Lambda^\pm) \mid d\varphi = 0\}$$

self-dual & anti-self-dual harmonic forms. Then

$$b_\pm(M) = \dim \mathcal{H}^\pm_g.$$

The subspaces $\mathcal{H}^\pm_g$ are conformally invariant:

Hodge theory:

$$H^2(M,\mathbb{R}) = \{\varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0,\ d\star\varphi = 0\}.$$

Since $\star$ is involution of RHS, $\Longrightarrow$

$$H^2(M,\mathbb{R}) = \mathcal{H}_g^+ \oplus \mathcal{H}_g^-,$$

where

$$\mathcal{H}_g^\pm = \{\varphi \in \Gamma(\Lambda^\pm) \mid d\varphi = 0\}$$

self-dual & anti-self-dual harmonic forms. Then

$$b_\pm(M) = \dim \mathcal{H}_g^\pm.$$

The subspaces $\mathcal{H}_g^\pm$ are conformally invariant:
Same for g and any $\widehat{g} = u^2 g$.

Hodge theory:

$$H^2(M,\mathbb{R}) = \{\varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0,\ d\star\varphi = 0\}.$$

Since $\star$ is involution of RHS, $\Longrightarrow$

$$H^2(M,\mathbb{R}) = \mathcal{H}^+_g \oplus \mathcal{H}^-_g,$$

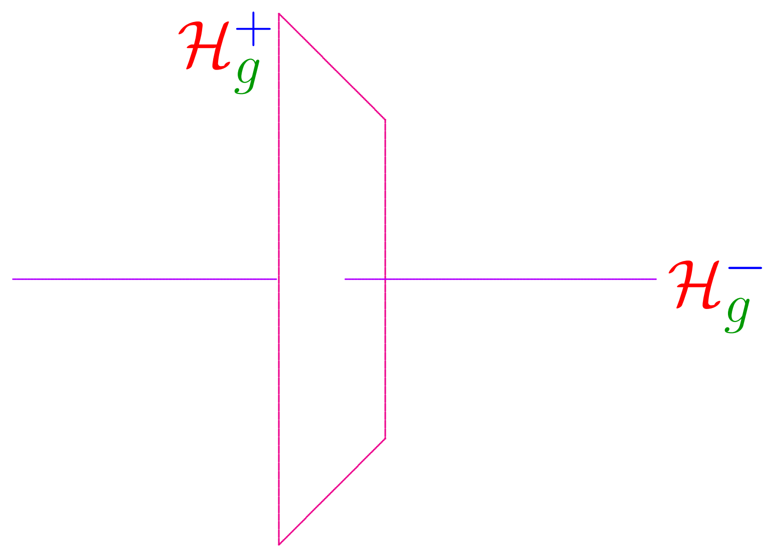
where

$$\mathcal{H}^\pm_g = \{\varphi \in \Gamma(\Lambda^\pm) \mid d\varphi = 0\}$$

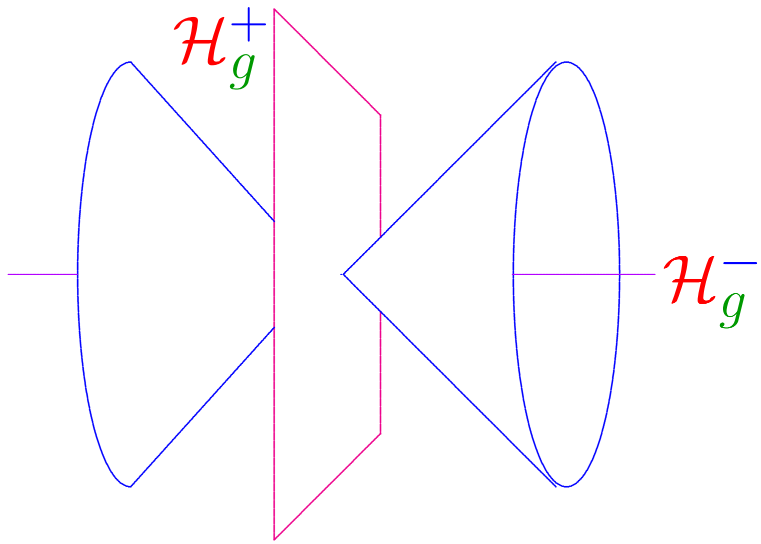
self-dual & anti-self-dual harmonic forms. Then

$$b_\pm(M) = \dim \mathcal{H}^\pm_g.$$

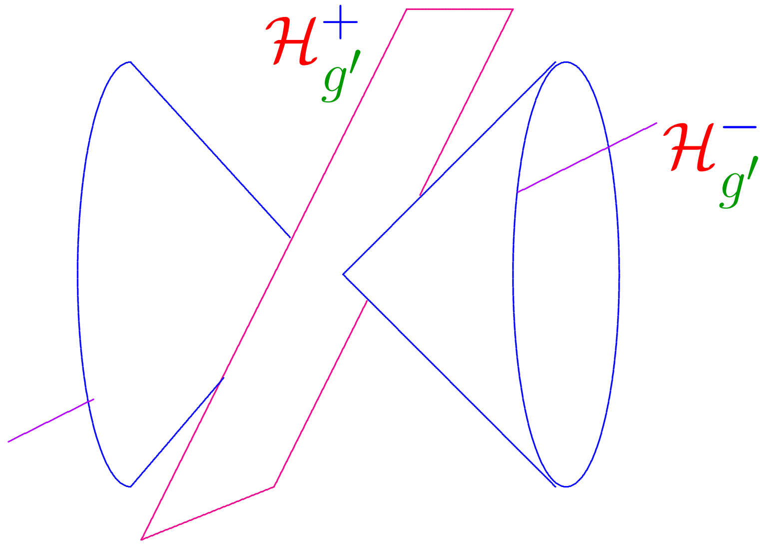
However, they are genuinely metric-dependent as soon as we allow for more general changes of g.



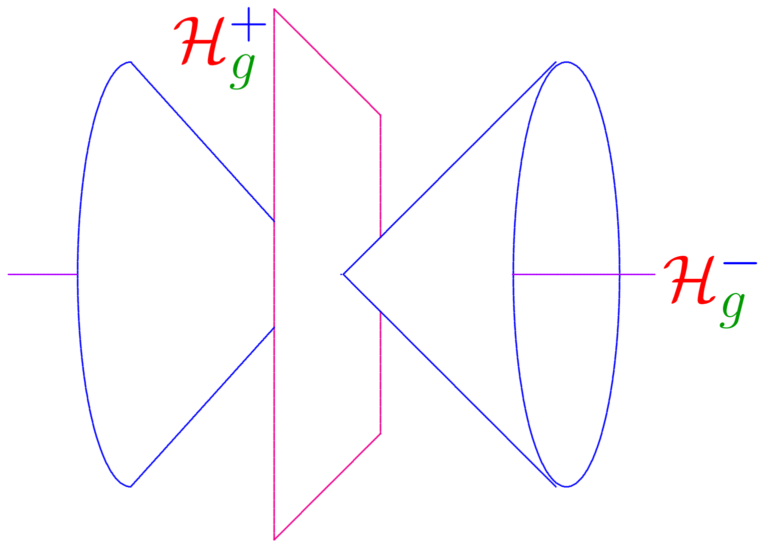
$$H^2(M, \mathbb{R})$$



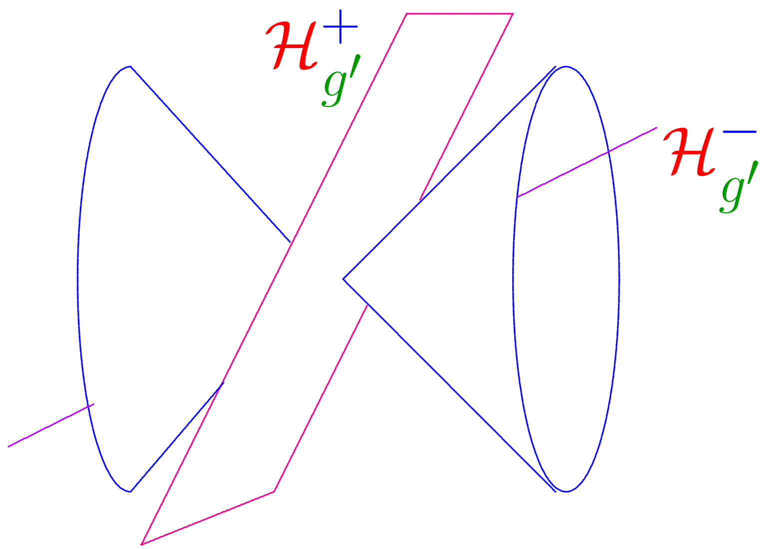
$$\{a \mid a \cdot a = 0\} \subset H^2(M, \mathbb{R})$$



$$\{a \mid a \cdot a = 0\} \subset H^2(M, \mathbb{R})$$



$$\{a \mid a \cdot a = 0\} \subset H^2(M, \mathbb{R})$$



$$\{a \mid a \cdot a = 0\} \subset H^2(M, \mathbb{R})$$

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

$$= \tfrac{1}{3}\langle p_1(TM), [M]\rangle$$

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|^2 - |W_-|^2 \right) d\mu$$

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|^2 - |W_-|^2 \right) d\mu$$

$$\mathscr{W}([g]) = \int_M |W_g|^2 d\mu_g$$

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

$$\begin{aligned}\mathscr{W}([g]) &= \int_M |W_g|^2 d\mu_g \\ &= \int_M \left(|W_+|^2 + |W_-|^2\right) d\mu_g\end{aligned}$$

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

$$\begin{aligned}\mathscr{W}([g]) &= \int_M |W_g|^2 d\mu_g \\ &= \int_M \left(|W_+|^2 + |W_-|^2\right) d\mu_g \\ &\geq \left|\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu_g\right|\end{aligned}$$

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

$$\begin{aligned}
\mathscr{W}([g]) &= \int_M |W_g|^2 d\mu_g \\
&= \int_M \left(|W_+|^2 + |W_-|^2\right) d\mu_g \\
&\geq \left|\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu_g\right| \\
&= 12\pi^2|\tau(M)|
\end{aligned}$$

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

$$\mathscr{W}([g]) \geq 12\pi^2\tau(M)$$

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

$$\mathscr{W}([g]) \geq 12\pi^2\tau(M)$$

with $= \iff W_- \equiv 0.$

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

$$\mathscr{W}([g]) \geq 12\pi^2\tau(M)$$

with $= \iff W_- \equiv 0.$ "self-dual"

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

$$\mathscr{W}([g]) \geq 12\pi^2\tau(M)$$

with $= \iff W_- \equiv 0.$ "self-dual"

$$\star W = W$$

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

$$\mathscr{W}([g]) \geq -12\pi^2\tau(M)$$

$$\text{with} = \Longleftrightarrow W_+ \equiv 0.$$

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

$$\mathscr{W}([g]) \geq -12\pi^2\tau(M)$$

with $= \iff W_+ \equiv 0$. "anti-self-dual"

$$\star W = -W$$

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

$$\mathscr{W}([g]) \geq -12\pi^2\tau(M)$$

with $= \iff W_+ \equiv 0.$ “anti-self-dual”

Reversing orientation $\rightsquigarrow$

self-duality $\longleftrightarrow$ anti-self-duality

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

$$\mathscr{W}([g]) \geq -12\pi^2\tau(M)$$

with $= \iff W_+ \equiv 0$. "anti-self-dual"

1985-1995: $\exists$ self-dual g on many M^4.

