

Desingularizations of
Conformally Kähler,
Einstein Orbifolds

Claude LeBrun
Stony Brook University

Geometry Seminar
Stanford University,
September 23, 2026

Most recent results:

Joint work with

Joint work with

Tristan Ozuch

Joint work with

Tristan Ozuch

Massachusetts Institute of Technology

Joint work with

Tristan Ozuch

Massachusetts Institute of Technology

e-print:

arXiv:2601.19215 [math.DG]

Definition. *A Riemannian metric g is said to be Einstein if it has constant Ricci curvature*

Definition. A Riemannian metric g is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Definition. A Riemannian metric g is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

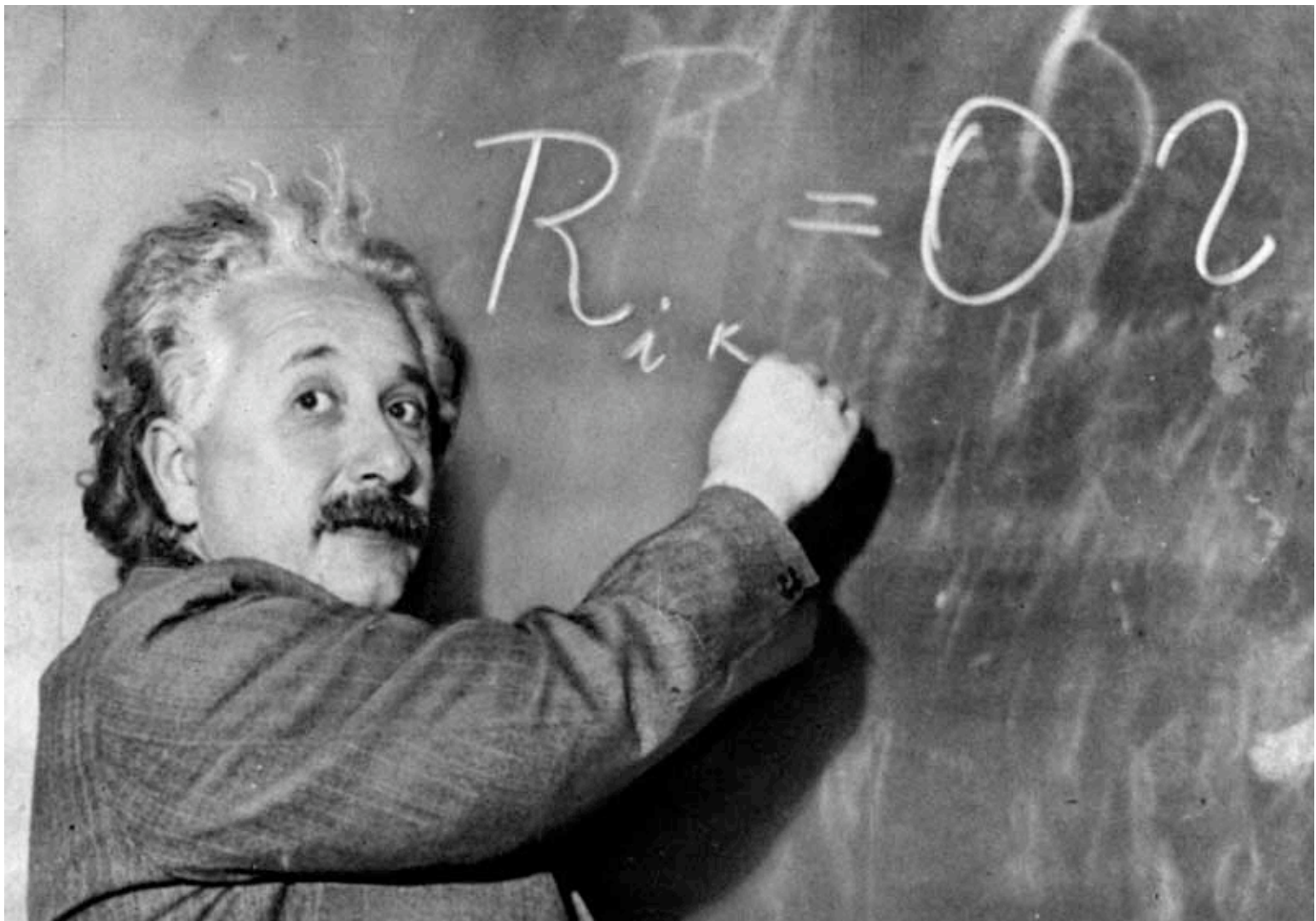
“...the greatest blunder of my life!”

