Einstein Metrics,

Weyl Curvature, &

Symplectic 4-Manifolds

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Stony Brook University

Geometric Analysis in Geometry and Topology
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Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature.
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$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$. 
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“...the greatest blunder of my life!”

— A. Einstein, to G. Gamow
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$$s = r^j_j = \mathcal{R}^{ij} ij.$$  

$$\frac{\text{vol}_g(B_{\varepsilon}(p))}{c_n \varepsilon^n} = 1 - s \frac{\varepsilon^2}{6(n+2)} + O(\varepsilon^4)$$
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Perhaps reasonable in other dimensions?
Recognition Problem:
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When $n = 4$, situation is more encouraging . . .
Moduli Spaces of Einstein metrics
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Berger,
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Berger, Hitchin,

\[ K3 = \text{underlying } M^4 \text{ of a generic quartic in } \mathbb{CP}^3. \]
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Four Dimensions is Exceptional!
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Key question:
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Berger, Hitchin, Besson-Courtois-Gallot, L.

Four Dimensions is Exceptional!

Key question:

For which \( M^4 \) is \( \mathcal{E}(M) \) connected?
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\[ \star^2 = 1. \]

$\Lambda^+$ self-dual 2-forms.  
$\Lambda^-$ anti-self-dual 2-forms.
Riemann curvature of $g$

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where

$s = \text{scalar curvature}$

$\hat{r} = \text{trace-free Ricci curvature}$

$W_+ = \text{self-dual Weyl curvature}$

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where

$s = \text{scalar curvature}$

$\mathring{\mathcal{r}} = \text{trace-free Ricci curvature}$

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In dimension 4, Einstein metrics
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Measures deviation $[g]$ from conformal flatness.
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Ricci-flat product $K3 \times T^m$ never critical!
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\[ \chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu \]
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for Euler-characteristic \( \chi(M) = \sum_j (-1)^j b_j(M). \)
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To prove that \( \mathcal{E}(M) \) connected, must control \( \mathcal{W}(g) \).
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For signature \( \tau(M) = b_+ - b_- \) of intersection form.
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\[ W(g) := \int_M \left( |W_+|^2 + |W_-|^2 \right) \ d\mu_g. \]

4-dimensional signature formula \(\implies\)

\[ W(g) \geq 12\pi^2 |\tau(M)| \]

With equality iff \(W_+ \equiv 0\) or \(W_- \equiv 0\).
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Connectedness of \( \mathcal{E}(M) \): must also control \( \int_M s^2 d\mu \).
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Connectedness of \( \mathcal{E}(M) \): more difficult when \( \lambda > 0 \).
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Standard Einstein metric on $\mathbb{C}P_2$ minimizes $\mathcal{W}(g)$. 
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What about standard Einstein metric on $S^2 \times S^2$?
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Also Kähler-Einstein, but \( W_\pm \neq 0 \).
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**Conjectured:** Global minimizer.
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Osamu Kobayashi (1985):

Proposed systematic study of invariant \( \inf \mathcal{W}(M) \).
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**Conjecture** (Generalized Kobayashi Conjecture).
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We will see later that $Y > 0$ does not seem essential.
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**Question.** If $(M^4, \omega)$ is a symplectic 4-manifold, when does $M^4$ admit an Einstein metric $g$ (perhaps unrelated to $\omega$)?
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**Question.** If $(M^4, \omega)$ is a symplectic 4-manifold, when does $M^4$ admit an Einstein metric $g$ (perhaps unrelated to $\omega$)? What if we also require $\lambda \geq 0$?
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Fortunately, a complete answer is available!
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$$M \overset{\text{diff}}{\approx} \begin{cases} \mathbb{CP}^2 \# k \mathbb{CP}^2, & 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K_3, \\ K_3/\mathbb{Z}_2, \\ T_4, \\ T_4/\mathbb{Z}_2, \\ T_4/\mathbb{Z}_3, \\ T_4/\mathbb{Z}_4, \\ T_4/\mathbb{Z}_6, \\ T_4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), \\ T_4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \\ T_4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{cases}$$
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S^2 \times S^2, \\
K3,
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K3, \\
K3/\mathbb{Z}_2, \\
T^4, \\
T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), \\
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Conventions:

\[ \overline{\mathbb{CP}_2} = \text{reverse oriented } \mathbb{CP}_2. \]
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Connected sum $\#$: 

\[
\begin{array}{c}
  \text{Diagram 1} \\
  \text{Diagram 2}
\end{array}
\]
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Del Pezzo surfaces,
K3 surface, Enriques surface,
Abelian surface, Hyper-elliptic surfaces.
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S^2 \times S^2, & \\
K3, & \\
K3/\mathbb{Z}_2, & \\
T^4, & \\
T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, & \\
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\end{cases}$$