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

$$\mathscr{W}([g]) \geq -12\pi^2\tau(M)$$

with $= \iff W_+ \equiv 0$. “anti-self-dual”

1985-1995: $\exists$ anti-self-dual g on many M^4.

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

$$\mathscr{W}([g]) \geq -12\pi^2\tau(M)$$

with $= \iff W_+ \equiv 0$. "anti-self-dual"

1985-1995: $\exists$ anti-self-dual g on many M^4.

Poon, L, Donaldson-Friedman, Taubes …

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

$$\mathscr{W}([g]) \geq -12\pi^2\tau(M)$$

with $= \iff W_+ \equiv 0.$ "anti-self-dual"

1985-1995: $\exists$ anti-self-dual g on many M^4.

Poon, L, Donaldson-Friedman, Taubes . . .

Often using complex geometry, via twistor spaces. . .

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

Proposition (Atiyah-Hitchin-Singer). *The Fubini-Study metric on $\mathbb{CP}_2$ is self-dual.*

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

Proposition (Atiyah-Hitchin-Singer). *The Fubini-Study metric on $\mathbb{CP}_2$ is self-dual. Consequently, minimizes Weyl functional.*

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

Proposition (Atiyah-Hitchin-Singer). *The Fubini-Study metric on $\mathbb{CP}_2$ is self-dual. Consequently, minimizes Weyl functional.*

Context: 1978 paper building on Penrose '76.

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

Proposition (Atiyah-Hitchin-Singer). *The Fubini-Study metric on $\mathbb{CP}_2$ is self-dual. Consequently, minimizes Weyl functional.*

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|^2 - |W_-|^2 \right) d\mu$$

Theorem (Poon '86). *Up conformal isometry, the Fubini-Study class is the unique self-dual conformal class on $\mathbb{CP}_2$ with $Y([g]) > 0$.*

$$Y([g]) = \inf_{\widehat{g}=u^2 g} \frac{\int_M s_{\widehat{g}} \, d\mu_{\widehat{g}}}{\sqrt{\int_M d\mu_{\widehat{g}}}} \; ;$$

$$Y([g]) = \inf_{\widehat{g}=u^2 g} \frac{\int_M s_{\widehat{g}} \, d\mu_{\widehat{g}}}{\sqrt{\int_M d\mu_{\widehat{g}}}} \; ;$$

If g has s of fixed sign, agrees with sign of $Y_{[g]}$.

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

Theorem (Poon '86). *Up conformal isometry, the Fubini-Study class is the unique self-dual conformal class on* $\mathbb{CP}_2$ *with* $Y([g]) > 0$.

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

Proposition (Atiyah-Hitchin-Singer '78). *The Fubini-Study metric on* $\mathbb{CP}_2$ *is self-dual. Consequently, minimizes Weyl functional.*

Osamu Kobayashi '86:

Osamu Kobayashi '86:

What about $S^2 \times S^2$?

Osamu Kobayashi '86:

What about $S^2 \times S^2$?

No self-dual metric!

Osamu Kobayashi '86:

What about $S^2 \times S^2$?

No self-dual metric!

Would be conformally flat, because $\tau = 0$.

Osamu Kobayashi '86:

What about $S^2 \times S^2$?

No self-dual metric!

Would be conformally flat, because $\tau = 0$.

Also $\pi_1 = 0$.

Osamu Kobayashi '86:

What about $S^2 \times S^2$?

No self-dual metric!

Would be conformally flat, because $\tau = 0$.

Also $\pi_1 = 0$.

Kuiper '49: $\therefore$ Round S^4! $\Rightarrow\Leftarrow$

Osamu Kobayashi '86:

What about $S^2 \times S^2$?

Osamu Kobayashi '86:

What about $S^2 \times S^2$?

Conjecture (Kobayashi). *The Kähler-Einstein product metric on* $S^2 \times S^2$ *minimizes the Weyl functional* $\mathscr{W}$.

Osamu Kobayashi '86:

What about $S^2 \times S^2$?

Conjecture (Kobayashi)**.** *The Kähler-Einstein product metric on* $S^2 \times S^2$ *minimizes the Weyl functional* $\mathscr{W}$.

Gave weak evidence:

Osamu Kobayashi '86:

What about $S^2 \times S^2$?

Conjecture (Kobayashi)**.** *The Kähler-Einstein product metric on* $S^2 \times S^2$ *minimizes the Weyl functional* $\mathscr{W}$.

Gave weak evidence:

Local minimum.

Osamu Kobayashi '86:

What about $S^2 \times S^2$?

Conjecture (Kobayashi). *The Kähler-Einstein product metric on* $S^2 \times S^2$ *minimizes the Weyl functional* $\mathscr{W}$.

Commonality between $\mathbb{CP}_2$ and $S^2 \times S^2$?

Osamu Kobayashi '86:

What about $S^2 \times S^2$?

Conjecture (Kobayashi). *The Kähler-Einstein product metric on* $S^2 \times S^2$ *minimizes the Weyl functional* $\mathscr{W}$.

Commonality between $\mathbb{CP}_2$ and $S^2 \times S^2$?

Kähler-Einstein, with $\lambda > 0$.

Natural Generalization:

Del Pezzo surfaces:

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points,
in general position,

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position,

Del Pezzo surfaces:

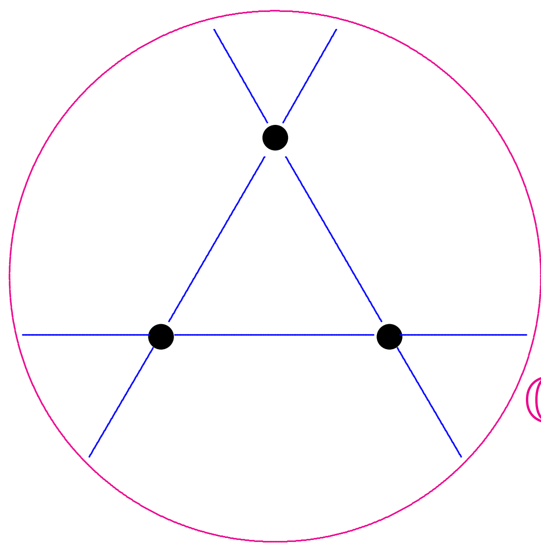
(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

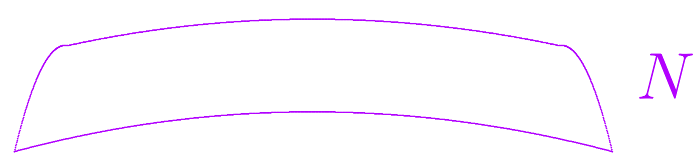
Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



Blowing up:

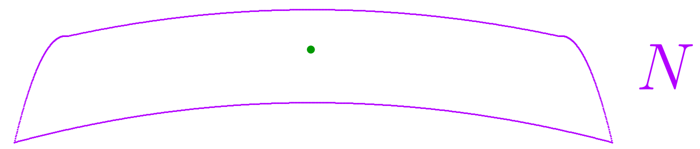
Blowing up:

If N is a complex surface,



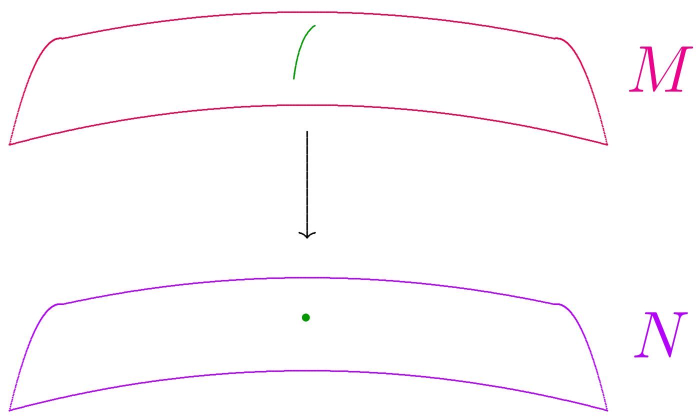
Blowing up:

If N is a complex surface, may replace $p \in N$



Blowing up:

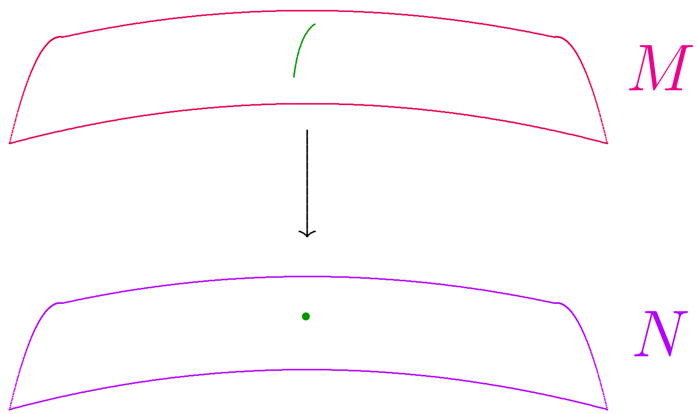
If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$



Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$



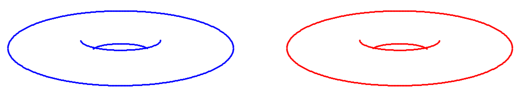
Conventions:

$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

Conventions:

$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

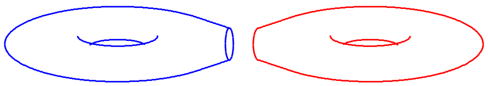
Connected sum #:



Conventions:

$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

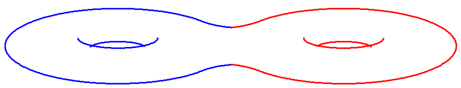
Connected sum #:



Conventions:

$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

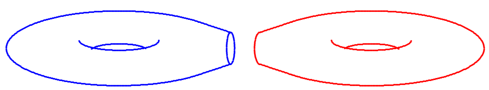
Connected sum $\#$:



Conventions:

$\overline{\mathbb{CP}}_2$ = reverse oriented $\mathbb{CP}_2$.