— A. Einstein, to G. Gamow

Definition. A Riemannian metric g is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.



Definition. A Riemannian metric g is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

As punishment ...

Definition. A Riemannian metric g is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Mathematicians call λ the Einstein constant.

Definition. A Riemannian metric g is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Mathematicians call λ the Einstein constant.

Has same sign as the scalar curvature

$$s = r_j^j = \mathcal{R}^{ij}_{ij}.$$

Definition. A Riemannian metric g is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

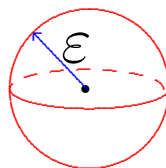
for some constant $\lambda \in \mathbb{R}$.

Mathematicians call λ the Einstein constant.

Has same sign as the *scalar curvature*

$$s = r_j^j = \mathcal{R}^{ij}_{ij}.$$

$$\frac{\text{vol}_g(B_\varepsilon(p))}{c_n \varepsilon^n} = 1 - s \frac{\varepsilon^2}{6(n+2)} + O(\varepsilon^4)$$



Dimension ≤ 3 :

Dimension ≤ 3 :

Einstein's equations are “locally trivial:”

Dimension ≤ 3 :

Einstein's equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

Dimension ≤ 3 :

Einstein's equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

$\implies$ Global rigidity results in these low dimensions.

Dimension ≥ 5 :

Dimension ≥ 5 :

On compact n -manifolds, $n \geq 5$, moduli space of Einstein metrics is often highly disconnected.

Dimension ≥ 5 :

On compact n -manifolds, $n \geq 5$, moduli space of Einstein metrics is often highly disconnected.

For example, the moduli space of Einstein metrics on S^5 has infinitely many connected components.

Dimension ≥ 5 :

On compact n -manifolds, $n \geq 5$, moduli space of Einstein metrics is often highly disconnected.

For example, the moduli space of Einstein metrics on S^5 has infinitely many connected components.

(Böhm,)

Dimension ≥ 5 :

On compact n -manifolds, $n \geq 5$, moduli space of Einstein metrics is often highly disconnected.

For example, the moduli space of Einstein metrics on S^5 has infinitely many connected components.

(Böhm, Collins-Székeleyhidi)

Dimension ≥ 5 :

On compact n -manifolds, $n \geq 5$, moduli space of Einstein metrics is often highly disconnected.

For example, the moduli space of Einstein metrics on S^5 has infinitely many connected components.

(Böhm, Collins-Székeleyhidi)

Recently generalized to all S^{2m+1} , $m \geq 2$.

Dimension ≥ 5 :

On compact n -manifolds, $n \geq 5$, moduli space of Einstein metrics is often highly disconnected.

For example, the moduli space of Einstein metrics on S^5 has infinitely many connected components.

(Böhm, Collins-Székeleyhidi)

Recently generalized to all S^{2m+1} , $m \geq 2$.

(Liu-Sano-Tasin)

Dimension ≥ 5 :

On compact n -manifolds, $n \geq 5$, moduli space of Einstein metrics is often highly disconnected.

For example, the moduli space of Einstein metrics on S^5 has infinitely many connected components.

Dimension ≥ 5 :

On compact n -manifolds, $n \geq 5$, moduli space of Einstein metrics is often highly disconnected.

For example, the moduli space of Einstein metrics on S^5 has infinitely many connected components.

There are smooth compact 8-manifolds that admit pairs of Einstein metrics with opposite signs for λ .

Dimension ≥ 5 :

On compact n -manifolds, $n \geq 5$, moduli space of Einstein metrics is often highly disconnected.

For example, the moduli space of Einstein metrics on S^5 has infinitely many connected components.