No others: Hitchin-Thorpe, Seiberg-Witten, …
$\mathbb{CP}_2 \# k \overline{\mathbb{CP}_2}$, $0 \leq k \leq 8$, 
$S^2 \times S^2$, 
$K3$, 
$K3/\mathbb{Z}_2$, 
$T^4$, 
$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3)$, or $T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4)$. 
Definitive list . . .

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$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6,$
$T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4).$
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\[ \mathbb{CP}_2 \# k \overline{\mathbb{CP}_2}, \quad 0 \leq k \leq 8, \]
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\[ T^4, \]
\[ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \]
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Below the line:

Every Einstein metric is Ricci-flat Kähler.
But we understand some cases better than others!

\[ \mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}, \quad 0 \leq k \leq 8, \]
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\[ K3, \]
\[ K3/\mathbb{Z}_2, \]
\[ T^4, \]
\[ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \]
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Every Einstein metric is Ricci-flat Kähler.

Moduli space \( \mathcal{E}(M) \)
But we understand some cases better than others!

\[ \mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}, \quad 0 \leq k \leq 8, \]
\[ S^2 \times S^2, \]
\[ K3, \]
\[ K3/\mathbb{Z}_2, \]
\[ T^4, \]
\[ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \]
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Every Einstein metric is Ricci-flat Kähler.

Moduli space \( \mathcal{E}(M) = \{ \text{Einstein } g \}/(\text{Diffeos} \times \mathbb{R}^+) \)
But we understand some cases better than others!

\[ \mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}, \quad 0 \leq k \leq 8, \]
\[ S^2 \times S^2, \]
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Every Einstein metric is Ricci-flat Kähler.

Moduli space \( \mathcal{E}(M) \) completely understood.
But we understand some cases better than others!

\[
\begin{align*}
\mathbb{CP}_2 \# k\overline{\mathbb{CP}_2}, & \quad 0 \leq k \leq 8, \\
S^2 \times S^2, & \\
K3, & \\
K3/\mathbb{Z}_2, & \\
T^4, & \\
T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, & \\
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\end{align*}
\]

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Moduli space \(\mathcal{E}(M)\) connected!
$\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8,$  
$S^2 \times S^2,$  
$K3,$  
$K3/\mathbb{Z}_2,$  
$T^4,$  
$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6,$  
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Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathcal{E}(M)$ connected!
Above the line:

\[ \mathbb{C}P_2 \# k\overline{\mathbb{C}P}_2, \quad 0 \leq k \leq 8, \]

\[ S^2 \times S^2, \]

\[ K3, \]

\[ K3/\mathbb{Z}_2, \]

\[ T^4, \]

\[ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \]

\[ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \]

Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space \( \mathcal{E}(M) \) connected!
Above the line:

Know an Einstein metric on each manifold.

\[
\begin{aligned}
\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}, & \quad 0 \leq k \leq 8, \\
S^2 \times S^2, \\
K3, \\
K3/\mathbb{Z}_2, \\
T^4, \\
T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\
T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4).
\end{aligned}
\]

Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space \( \mathcal{E}(M) \) connected!
Above the line:

Moduli space $\mathcal{E}(M) \neq \emptyset$.

$\mathbb{C}P_2 \# k\overline{\mathbb{C}P_2}$, $0 \leq k \leq 8,$

$S^2 \times S^2,$

$K3,$

$K3/\mathbb{Z}_2,$

$T^4,$

$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6,$

$T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3),$ or $T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4)$.

Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathcal{E}(M)$ connected!
Above the line:

Moduli space $\mathcal{E}(M) \neq \emptyset$. But is it connected?

$\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}, \quad 0 \leq k \leq 8,$

$S^2 \times S^2,$

$K3,$

$K3/\mathbb{Z}_2,$

$T^4,$

$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6,$

$T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3),$ or $T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4).$

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for some Kähler metric \( h \) and a positive function \( u \).