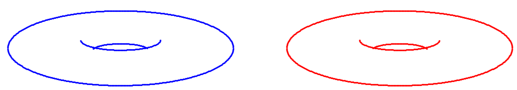
Connected sum $\#$:



Conventions:

$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

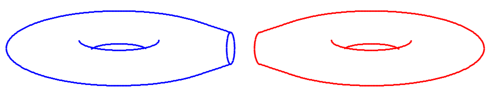
Connected sum #:



Conventions:

$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

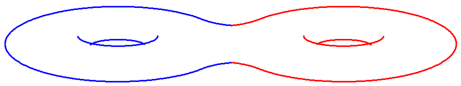
Connected sum #:



Conventions:

$\overline{\mathbb{CP}}_2$ = reverse oriented $\mathbb{CP}_2$.

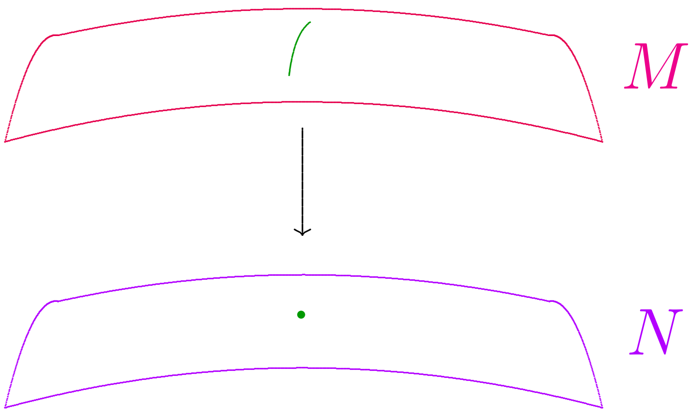
Connected sum #:



Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

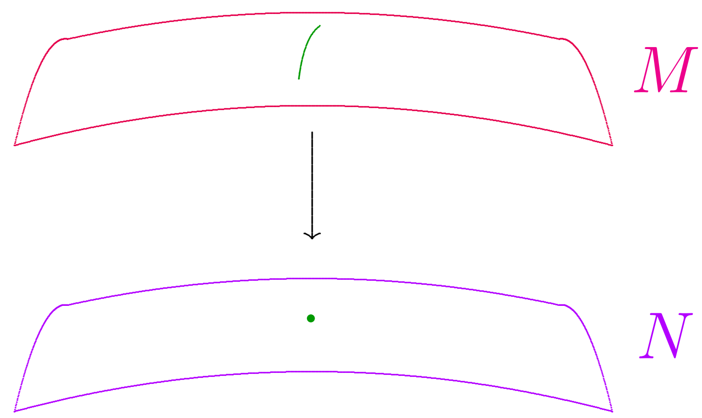


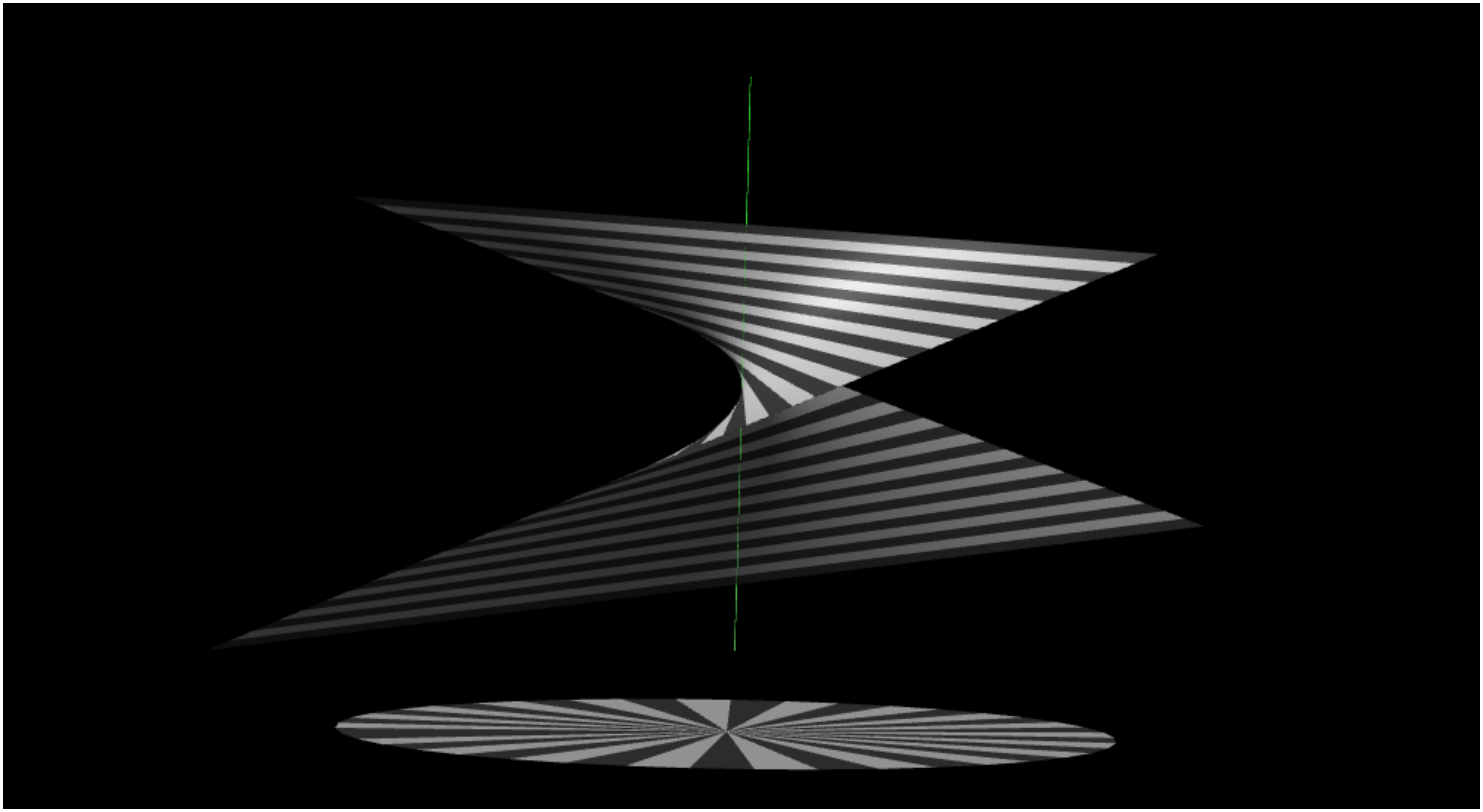
Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}_2}$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.



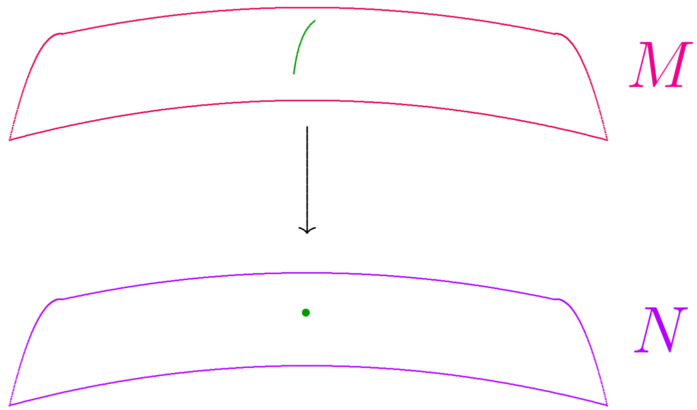


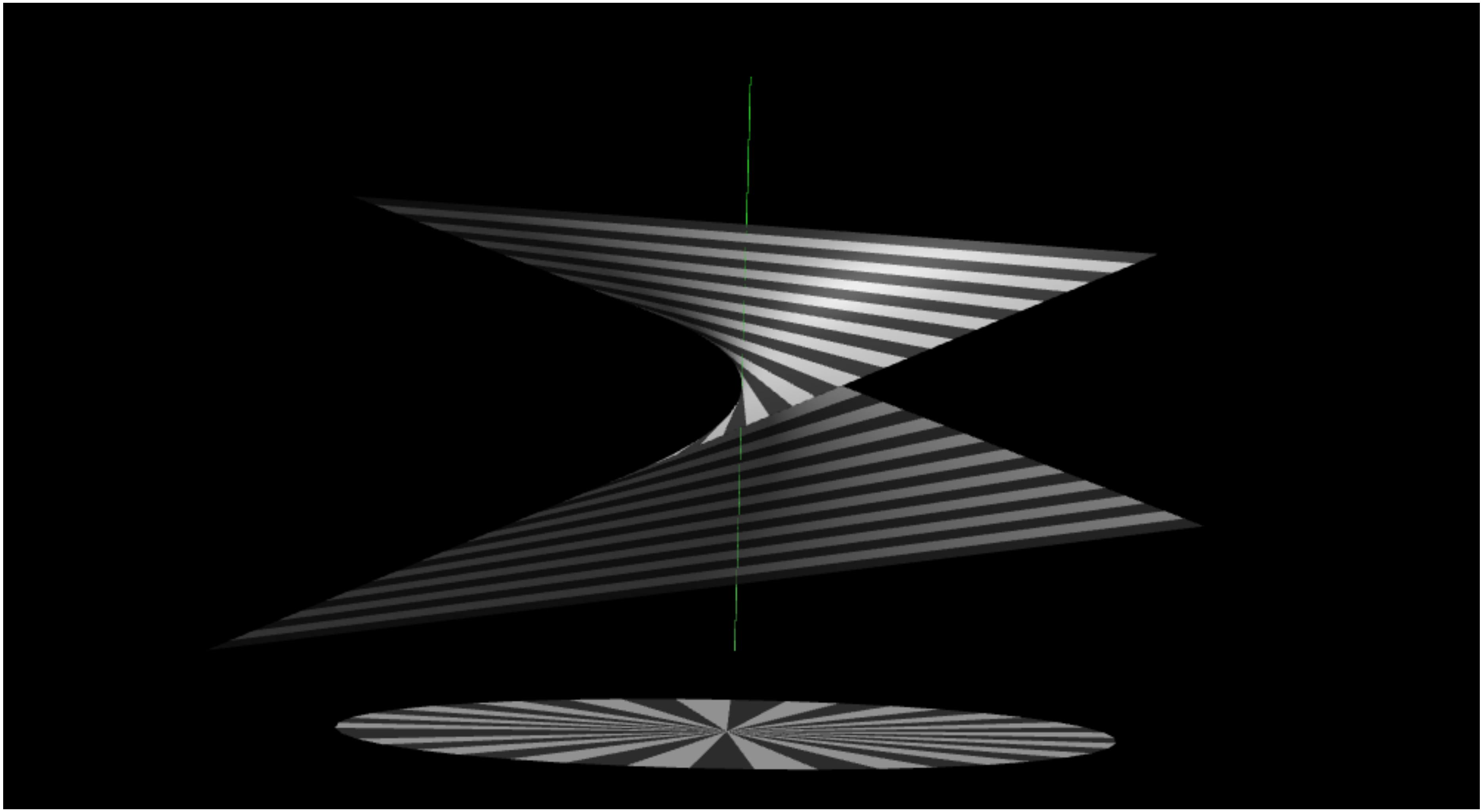
Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.