There are smooth compact 8-manifolds that admit pairs of Einstein metrics with opposite signs for λ .

(Catanesi-LeBrun,)

Dimension ≥ 5 :

On compact n -manifolds, $n \geq 5$, moduli space of Einstein metrics is often highly disconnected.

For example, the moduli space of Einstein metrics on S^5 has infinitely many connected components.

There are smooth compact 8-manifolds that admit pairs of Einstein metrics with opposite signs for λ .

(Catanese-LeBrun, Rasdeaconu-Şuvaina)

Dimension ≥ 5 :

On compact n -manifolds, $n \geq 5$, moduli space of Einstein metrics is often highly disconnected.

For example, the moduli space of Einstein metrics on S^5 has infinitely many connected components.

There are smooth compact 8-manifolds that admit pairs of Einstein metrics with opposite signs for λ .

(Catanese-LeBrun, Rasdeaconu-Şuvaina)

Similarly in dimension $n = 4k$, $k \geq 2$.

Dimension ≥ 5 :

On compact n -manifolds, $n \geq 5$, moduli space of Einstein metrics is often highly disconnected.

Dimension ≥ 5 :

On compact n -manifolds, $n \geq 5$, moduli space of Einstein metrics is often highly disconnected.

In high dimensions, the Einstein condition allows for a surprising degree of flexibility!

Four Dimensions is Exceptional

Four Dimensions is Exceptional

Four Dimensions is Exceptional

“Goldilocks Zone” for Einstein metrics:

Four Dimensions is Exceptional

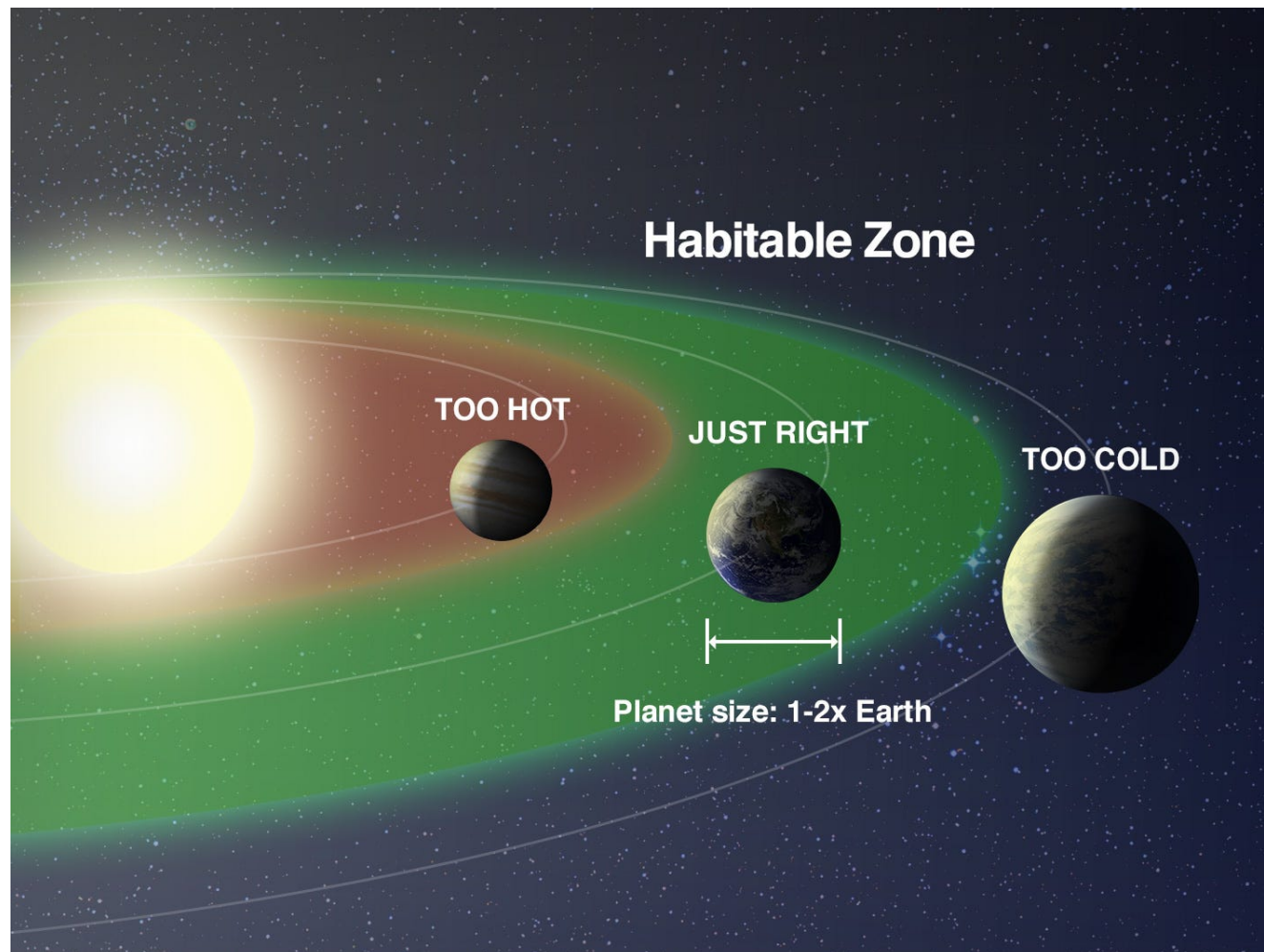
“Goldilocks Zone” for Einstein metrics:

Not too rigid. Not too flexible. Just right!

Four Dimensions is Exceptional

“Goldilocks Zone” for Einstein metrics:

Not too rigid. Not too flexible. Just right!



Dimension 4:

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus T^4 is flat.*

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus T^4 is flat.*

$\implies$ Moduli space of Einstein metrics is connected.

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus T^4 is flat.*

$\implies$ Moduli space of Einstein metrics is connected.

Theorem (Hitchin). *Any Einstein metric on $K3$ is Ricci-flat Kähler.*

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus T^4 is flat.*

$\implies$ Moduli space of Einstein metrics is connected.

Theorem (Hitchin). *Any Einstein metric on $K3$ is Ricci-flat Kähler.*



Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus T^4 is flat.*

$\implies$ Moduli space of Einstein metrics is connected.

Theorem (Hitchin). *Any Einstein metric on $K3$ is Ricci-flat Kähler.*

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus T^4 is flat.*

$\implies$ Moduli space of Einstein metrics is connected.

Theorem (Hitchin). *Any Einstein metric on $K3$ is Ricci-flat Kähler.*

$\implies$ Moduli space of Einstein metrics is connected.

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus T^4 is flat.*

$\implies$ Moduli space of Einstein metrics is connected.

Theorem (Hitchin). *Any Einstein metric on $K3$ is Ricci-flat Kähler.*

$\implies$ Moduli space of Einstein metrics is connected.
(Kodaira, Yau, Siu, Kobayashi-Todorov)

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus T^4 is flat.*

$\implies$ Moduli space of Einstein metrics is connected.

Theorem (Hitchin). *Any Einstein metric on $K3$ is Ricci-flat Kähler.*

$\implies$ Moduli space of Einstein metrics is connected.
(Kodaira, Yau, Siu, Kobayashi-Todorov)

Theorem (Besson-Courtois-Gallot). *There is only one Einstein metric on compact hyperbolic 4-manifold $\mathcal{H}^4/\Gamma$, up to scale and diffeos.*

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus T^4 is flat.*

$\implies$ Moduli space of Einstein metrics is connected.

Theorem (Hitchin). *Any Einstein metric on $K3$ is Ricci-flat Kähler.*

$\implies$ Moduli space of Einstein metrics is connected.
(Kodaira, Yau, Siu, Kobayashi-Todorov)

Theorem (Besson-Courtois-Gallot). *There is only one Einstein metric on compact hyperbolic 4-manifold $\mathcal{H}^4/\Gamma$, up to scale and diffeos.*

Theorem (LeBrun). *There is only one Einstein metric on compact complex-hyperbolic 4-manifold $\mathbb{C}\mathcal{H}_2/\Gamma$, up to scale and diffeos.*

What's so special about dimension 4?