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These live on Del Pezzo surfaces, which are, in particular, oriented 4-manifolds with \( b_+ = 1 \).
Smooth compact $M^4$ has invariants $b_{\pm}(M)$, defined in terms of intersection pairing
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\[ H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) \rightarrow \mathbb{R} \]

\[ ( [\varphi], [\psi] ) \mapsto \int_M \varphi \wedge \psi \]
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$$\begin{pmatrix}
+1 \\
\vdots \\
+1
\end{pmatrix}$$
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Diagonalize:

$$\begin{bmatrix}
  +1 & & & \\
  \vdots & \ddots & \ddots & \\
  b_+ (M) & \cdots & +1 & \\
  b_- (M) & \{ & -1 & \cdots \} & -1
\end{bmatrix}.$$
Hodge theory:

\[ H^2(M, \mathbb{R}) = \{ \varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0, \ d\star \varphi = 0 \}. \]
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Since \( \ast \) is involution of RHS, \( \implies \)

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Since \( \star \) is involution of RHS, \( \implies \)

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self-dual & anti-self-dual harmonic forms. Then

\[ b_\pm(M) = \dim \mathcal{H}_g^\pm. \]

This decomposition is conformally invariant,

but does vary as we change \( [g] \).
\( \mathcal{H}_g^+ \subset H^2(M, \mathbb{R}) \)
\{ a \mid a \cdot a = 0 \} \subset H^2(M, \mathbb{R})
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Let us now focus on metrics $g$. 
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Let us now focus on metrics $g$ for which

$$W_+(\omega, \omega) > 0$$
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Up to scale, $\forall \, g$, $\exists !$ self-dual harmonic 2-form $\omega$:

$$d\omega = 0, \quad \star \omega = \omega.$$ 

This allows us to associate the scalar quantity $W_+(\omega, \omega)$ with any metric $g$ on such a manifold.

Let us now focus on metrics $g$ for which

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everywhere on $M$. 
$W_+(\omega,\omega)$ is non-trivially related
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$W_+ (\omega, \omega)$ is non-trivially related to scalar curv $s$, via Weitzenböck for harmonic self-dual 2-form $\omega$: 
$W_+^+(\omega, \omega)$ is non-trivially related to scalar curv $s$, via Weitzenböck for harmonic self-dual 2-form $\omega$:

$$0 = \nabla^* \nabla \omega - 2W_+^+(\omega, \cdot) + \frac{s}{3} \omega$$
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on average. But we will need this everywhere.
However, $W_+(\omega, \omega)$ conformally invariant, with weight:
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However, $W_+(\omega, \omega)$ conformally invariant, with weight:

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Much simpler than scalar curvature!
However, $W_+ (\omega, \omega)$ conformally invariant, with weight:

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Much simpler than scalar curvature!

In particular, if $g$ satisfies

$$W_+(\omega, \omega) > 0$$

so does every other metric $\tilde{g}$ in conformal class $[g]$. 
Theorem A.

Let $(M,h)$ be a smooth compact 4-dimensional Einstein manifold with $b^+ = 1$. If $h$ satisfies everywhere on $M$, then $h$ is conformally Kähler and has Einstein constant $\lambda > 0$. Moreover, $M$ is diffeomorphic to a Del Pezzo surface. Conversely, every Del Pezzo surface admits Einstein metrics with these properties.
Theorem A. Let \((M, \mathfrak{g})\) be a smooth compact 4-dimensional Einstein manifold with \(b^+ = 1\). If \(h\) satisfies everywhere on \(M\), then \(h\) is conformally Kähler and has Einstein constant \(\lambda > 0\). Moreover, \(M\) is diffeomorphic to a Del Pezzo surface. Conversely, every Del Pezzo surface admits Einstein metrics with these properties.
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In fact, all known Einstein metrics on Del Pezzo surfaces have these properties.
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In fact, all known Einstein metrics on Del Pezzo surfaces have these properties. They are

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- the CLW metric on \(\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}\).
Del Pezzo surfaces:
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$$(M^4, J)$$ for which $c_1$ is a Kähler class $[\omega]$. 
Del Pezzo surfaces:

\[(M^4, J)\] for which \(c_1\) is a Kähler class \([\omega]\).
Shorthand: “\(c_1 > 0\).”
Del Pezzo surfaces:

\((M^4, J)\) for which \(c_1\) is a Kähler class \([\omega]\).

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Blow-up of \(\mathbb{CP}_2\) at \(k\) distinct points, in general position,
Del Pezzo surfaces:

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Shorthand: “\(c_1 > 0\).”