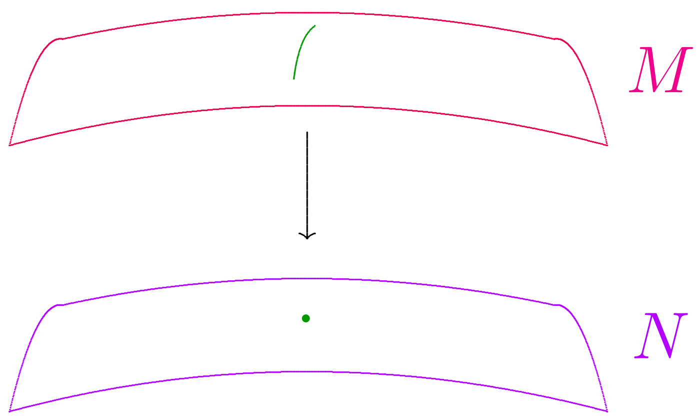


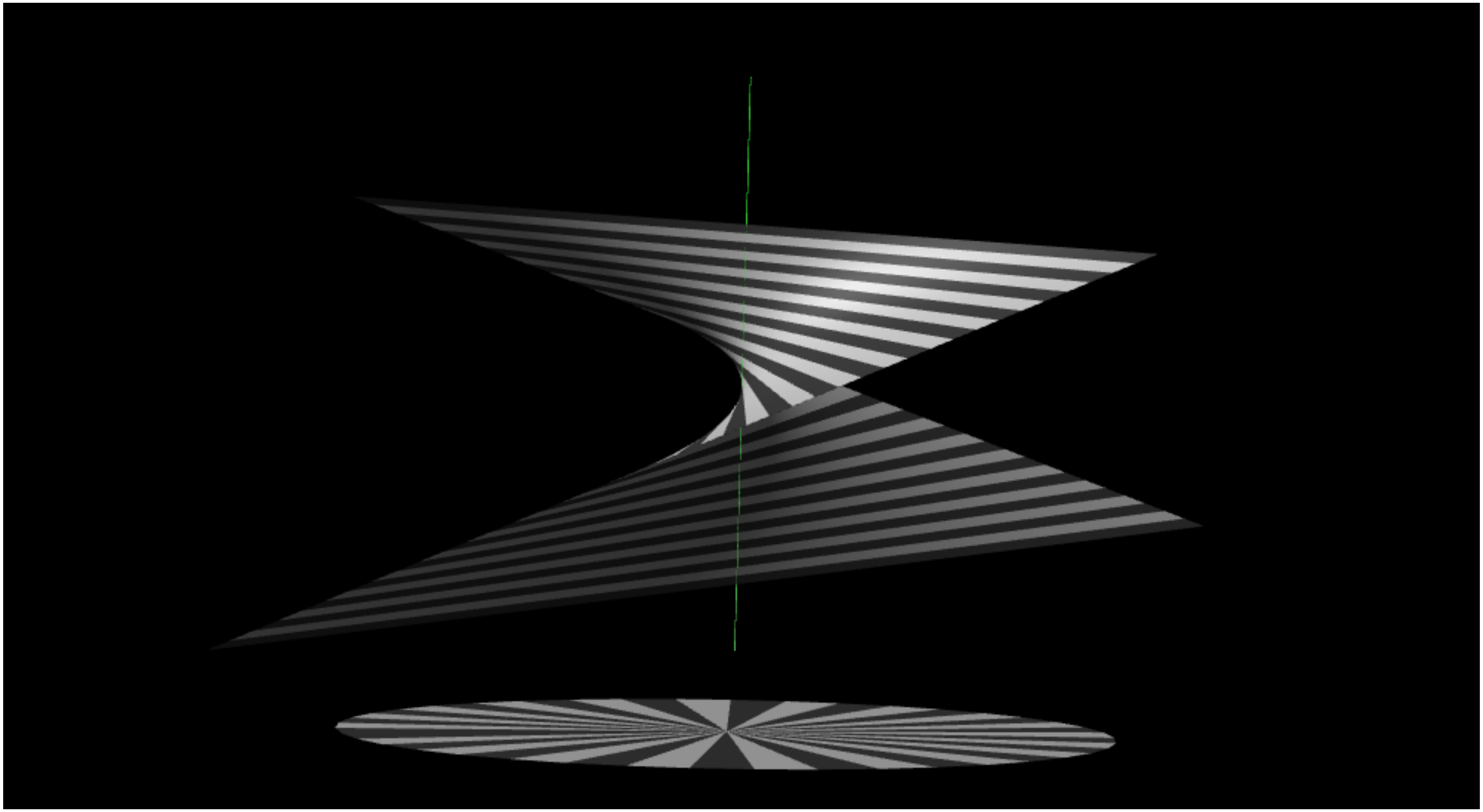
Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.



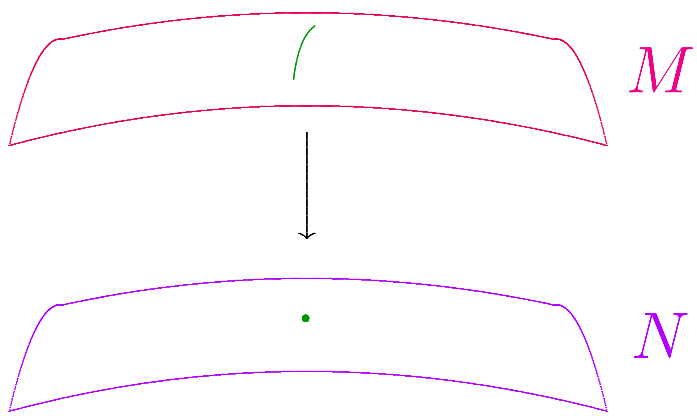


Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}_2}$$

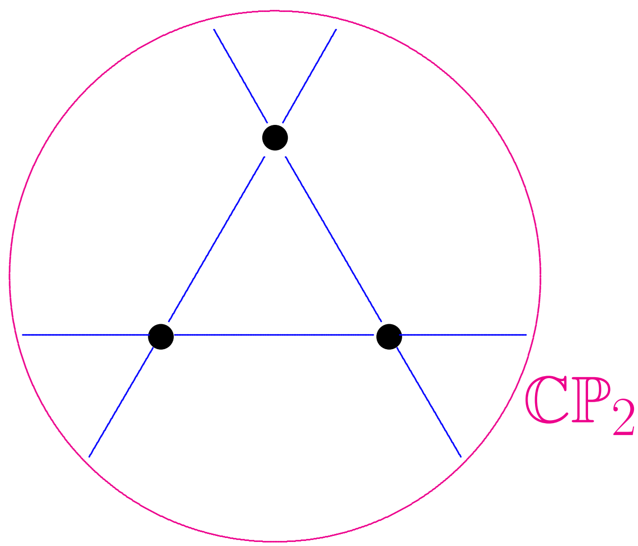
in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.



Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

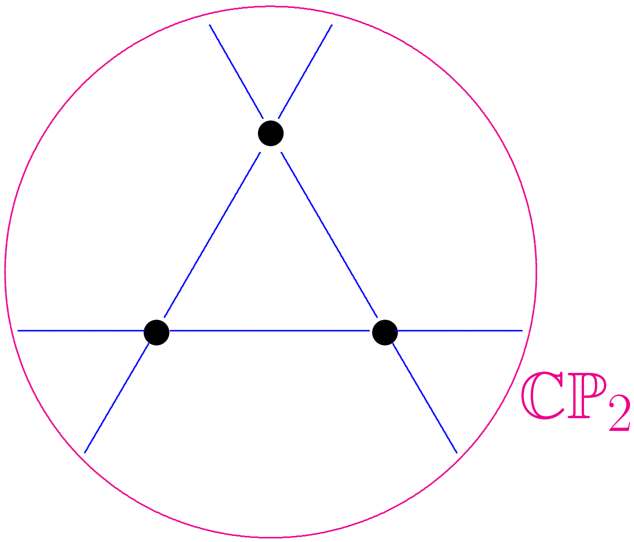
Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