What's so special about dimension 4?

The Lie group $SO(4)$ is *not simple*

What's so special about dimension 4?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

What's so special about dimension 4?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g) ,

What's so special about dimension 4?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g) , $\implies$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

What's so special about dimension 4?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g) , $\implies$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^\pm$ are (± 1) -eigenspaces of

$$\star : \Lambda^2 \rightarrow \Lambda^2,$$

What's so special about dimension 4?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g) , $\implies$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^\pm$ are (± 1) -eigenspaces of

$$\star : \Lambda^2 \rightarrow \Lambda^2,$$

$$\star^2 = 1.$$

Λ^+ self-dual 2-forms.

Λ^- anti-self-dual 2-forms.

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$$

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$$

splits into 4 irreducible pieces:

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$$

splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^+	$W_+ + \frac{s}{12}$	$\overset{\circ}{r}$
Λ^-	$\overset{\circ}{r}$	$W_- + \frac{s}{12}$

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$$

splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^+	$W_+ + \frac{s}{12}$	$\overset{\circ}{r}$
Λ^-	$\overset{\circ}{r}$	$W_- + \frac{s}{12}$

where

s = scalar curvature

$\overset{\circ}{r}$ = trace-free Ricci curvature

W_+ = self-dual Weyl curvature

W_- = anti-self-dual Weyl curvature

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$$

splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^+	$W_+ + \frac{s}{12}$	$\overset{\circ}{r}$
Λ^-	$\overset{\circ}{r}$	$W_- + \frac{s}{12}$

where

s = scalar curvature

$\overset{\circ}{r}$ = trace-free Ricci curvature

W_+ = self-dual Weyl curvature (*conformally invariant*)

W_- = anti-self-dual Weyl curvature //

Which 4-manifolds admit Einstein metrics?

Symplectic 4-manifolds:

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω .

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω .

$$d\omega = 0, \quad \lrcorner \omega : TM \xrightarrow{\cong} T^*M.$$

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω .

$$\omega = dx \wedge dy + dz \wedge dt$$

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω .

Seiberg-Witten theory provides powerful constraints on the differential topology of such manifolds.

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω .

Seiberg-Witten theory provides powerful constraints on the differential topology of such manifolds.

Some Suggestive Questions.

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω .

Seiberg-Witten theory provides powerful constraints on the differential topology of such manifolds.

Some Suggestive Questions. *If (M^4, ω) is a symplectic 4-manifold,*

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω .

Seiberg-Witten theory provides powerful constraints on the differential topology of such manifolds.

Some Suggestive Questions. *If (M^4, ω) is a symplectic 4-manifold, when does M^4 admit an Einstein metric g*

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω .

Seiberg-Witten theory provides powerful constraints on the differential topology of such manifolds.

Some Suggestive Questions. *If (M^4, ω) is a symplectic 4-manifold, when does M^4 admit an Einstein metric g (a priori unrelated to ω)?*

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω .

Seiberg-Witten theory provides powerful constraints on the differential topology of such manifolds.

Some Suggestive Questions. *If (M^4, ω) is a symplectic 4-manifold, when does M^4 admit an Einstein metric g (a priori unrelated to ω)? What if we also require $\lambda > 0$?*

Theorem (CLW '08). *Suppose that M is a smooth compact oriented 4-manifold which carries some symplectic form ω .*

Theorem (CLW '08). *Suppose that M is a smooth compact oriented 4-manifold which carries some symplectic form ω . Then M admits an (a priori unrelated) Einstein metric g with $\lambda > 0$*

Theorem (CLW '08). *Suppose that M is a smooth compact oriented 4-manifold which carries some symplectic form ω . Then M admits an (a priori unrelated) Einstein metric g with $\lambda > 0$*

$$\iff M \overset{\text{diff}}{\approx}$$

Theorem (CLW '08). *Suppose that M is a smooth compact oriented 4-manifold which carries some symplectic form ω . Then M admits an (a priori unrelated) Einstein metric g with $\lambda > 0$*

$$\iff M \overset{\text{diff}}{\approx} \left\{ \mathbb{CP}_2 \#^k \overline{\mathbb{CP}}_2, \right.$$