Blow-up of \(\mathbb{CP}_2\) at \(k\) distinct points, \(0 \leq k \leq 8\), in general position,
Del Pezzo surfaces:

$(M^4, J)$ for which $c_1$ is a Kähler class $[\omega]$. Shorthand: “$c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at $k$ distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$. 
Del Pezzo surfaces:

$(M^4, J)$ for which $c_1$ is a Kähler class $[\omega]$.

Shorthand: “$c_1 > 0.$”

Blow-up of $\mathbb{C}P^2$ at $k$ distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{C}P^1 \times \mathbb{C}P^1$. 

![Diagram of CP^2 with points and lines]
Blowing up:
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If $N$ is a complex surface,
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If $N$ is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

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![Diagram of blow-up process](image)
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Del Pezzo surfaces:

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Blow-up of $\mathbb{CP}_2$ at $k$ distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.  

[Diagram of $\mathbb{CP}_2$ with points labeled and lines connecting them.]

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Del Pezzo surfaces:

\((M^4, J)\) for which \(c_1\) is a Kähler class \([\omega]\).
Shorthand: “\(c_1 > 0.\)”

Blow-up of \(\mathbb{CP}_2\) at \(k\) distinct points, \(0 \leq k \leq 8\), in general position, or \(\mathbb{CP}_1 \times \mathbb{CP}_1\).

No 3 on a line,
Del Pezzo surfaces:

\((M^4, J)\) for which \(c_1\) is a Kähler class \([\omega]\).

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Blow-up of \(\mathbb{CP}_2\) at \(k\) distinct points, \(0 \leq k \leq 8\), in general position, or \(\mathbb{CP}_1 \times \mathbb{CP}_1\).

No 3 on a line, no 6 on conic,
Del Pezzo surfaces:

\((M^4, J)\) for which \(c_1\) is a Kähler class \([\omega]\).

Shorthand: “\(c_1 > 0\).”

Blow-up of \(\mathbb{CP}^2\) at \(k\) distinct points, \(0 \leq k \leq 8\), in general position, or \(\mathbb{CP}_1 \times \mathbb{CP}_1\).

No 3 on a line, no 6 on conic, no 8 on nodal cubic.
Del Pezzo surfaces:

\((M^4, J)\) for which \(c_1\) is a Kähler class \([\omega]\).
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Blow-up of \(\mathbb{CP}_2\) at \(k\) distinct points, \(0 \leq k \leq 8\),
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Theorem. Each Del Pezzo \((M^4, J)\) admits a compatible conformally Kähler Einstein metric, and this metric is unique up to automorphisms.
Del Pezzo surfaces:

\((M^4, J)\) for which \(c_1\) is a Kähler class [\(\omega\)].

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**Theorem.** Each Del Pezzo \((M^4, J)\) admits a compatible conformally Kähler Einstein metric, and this metric is unique up to automorphisms.

Existence: Tian, Odaka-Spotti-Sun, Chen-L-Weber…

Uniqueness: Bando-Mabuchi, L 2012…
Del Pezzo surfaces:

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Blow-up of \(\mathbb{CP}_2\) at \(k\) distinct points, \(0 \leq k \leq 8\),
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For each topological type:
Del Pezzo surfaces:

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Blow-up of \(\mathbb{CP}_2\) at \(k\) distinct points, \(0 \leq k \leq 8\), in general position, or \(\mathbb{CP}_1 \times \mathbb{CP}_1\).

For each topological type:

Moduli space of such \((M^4, J)\) is connected.
Del Pezzo surfaces:

$(M^4, J)$ for which $c_1$ is a Kähler class $[\omega]$. Shorthand: “$c_1 > 0$.”

Blow-up of $\mathbb{C}P_2$ at $k$ distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{C}P_1 \times \mathbb{C}P_1$.

For each topological type:

Moduli space of such $(M^4, J)$ is connected.

Just a point if $b_2(M) \leq 5$.  

For $M^4$ a Del Pezzo surface, set
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Corollary. $\mathcal{E}^+_{\omega}(M)$ is exactly one connected component of $\mathcal{E}(M)$. 
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- open condition;
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- most such classes have \( Y([g]) < 0 \).
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Now works in a setting where \( Y \to -\infty \) allowed.
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Strong evidence for O. Kobayashi’s conjecture.
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Tempting to conjecture that these minimize, too!
小林先生、お誕生日おめでとうございます。
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以上です。