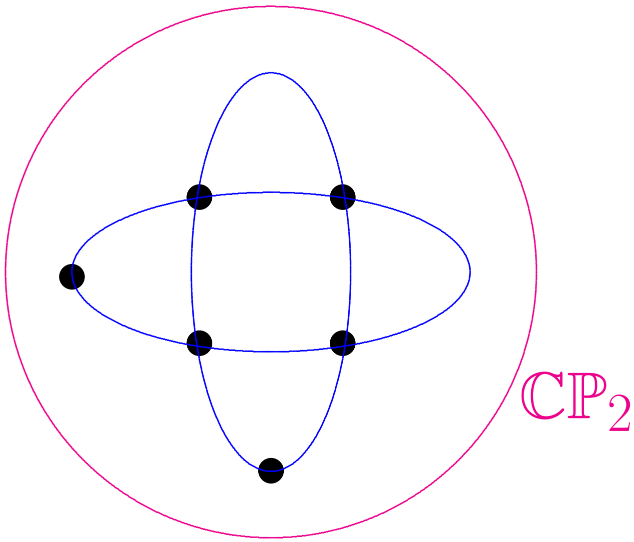


No 3 on a line,

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

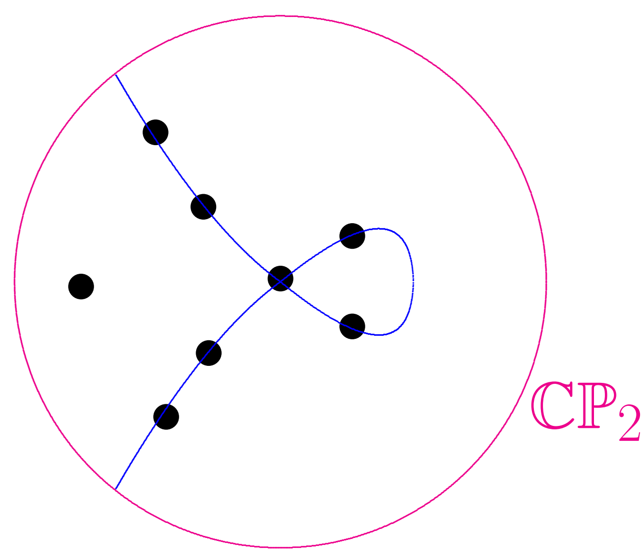


No 3 on a line, no 6 on conic,

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



No 3 on a line, no 6 on conic, no 8 on nodal cubic.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo* (M^4, J) *admits a* J*-compatible*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo* (M^4, J) *admits a* J*-compatible conformally Kähler,*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Existence: Page

1978

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński,

1983

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński, Siu,

1986

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo* (M^4, J) *admits a* J*-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński, Siu, Tian-Yau,

1987

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński, Siu, Tian-Yau, Tian,

1990

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński, Siu, Tian-Yau, Tian, Odaka-Spotti-Sun,

2016

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński, Siu, Tian-Yau, Tian, Odaka-Spotti-Sun, Chen-L-Weber.

2008

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Uniqueness: Bando-Mabuchi '87

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Uniqueness: Bando-Mabuchi '87, L '12.

One reason this seems satisfying...

Theorem (CLW '08). *Suppose that M is a smooth compact oriented 4-manifold which carries some symplectic form ω.*

Theorem (CLW '08). *Suppose that* M *is a smooth compact oriented* 4*-manifold which carries some symplectic form* ω*. Then* M *admits an (unrelated) Einstein metric* g *with* $\lambda > 0$

Theorem (CLW '08). *Suppose that M is a smooth compact oriented* 4*-manifold which carries some symplectic form* ω. *Then M admits an (unrelated) Einstein metric g with* $\lambda > 0$

$$\Longleftrightarrow \quad M \approx \begin{cases} \mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2, & 0 \leq k \leq 8, \\ \quad or & \\ S^2 \times S^2 & \end{cases}$$

Theorem (CLW '08). *Suppose that M is a smooth compact oriented 4-manifold which carries some symplectic form ω. Then M admits an (unrelated) Einstein metric g with $\lambda > 0$*

$$\Longleftrightarrow \quad M \approx \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, & 0 \leq k \leq 8, \\ \quad or \\ S^2 \times S^2 \end{cases}$$

Diffeotypes: exactly the Del Pezzo surfaces.

Theorem (CLW '08). *Suppose that M is a smooth compact oriented 4-manifold which carries some symplectic form ω. Then M admits an (unrelated) Einstein metric g with $\lambda > 0$*

$$\Longleftrightarrow \quad M \approx \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, & 0 \leq k \leq 8, \\ \quad or & \\ S^2 \times S^2 & \end{cases}$$

Diffeotypes: exactly the Del Pezzo surfaces.

For known g, can take ω related to g.

Theorem (CLW '08). *Suppose that M is a smooth compact oriented* 4*-manifold which carries some symplectic form ω. Then M admits an (unrelated) Einstein metric g with $\lambda > 0$*

$$\Longleftrightarrow \quad M \approx \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, & 0 \leq k \leq 8, \\ \quad \textit{or} & \\ S^2 \times S^2 & \end{cases}$$

Diffeotypes: exactly the Del Pezzo surfaces.

For known g, can take ω related to g.

But this is not needed in above result.

Osamu Kobayashi '86:

What about $S^2 \times S^2$?

Conjecture (Kobayashi)**.** *The Kähler-Einstein product metric on* $S^2 \times S^2$ *minimizes the Weyl functional* $\mathscr{W}$.

Natural Generalization:

Natural Generalization:

Conjecture. *On any del Pezzo surface* (M^4, J), *conformally Kähler, Einstein metric minimizes the Weyl functional* $\mathscr{W}$.

Natural Generalization:

Conjecture. *On any del Pezzo surface* (M^4, J)*, conformally Kähler, Einstein metric minimizes the Weyl functional* $\mathscr{W}$.

Persuasive partial results.

Natural Generalization:

Conjecture. *On any del Pezzo surface* (M^4, J), *conformally Kähler, Einstein metric minimizes the Weyl functional* $\mathscr{W}$.

Persuasive partial results.

But problem still not settled.

Theorem (Gursky '98). *Let M be a smooth compact 4-manifold with $b_+(M) \neq 0$.*

Theorem (Gursky '98). *Let* M *be a smooth compact* 4*-manifold with* $b_+(M) \neq 0$*. Then any conformal class* $[g]$

Theorem (Gursky '98). *Let M be a smooth compact 4-manifold with $b_+(M) \neq 0$. Then any conformal class $[g]$ with $Y([g]) > 0$*

$$Y([g]) = \inf_{\widehat{g}=u^2 g} \frac{\int_M s_{\widehat{g}} \, d\mu_{\widehat{g}}}{\sqrt{\int_M d\mu_{\widehat{g}}}} \; ;$$

Theorem (Gursky '98). *Let M be a smooth compact* 4*-manifold with* $b_+(M) \neq 0$. *Then any conformal class* $[g]$ *with* $Y([g]) > 0$ *satisfies*

Theorem (Gursky '98). *Let M be a smooth compact 4-manifold with $b_+(M) \neq 0$. Then any conformal class $[g]$ with $Y([g]) > 0$ satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

Theorem (Gursky '98). *Let* M *be a smooth compact* 4*-manifold with* $b_+(M) \neq 0$. *Then any conformal class* $[g]$ *with* $Y([g]) > 0$ *satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow$ $[g]$ *contains Kähler-Einstein* $\widehat{g}$ *with* $s > 0$.

Theorem (Gursky '98). *Let M be a smooth compact 4-manifold with $b_+(M) \neq 0$. Then any conformal class $[g]$ with $Y([g]) > 0$ satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow$ $[g]$ contains Kähler-Einstein $\widehat{g}$ with $s > 0$.

In particular, any K-E g with $s > 0$ minimizes restriction of $\mathscr{W}$ to $s > 0$ metrics.

Theorem (Gursky '98). *Let M be a smooth compact 4-manifold with $b_+(M) \neq 0$. Then any conformal class $[g]$ with $Y([g]) > 0$ satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow$ $[g]$ contains Kähler-Einstein $\widehat{g}$ with $s > 0$.

In particular, any K-E g with $s > 0$ minimizes restriction of $\mathscr{W}$ to $s > 0$ metrics.

Big step in direction of Kobayashi's conjecture.

Theorem (Gursky '98). *Let M be a smooth compact 4-manifold with $b_+(M) \neq 0$. Then any conformal class $[g]$ with $Y([g]) > 0$ satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow$ $[g]$ contains Kähler-Einstein $\widehat{g}$ with $s > 0$.

In particular, any K-E g with $s > 0$ minimizes restriction of $\mathscr{W}$ to $s > 0$ metrics.

Big step in direction of Kobayashi's conjecture.

Applies in much greater generality.

Theorem (Gursky '98). *Let M be a smooth compact 4-manifold with $b_+(M) \neq 0$. Then any conformal class $[g]$ with $Y([g]) > 0$ satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow$ $[g]$ contains Kähler-Einstein $\widehat{g}$ with $s > 0$.

In particular, any K-E g with $s > 0$ minimizes restriction of $\mathscr{W}$ to $s > 0$ metrics.

Big step in direction of Kobayashi's conjecture.

But says nothing about $Y([g]) < 0$ realm.

Theorem (Gursky '98). *Let M be a smooth compact 4-manifold with $b_+(M) \neq 0$. Then any conformal class $[g]$ with $Y([g]) > 0$ satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow$ $[g]$ contains Kähler-Einstein $\widehat{g}$ with $s > 0$.

In particular, any K-E g with $s > 0$ minimizes restriction of $\mathscr{W}$ to $s > 0$ metrics.

Big step in direction of Kobayashi's conjecture.

But says nothing about "most" conformal classes.

Theorem (Gursky '98). *Let M be a smooth compact 4-manifold with $b_+(M) \neq 0$. Then any conformal class $[g]$ with $Y([g]) > 0$ satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow$ $[g]$ contains Kähler-Einstein $\widehat{g}$ with $s > 0$.

Method: Weitzenböck formula

Theorem (Gursky '98). *Let M be a smooth compact 4-manifold with $b_+(M) \neq 0$. Then any conformal class $[g]$ with $Y([g]) > 0$ satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow$ $[g]$ contains Kähler-Einstein $\widehat{g}$ with $s > 0$.

Method: Weitzenböck formula

$$0 = \frac{1}{2}\Delta|\omega|^2 + |\nabla\omega|^2 - 2W_+(\omega, \omega) + \frac{s}{3}|\omega|^2$$

for self-dual harmonic 2-form ω.