Theorem (CLW '08). *Suppose that M is a smooth compact oriented 4-manifold which carries some symplectic form ω . Then M admits an (a priori unrelated) Einstein metric g with $\lambda > 0$*

$$\iff M \overset{\text{diff}}{\approx} \left\{ \mathbb{CP}_2 \#^k \overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8, \right.$$

Theorem (CLW '08). *Suppose that M is a smooth compact oriented 4-manifold which carries some symplectic form ω . Then M admits an (a priori unrelated) Einstein metric g with $\lambda > 0$*

$$\iff M \overset{\text{diff}}{\approx} \begin{cases} \mathbb{CP}_2 \#^k \overline{\mathbb{CP}}_2, & 0 \leq k \leq 8, \\ \text{or} \\ S^2 \times S^2 \end{cases}$$

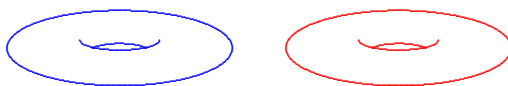
Conventions:

$\overline{\mathbb{CP}}_2$ = reverse oriented $\mathbb{CP}_2$.

Conventions:

$\overline{\mathbb{CP}}_2$ = reverse oriented $\mathbb{CP}_2$.

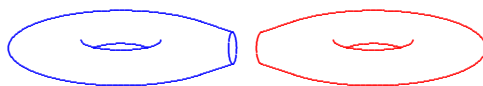
Connected sum #:



Conventions:

$\overline{\mathbb{CP}}_2$ = reverse oriented $\mathbb{CP}_2$.

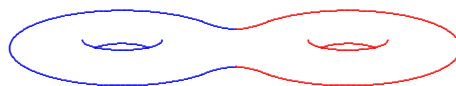
Connected sum #:



Conventions:

$\overline{\mathbb{CP}}_2$ = reverse oriented $\mathbb{CP}_2$.

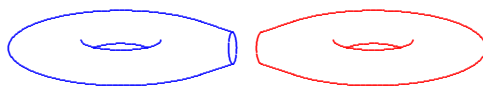
Connected sum #:



Conventions:

$\overline{\mathbb{CP}}_2$ = reverse oriented $\mathbb{CP}_2$.

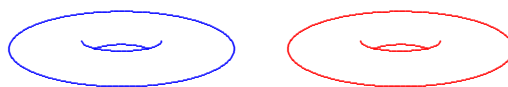
Connected sum #:



Conventions:

$\overline{\mathbb{CP}}_2$ = reverse oriented $\mathbb{CP}_2$.

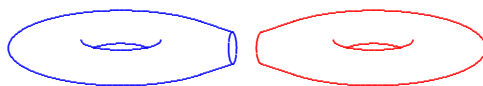
Connected sum #:



Conventions:

$\overline{\mathbb{CP}}_2$ = reverse oriented $\mathbb{CP}_2$.

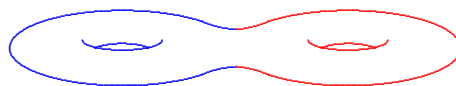
Connected sum #:



Conventions:

$\overline{\mathbb{CP}}_2$ = reverse oriented $\mathbb{CP}_2$.

Connected sum #:



Theorem (CLW '08). *Suppose that M is a smooth compact oriented 4-manifold which carries some symplectic form ω . Then M admits an (unrelated) Einstein metric g with $\lambda > 0$*

$$\iff M \approx \begin{cases} \mathbb{CP}_2 \#^k \overline{\mathbb{CP}}_2, & 0 \leq k \leq 8, \\ \text{or} \\ S^2 \times S^2 \end{cases}$$

Theorem (CLW '08). *Suppose that M is a smooth compact oriented 4-manifold which carries some symplectic form ω . Then M admits an (unrelated) Einstein metric g with $\lambda > 0$*

$$\iff M \approx \begin{cases} \mathbb{CP}_2 \#^k \overline{\mathbb{CP}}_2, & 0 \leq k \leq 8, \\ \text{or} \\ S^2 \times S^2 \end{cases}$$

Diffeotypes: exactly the Del Pezzo surfaces.