Theorem (Gursky '98). *Let* M *be a smooth compact* 4*-manifold with* $b_+(M) \neq 0$. *Then any conformal class* $[g]$ *with* $Y([g]) > 0$ *satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow$ $[g]$ *contains Kähler-Einstein* $\widehat{g}$ *with* $s > 0$.

Method: Weitzenböck formula

$$\Longrightarrow \exists \widehat{g} = u^2 g \quad \text{s.t.} \quad \widehat{\mathfrak{s}} := \widehat{s} - 2\sqrt{6}\widehat{|W_+|} \leq 0.$$

A different use of self-dual harmonic forms yields a complementary result.

A different use of self-dual harmonic forms yields a complementary result.

Definition. *A conformal class* $[g]$ *on a compact oriented* 4*-manifold* M *is said to be of symplectic type*

A different use of self-dual harmonic forms yields a complementary result.

Definition. *A conformal class* $[g]$ *on a compact oriented* 4*-manifold* M *is said to be of symplectic type* $\Leftrightarrow$

A different use of self-dual harmonic forms yields a complementary result.

Definition. *A conformal class* $[g]$ *on a compact oriented* 4*-manifold* M *is said to be of symplectic type* $\Leftrightarrow \exists$ *self-dual harmonic* 2*-form* ω *on* (M, g)

A different use of self-dual harmonic forms yields a complementary result.

Definition. *A conformal class* $[g]$ *on a compact oriented* 4*-manifold* M *is said to be of symplectic type* $\Leftrightarrow \exists$ *self-dual harmonic* 2*-form* ω *on* (M, g) *such that* $\omega \neq 0$ *everywhere.*

A different use of self-dual harmonic forms yields a complementary result.

Definition. *A conformal class* $[g]$ *on a compact oriented* 4*-manifold* M *is said to be of symplectic type* $\Leftrightarrow \exists$ *self-dual harmonic* 2*-form* ω *on* (M, g) *such that* $\omega \neq 0$ *everywhere.*

Implies ω is orientation-compatible symplectic form.

A different use of self-dual harmonic forms yields a complementary result.

Definition. *A conformal class* $[g]$ *on a compact oriented* 4*-manifold* M *is said to be of symplectic type* $\Leftrightarrow \exists$ *self-dual harmonic* 2*-form* ω *on* (M, g) *such that* $\omega \neq 0$ *everywhere.*

Implies ω is orientation-compatible symplectic form.

Every symplectic 4-manifold arises this way.

A different use of self-dual harmonic forms yields a complementary result.

Definition. *A conformal class* $[g]$ *on a compact oriented* 4*-manifold* M *is said to be of symplectic type* $\Leftrightarrow \exists$ *self-dual harmonic* 2*-form* ω *on* (M, g) *such that* $\omega \neq 0$ *everywhere.*

Implies ω is orientation-compatible symplectic form.

Every symplectic 4-manifold arises this way.

Choose $g \in [g]$ so that $|\omega| \equiv \sqrt{2}$.

A different use of self-dual harmonic forms yields a complementary result.

Definition. *A conformal class* $[g]$ *on a compact oriented* 4*-manifold* M *is said to be of symplectic type* $\Leftrightarrow \exists$ *self-dual harmonic* 2*-form* ω *on* (M, g) *such that* $\omega \neq 0$ *everywhere.*

Implies ω is orientation-compatible symplectic form.

Every symplectic 4-manifold arises this way.

Choose $g \in [g]$ so that $|\omega| \equiv \sqrt{2}$.

Then (M, g, ω) is almost-Kähler manifold:

A different use of self-dual harmonic forms yields a complementary result.

Definition. *A conformal class* $[g]$ *on a compact oriented* 4*-manifold* M *is said to be of symplectic type* $\Leftrightarrow \exists$ *self-dual harmonic* 2*-form* ω *on* (M, g) *such that* $\omega \neq 0$ *everywhere.*

Implies ω is orientation-compatible symplectic form.

Every symplectic 4-manifold arises this way.

Choose $g \in [g]$ so that $|\omega| \equiv \sqrt{2}$.

Then (M, g, ω) is almost-Kähler manifold:

$$\exists J \quad s.t. \quad \omega = g(J\cdot, \cdot)$$

A different use of self-dual harmonic forms yields a complementary result.

Definition. *A conformal class* $[g]$ *on a compact oriented* 4*-manifold* M *is said to be of symplectic type* $\Leftrightarrow \exists$ *self-dual harmonic* 2*-form* ω *on* (M, g) *such that* $\omega \neq 0$ *everywhere.*

Open condition in C^2 topology on metrics.

A different use of self-dual harmonic forms yields a complementary result.

Definition. *A conformal class* $[g]$ *on a compact oriented* 4*-manifold* M *is said to be of symplectic type* $\Leftrightarrow \exists$ *self-dual harmonic* 2*-form* ω *on* (M, g) *such that* $\omega \neq 0$ *everywhere.*

Open condition in C^2 topology on metrics.

(Harmonic forms depend continuously on metric.)

Theorem (L '15). *Let M be the underlying smooth oriented* 4*-manifold of a del Pezzo surface.*

Theorem (L '15)**.** *Let M be the underlying smooth oriented* 4*-manifold of a del Pezzo surface. Then any conformal class $[g]$ of symplectic type on M satisfies*

Theorem (L '15)**.** *Let M be the underlying smooth oriented 4-manifold of a del Pezzo surface. Then any conformal class $[g]$ of symplectic type on M satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

Theorem (L '15). *Let M be the underlying smooth oriented 4-manifold of a del Pezzo surface. Then any conformal class $[g]$ of symplectic type on M satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow [g]$ contains a Kähler-Einstein metric g.

Theorem (L '15). *Let M be the underlying smooth oriented 4-manifold of a del Pezzo surface. Then any conformal class $[g]$ of symplectic type on M satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow [g]$ contains a Kähler-Einstein metric g.

This recovers Gursky's inequality

Theorem (L '15). *Let M be the underlying smooth oriented 4-manifold of a del Pezzo surface. Then any conformal class $[g]$ of symplectic type on M satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow [g]$ contains a Kähler-Einstein metric g.

This recovers Gursky's inequality — but for a different open set of conformal classes!

Theorem (L '15). *Let M be the underlying smooth oriented* 4*-manifold of a* del Pezzo *surface. Then any conformal class $[g]$ of symplectic type on M satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow [g]$ contains a Kähler-Einstein metric g.

$\exists$ conformal classes of symplectic type with

$$Y([g_j]) \to -\infty.$$

Theorem (L '15). *Let M be the underlying smooth oriented 4-manifold of a* del Pezzo surface. *Then any conformal class $[g]$ of symplectic type on M satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow [g]$ contains a Kähler-Einstein metric g.

$\exists$ conformal classes of symplectic type with

$$Y([g_j]) \to -\infty.$$

Inequality not limited to the positive Yamabe realm!

Theorem (L '15). *Let M be the underlying smooth oriented* 4*-manifold of a del Pezzo surface. Then any conformal class $[g]$ of symplectic type on M satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow [g]$ contains a Kähler-Einstein metric g.

Same technique covers conformally Kähler, Einstein cases among classes with fixed T^2 symmetry.

Theorem (L '15)**.** *Let M be the underlying smooth oriented* 4*-manifold of a del Pezzo surface. Then any conformal class $[g]$ of symplectic type on M satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow [g]$ contains a Kähler-Einstein metric g.

Method: Almost-Kähler geometry:

$$\int_M \left[\frac{2s}{3} + W_+(\omega, \omega)\right] d\mu = 4\pi c_1 \bullet [\omega]$$

Theorem (L '15). *Let M be the underlying smooth oriented* 4*-manifold of a del Pezzo surface. Then any conformal class $[g]$ of symplectic type on M satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow [g]$ contains a Kähler-Einstein metric g.

Method: Almost-Kähler geometry:

$$3 \int_M W_+(\omega, \omega)\, d\mu \geq 4\pi c_1 \bullet [\omega]$$

Theorem (L '15). *Let M be the underlying smooth oriented* 4*-manifold of a del Pezzo surface. Then any conformal class* $[g]$ *of symplectic type on* M *satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow [g]$ *contains a Kähler-Einstein metric* g*.*

However, only works for M del Pezzo.

Theorem (L '15)**.** *Let M be the underlying smooth oriented* 4*-manifold of a* del Pezzo surface. *Then any conformal class $[g]$ of symplectic type on M satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow$ $[g]$ contains a Kähler-Einstein metric g.

However, only works for M del Pezzo.

This is apparently not an accident!

Kobayashi's conjecture concerned $S^2 \times S^2$.

Kobayashi's conjecture concerned $S^2 \times S^2$.

But Gursky's theorem also works for $(S^2 \times S^2)\#(S^2 \times S^2)$.

Kobayashi's conjecture concerned $S^2 \times S^2$.

But Gursky's theorem also works for $(S^2 \times S^2)\#(S^2 \times S^2)$.

And for iterated connect-sums $(S^2 \times S^2)\# \cdots \#(S^2 \times S^2)$.

Kobayashi's conjecture concerned $S^2 \times S^2$.

But Gursky's theorem also works for $(S^2 \times S^2)\#(S^2 \times S^2)$.

And for iterated connect-sums $(S^2 \times S^2)\# \cdots \#(S^2 \times S^2)$.

What happens here in the Yamabe-negative realm?

Theorem A.

Theorem A. *Let* N *be a smooth simply-connected*

Theorem A. *Let N be a smooth simply-connected compact 4-manifold*

Theorem A. *Let N be a smooth simply-connected compact 4-manifold that admits a metric g_0*

Theorem A. *Let* N *be a smooth simply-connected compact* 4*-manifold that admits a metric* g_0 *of scalar curvature* $s_{g_0} > 0$.

Theorem A. *Let* N *be a smooth simply-connected compact* 4*-manifold that admits a metric* g_0 *of scalar curvature* $s_{g_0} > 0$*. Then, for any sufficiently large integer* $m \gg 0$,

Theorem A. *Let* N *be a smooth simply-connected compact* 4*-manifold that admits a metric* g_0 *of scalar curvature* $s_{g_0} > 0$*. Then, for any sufficiently large integer* $m \geq m_0$*,*

Theorem A. *Let N be a smooth simply-connected compact 4-manifold that admits a metric g_0 of scalar curvature $s_{g_0} > 0$. Then, for any sufficiently large integer $m \geq m_0$, the smooth simply connected 4-manifold*

Theorem A. *Let N be a smooth simply-connected compact 4-manifold that admits a metric g_0 of scalar curvature $s_{g_0} > 0$. Then, for any sufficiently large integer $m \geq m_0$, the smooth simply connected 4-manifold*

$$M = N \# m(S^2 \times S^2)$$

Theorem A. *Let N be a smooth simply-connected compact 4-manifold that admits a metric g_0 of scalar curvature $s_{g_0} > 0$. Then, for any sufficiently large integer $m \geq m_0$, the smooth simply connected 4-manifold*

$$\begin{aligned} M &= N \# m(S^2 \times S^2) \\ &:= N \# \underbrace{(S^2 \times S^2) \# \cdots \# (S^2 \times S^2)}_{m} \end{aligned}$$

Theorem A. *Let N be a smooth simply-connected compact 4-manifold that admits a metric g_0 of scalar curvature $s_{g_0} > 0$. Then, for any sufficiently large integer $m \geq m_0$, the smooth simply connected 4-manifold*

$$\begin{aligned} M &= N \# m(S^2 \times S^2) \\ &:= N \# \underbrace{(S^2 \times S^2) \# \cdots \# (S^2 \times S^2)}_{m} \end{aligned}$$

admits metrics with $s > 0$,

Theorem A. *Let N be a smooth simply-connected compact 4-manifold that admits a metric g_0 of scalar curvature $s_{g_0} > 0$. Then, for any sufficiently large integer $m \geq m_0$, the smooth simply connected 4-manifold*

$$\begin{aligned} M &= N \# m(S^2 \times S^2) \\ &:= N \# \underbrace{(S^2 \times S^2) \# \cdots \# (S^2 \times S^2)}_{m} \end{aligned}$$

admits metrics with $s > 0$, but nonetheless also admits conformal classes $[g]$ such that

Theorem A. *Let N be a smooth simply-connected compact 4-manifold that admits a metric g_0 of scalar curvature $s_{g_0} > 0$. Then, for any sufficiently large integer $m \geq m_0$, the smooth simply connected 4-manifold*

$$\begin{aligned} M &= N \# m(S^2 \times S^2) \\ &:= N \# \underbrace{(S^2 \times S^2) \# \cdots \# (S^2 \times S^2)}_{m} \end{aligned}$$

admits metrics with $s > 0$, but nonetheless also admits conformal classes $[g]$ such that

$$\int_M |W_+|^2 d\mu < \frac{4\pi^2}{3}(2\chi + 3\tau)(M).$$

Theorem A. *Let N be a smooth simply-connected compact 4-manifold that admits a metric g_0 of scalar curvature $s_{g_0} > 0$. Then, for any sufficiently large integer $m \geq m_0$, the smooth simply connected 4-manifold*

$$\begin{aligned} M &= N \# m(S^2 \times S^2) \\ &:= N \# \underbrace{(S^2 \times S^2) \# \cdots \# (S^2 \times S^2)}_{m} \end{aligned}$$

admits metrics with $s > 0$, but nonetheless also admits conformal classes $[g]$ such that

$$\int_M |W_+|^2 d\mu < \frac{4\pi^2}{3}(2\chi + 3\tau)(M).$$

That is, Gursky's estimate fails for some $s < 0$ metrics

Theorem A. *Let N be a smooth simply-connected compact 4-manifold that admits a metric g_0 of scalar curvature $s_{g_0} > 0$. Then, for any sufficiently large integer $m \geq m_0$, the smooth simply connected 4-manifold*

$$\begin{aligned} M &= N\# m(S^2\times S^2) \\ &:= N\# \underbrace{(S^2\times S^2)\#\cdots\#(S^2\times S^2)}_{m} \end{aligned}$$

admits metrics with $s > 0$, but nonetheless also admits conformal classes $[g]$ such that

$$\int_M |W_+|^2 d\mu < \frac{4\pi^2}{3}(2\chi + 3\tau)(M).$$

That is, Gursky's estimate fails for some $s < 0$ metrics once we "stabilize" by taking the connected sum

Theorem A. *Let N be a smooth simply-connected compact 4-manifold that admits a metric g_0 of scalar curvature $s_{g_0} > 0$. Then, for any sufficiently large integer $m \geq m_0$, the smooth simply connected 4-manifold*

$$\begin{aligned} M &= N\#m(S^2\times S^2) \\ &:= N\#\underbrace{(S^2\times S^2)\#\cdots\#(S^2\times S^2)}_{m} \end{aligned}$$

admits metrics with $s > 0$, but nonetheless also admits conformal classes $[g]$ such that

$$\int_M |W_+|^2 d\mu < \frac{4\pi^2}{3}(2\chi + 3\tau)(M).$$

That is, Gursky's estimate fails for some $s < 0$ metrics once we "stabilize" by taking the connected sum with enough copies of $S^2\times S^2$.

Key facts used in proof:

Key facts used in proof:

Wall '66: If Y and Z are homotopy-equivalent simply-connected smooth compact 4-manifolds,

Key facts used in proof:

Wall '66: If Y and Z are homotopy-equivalent simply-connected smooth compact 4-manifolds,

then $Y \# \ell (S^2 \times S^2) \approx Z \# \ell (S^2 \times S^2)$ for all $\ell \gg 0$.

Key facts used in proof:

Wall '66: If Y and Z are homotopy-equivalent simply-connected smooth compact 4-manifolds,

then $Y\#\ell(S^2\times S^2) \approx Z\#\ell(S^2\times S^2)$ for all $\ell \gg 0$.

Since a spin 4-manifold with $s > 0$ must have $\tau = 0$, we therefore just need the following two lemmas:

Lemma. *For any sufficiently large integer* m*,*

Lemma. *For any sufficiently large integer m,*

$$M = m(S^2 \times S^2) := \underbrace{(S^2 \times S^2) \# \cdots \# (S^2 \times S^2)}_{m}$$

Lemma. *For any sufficiently large integer* m,

$$M = m(S^2 \times S^2) := \underbrace{(S^2 \times S^2) \# \cdots \# (S^2 \times S^2)}_{m}$$

admits Riemannian metrics g *such that*

Lemma. *For any sufficiently large integer* m,

$$M = m(S^2 \times S^2) := \underbrace{(S^2 \times S^2) \# \cdots \# (S^2 \times S^2)}_{m}$$

admits Riemannian metrics g *such that*

$$\int_M |W_+|^2 d\mu < \frac{4\pi^2}{3}(2\chi + 3\tau)(M).$$

Lemma. *Similarly, for any any sufficiently large integer* m

Lemma. *Similarly, for any any sufficiently large integer* m *and any integer* n

Lemma. *Similarly, for any any sufficiently large integer* m *and any integer* n *such that* $\frac{n}{m}$ *is sufficiently close to* 1*,*

Lemma. *Similarly, for any any sufficiently large integer* m *and any integer* n *such that* $\frac{n}{m}$ *is sufficiently close to* 1*, the non-spin manifold*

Lemma. *Similarly, for any any sufficiently large integer* m *and any integer* n *such that* $\frac{n}{m}$ *is sufficiently close to* 1*, the non-spin manifold*

$$M = \underbrace{\mathbb{CP}_2 \# \cdots \# \mathbb{CP}_2}_{m} \# \underbrace{\overline{\mathbb{CP}}_2 \# \cdots \# \overline{\mathbb{CP}}_2}_{n}$$

Lemma. *Similarly, for any any sufficiently large integer* m *and any integer* n *such that* $\frac{n}{m}$ *is sufficiently close to* 1*, the non-spin manifold*

$$M = \underbrace{\mathbb{CP}_2 \# \cdots \# \mathbb{CP}_2}_{m} \# \underbrace{\overline{\mathbb{CP}}_2 \# \cdots \# \overline{\mathbb{CP}}_2}_{n}$$

admits Riemannian metrics g *such that*

Lemma. *Similarly, for any any sufficiently large integer* m *and any integer* n *such that* $\frac{n}{m}$ *is sufficiently close to* 1*, the non-spin manifold*

$$M = \underbrace{\mathbb{CP}_2 \# \cdots \# \mathbb{CP}_2}_{m} \# \underbrace{\overline{\mathbb{CP}}_2 \# \cdots \# \overline{\mathbb{CP}}_2}_{n}$$

admits Riemannian metrics g *such that*

$$\int_M |W_+|^2 d\mu < \frac{4\pi^2}{3}(2\chi + 3\tau)(M).$$

Lemma. *Similarly, for any any sufficiently large integer* m *and any integer* n *such that* $\frac{n}{m}$ *is sufficiently close to* 1*, the non-spin manifold*

$$M = \underbrace{\mathbb{CP}_2 \# \cdots \# \mathbb{CP}_2}_{m} \# \underbrace{\overline{\mathbb{CP}}_2 \# \cdots \# \overline{\mathbb{CP}}_2}_{n}$$

admits Riemannian metrics g *such that*

$$\int_M |W_+|^2 d\mu < \frac{4\pi^2}{3}(2\chi + 3\tau)(M).$$

$$\left|\frac{n}{m} - 1\right| < \frac{1}{5}$$

Lemma. *Similarly, for any any sufficiently large integer* m *and any integer* n *such that* $\frac{n}{m}$ *is sufficiently close to* 1*, the non-spin manifold*

$$M = \underbrace{\mathbb{CP}_2 \# \cdots \# \mathbb{CP}_2}_{m} \# \underbrace{\overline{\mathbb{CP}}_2 \# \cdots \# \overline{\mathbb{CP}}_2}_{n}$$

admits Riemannian metrics g *such that*

$$\int_M |W_+|^2 d\mu < \frac{4\pi^2}{3}(2\chi + 3\tau)(M).$$

Key facts used in proof:

Key facts used in proof:

Wall '66: If Y and Z are homotopy-equivalent simply-connected smooth compact 4-manifolds,

Key facts used in proof:

Wall '66: If Y and Z are homotopy-equivalent simply-connected smooth compact 4-manifolds,

then $Y \# \ell(S^2 \times S^2) \approx Z \# \ell(S^2 \times S^2)$ for all $\ell \gg 0$.

Key facts used in proof:

Wall '66: If Y and Z are homotopy-equivalent simply-connected smooth compact 4-manifolds,

then $Y\#\ell(S^2\times S^2)\approx Z\#\ell(S^2\times S^2)$ for all $\ell\gg 0$.

In proof, we apply this to

$$M=(k+\ell)(X\#\overline{X})\#(k+2\ell)(S^2\times S^2)$$

where X simply-connected minimal complex surface of general type with $\tau(X)>0$.

Key facts used in proof:

Wall '66: If Y and Z are homotopy-equivalent simply-connected smooth compact 4-manifolds,

then $Y\#\ell(S^2\times S^2) \approx Z\#\ell(S^2\times S^2)$ for all $\ell \gg 0$.

In proof, we apply this to

$$M = (k+\ell)(X\#\overline{X})\#(k+2\ell)(S^2\times S^2)$$

where X simply-connected minimal complex surface of general type with $\tau(X) > 0$.

Such X now known to exist in profusion!

Key facts used in proof:

Wall '66: If Y and Z are homotopy-equivalent simply-connected smooth compact 4-manifolds,

then $Y\#\ell(S^2\times S^2) \approx Z\#\ell(S^2\times S^2)$ for all $\ell \gg 0$.

In proof, we apply this to

$$M = (k+\ell)(X\#\overline{X})\#(k+2\ell)(S^2\times S^2)$$

where X simply-connected minimal complex surface of general type with $\tau(X) > 0$.

Such X now known to exist in profusion!

Roulleau-Urzúa '15: $\exists$ sequences with $\tau/\chi \to 1/3$.

Key facts used in proof:

Wall '66: If Y and Z are homotopy-equivalent simply-connected smooth compact 4-manifolds,

then $Y\#\ell(S^2\times S^2) \approx Z\#\ell(S^2\times S^2)$ for all $\ell \gg 0$.

In proof, we apply this to

$$M = (k+\ell)(X\#\overline{X})\#(k+2\ell)(S^2\times S^2)$$

where X simply-connected minimal complex surface of general type with $\tau(X) > 0$.

Such X now known to exist in profusion!

Roulleau-Urzúa '15: $\exists$ sequences with $\tau/\chi \to 1/3$.

$\to$ Miyaoka-Yau line! Can choose spin or non-spin!

Key facts used in proof:

Wall '66: If Y and Z are homotopy-equivalent simply-connected smooth compact 4-manifolds,

then $Y \# \ell(S^2 \times S^2) \approx Z \# \ell(S^2 \times S^2)$ for all $\ell \gg 0$.

In proof, we apply this to

$$M = (k+\ell)(X \# \overline{X}) \# (k+2\ell)(S^2 \times S^2)$$

where X simply-connected minimal complex surface of general type with $\tau(X) > 0$.

Equip pluricanonical model $\check{X}$ with orbifold Kähler-Einstein metric.

Key facts used in proof:

Wall '66: If Y and Z are homotopy-equivalent simply-connected smooth compact 4-manifolds,

then $Y \# \ell(S^2 \times S^2) \approx Z \# \ell(S^2 \times S^2)$ for all $\ell \gg 0$.

In proof, we apply this to

$$M = (k+\ell)(X \# \overline{X}) \# (k+2\ell)(S^2 \times S^2)$$

where X simply-connected minimal complex surface of general type with $\tau(X) > 0$.

Equip pluricanonical model $\check{X}$ with orbifold Kähler-Einstein metric. Flatten on small balls.

Key facts used in proof:

Wall '66: If Y and Z are homotopy-equivalent simply-connected smooth compact 4-manifolds,

then $Y\#\ell(S^2\times S^2) \approx Z\#\ell(S^2\times S^2)$ for all $\ell \gg 0$.

In proof, we apply this to

$$M = (k+\ell)(X\#\overline{X})\#(k+2\ell)(S^2\times S^2)$$

where X simply-connected minimal complex surface of general type with $\tau(X) > 0$.

Equip pluricanonical model $\check{X}$ with orbifold Kähler-Einstein metric. Flatten on small balls. Glue in gravitational instantons to desingularize.

Key facts used in proof:

Wall '66: If Y and Z are homotopy-equivalent simply-connected smooth compact 4-manifolds,

then $Y \# \ell(S^2 \times S^2) \approx Z \# \ell(S^2 \times S^2)$ for all $\ell \gg 0$.

In proof, we apply this to

$$M = (k+\ell)(X \# \overline{X}) \# (k+2\ell)(S^2 \times S^2)$$

where X simply-connected minimal complex surface of general type with $\tau(X) > 0$.

Equip pluricanonical model $\check{X}$ with orbifold Kähler-Einstein metric. Flatten on small balls. Glue in gravitational instantons to desingularize. Delete small flat balls.

Key facts used in proof:

Wall '66: If Y and Z are homotopy-equivalent simply-connected smooth compact 4-manifolds,

then $Y \# \ell(S^2 \times S^2) \approx Z \# \ell(S^2 \times S^2)$ for all $\ell \gg 0$.

In proof, we apply this to

$$M = (k+\ell)(X \# \overline{X}) \# (k+2\ell)(S^2 \times S^2)$$

where X simply-connected minimal complex surface of general type with $\tau(X) > 0$.

Equip pluricanonical model $\check{X}$ with orbifold Kähler-Einstein metric. Flatten on small balls. Glue in gravitational instantons to desingularize. Delete small flat balls. Glue by conformal reflections. . .

Theorem A. *Let* N *be a smooth simply-connected compact* 4*-manifold that admits a metric* g_0 *of scalar curvature* $s_{g_0} > 0$*. Then, for any sufficiently large integer* $m \geq m_0$*, the smooth simply connected* 4*-manifold*

$$\begin{aligned} M &= N\# m(S^2\times S^2) \\ &:= N\#\underbrace{(S^2\times S^2)\#\cdots\#(S^2\times S^2)}_{m} \end{aligned}$$

admits metrics with $s > 0$*, but nonetheless also admits conformal classes* $[g]$ *such that*

$$\int_M |W_+|^2 d\mu < \frac{4\pi^2}{3}(2\chi + 3\tau)(M).$$

Our discussion today has touched on

Our discussion today has touched on

- Question of balance;

Our discussion today has touched on

- Question of balance;
- Almost-Kähler Manifolds; and

Our discussion today has touched on

- Question of balance;
- Almost-Kähler Manifolds; and
- Harmonic self-dual Weyl curvature.

Our discussion today has touched on

- Question of balance;
- Almost-Kähler Manifolds; and
- Harmonic self-dual Weyl curvature.

Dessert course:

Our discussion today has touched on

- Question of balance;
- Almost-Kähler Manifolds; and
- Harmonic self-dual Weyl curvature.

Dessert course:

A final result involving these ideas.

Theorem B. *If* (M, g, ω) *is a compact almost-Kähler* 4*-manifold*

Theorem B. *If* (M, g, ω) *is a compact almost-Kähler* 4*-manifold such that* $\delta W_+ = 0$*,*

Theorem B. *If* (M, g, ω) *is a compact almost-Kähler* 4*-manifold such that* $\delta W_+ = 0$*, where* δ *denotes the divergence operator, then*

Theorem B. *If* (M, g, ω) *is a compact almost-Kähler* 4*-manifold such that* $\delta W_+ = 0$*, where* δ *denotes the divergence operator, then*

$$\int_M \frac{s^2}{24} d\mu_g \geq \int_M |W_+|^2 d\mu_g \ ,$$

Theorem B. *If* (M, g, ω) *is a compact almost-Kähler* 4*-manifold such that* $\delta W_+ = 0$*, where* δ *denotes the divergence operator, then*

$$\int_M \frac{s^2}{24} d\mu_g \geq \int_M |W_+|^2 d\mu_g \ ,$$

with equality $\Leftrightarrow$ (M, g, ω) *is Kähler.*

Theorem B. *If* (M, g, ω) *is a compact almost-Kähler* 4*-manifold such that* $\delta W_+ = 0$*, where* δ *denotes the divergence operator, then*

$$\int_M \frac{s^2}{24} d\mu_g \geq \int_M |W_+|^2 d\mu_g \ ,$$

with equality $\Leftrightarrow$ (M, g, ω) *is Kähler. By contrast, if* (M, g, ω) *instead has scalar curvature* $s \geq 0$*, then*

Theorem B. *If* (M, g, ω) *is a compact almost-Kähler* 4*-manifold such that* $\delta W_+ = 0$*, where* δ *denotes the divergence operator, then*

$$\int_M \frac{s^2}{24} d\mu_g \geq \int_M |W_+|^2 d\mu_g \ ,$$

with equality $\Leftrightarrow$ (M, g, ω) *is Kähler. By contrast, if* (M, g, ω) *instead has scalar curvature* $s \geq 0$*, then*

$$\int_M |W_+|^2 d\mu_g \geq \int_M \frac{s^2}{24} d\mu_g,$$

Theorem B. *If* (M, g, ω) *is a compact almost-Kähler* 4*-manifold such that* $\delta W_+ = 0$*, where* δ *denotes the divergence operator, then*

$$\int_M \frac{s^2}{24} d\mu_g \geq \int_M |W_+|^2 d\mu_g \ ,$$

with equality $\Leftrightarrow$ (M, g, ω) *is Kähler. By contrast, if* (M, g, ω) *instead has scalar curvature* $s \geq 0$*, then*

$$\int_M |W_+|^2 d\mu_g \geq \int_M \frac{s^2}{24} d\mu_g,$$

again with equality $\Leftrightarrow$ (M, g, ω) *is Kähler.*

Theorem B. *If* (M, g, ω) *is a compact almost-Kähler* 4*-manifold such that* $\delta W_+ = 0$*, where* δ *denotes the divergence operator, then*

$$\int_M \frac{s^2}{24} d\mu_g \geq \int_M |W_+|^2 d\mu_g \ ,$$

with equality $\Leftrightarrow$ (M, g, ω) *is Kähler. By contrast, if* (M, g, ω) *instead has scalar curvature* $s \geq 0$*, then*

$$\int_M |W_+|^2 d\mu_g \geq \int_M \frac{s^2}{24} d\mu_g,$$

again with equality $\Leftrightarrow$ (M, g, ω) *is Kähler.*

In particular, any compact almost-Kähler 4*-manifold* (M, g, ω) *with* $\delta W_+ = 0$ *and* $s \geq 0$ *is Kähler.*

Merci de m'avoir invité!

Merci de m’avoir invité!

Je suis ravi d'être ici!

À la prochaine, j'espère!