Del Pezzo surfaces:

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points,
in general position,

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position,

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

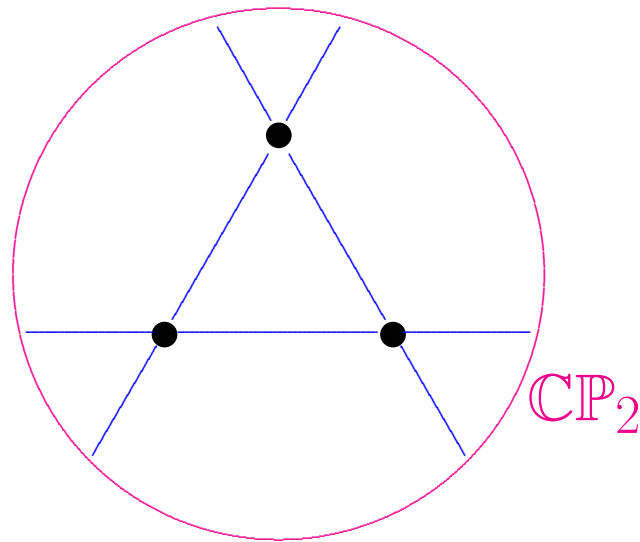
Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



Blowing up:

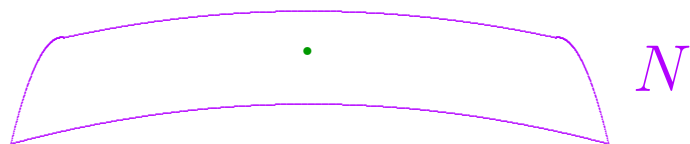
Blowing up:

If N is a complex surface,



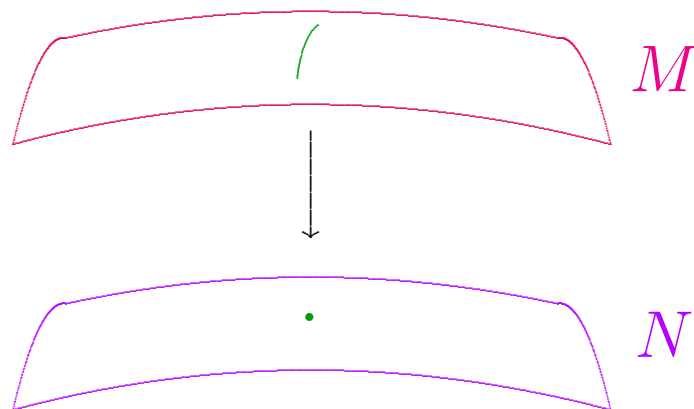
Blowing up:

If N is a complex surface, may replace $p \in N$



Blowing up:

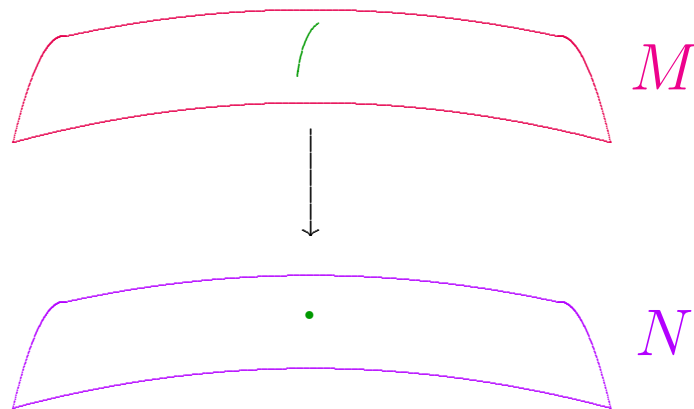
If N is a complex surface, may replace $p \in N$
with $\mathbb{CP}_1$



Blowing up:

If N is a complex surface, may replace $p \in N$
with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

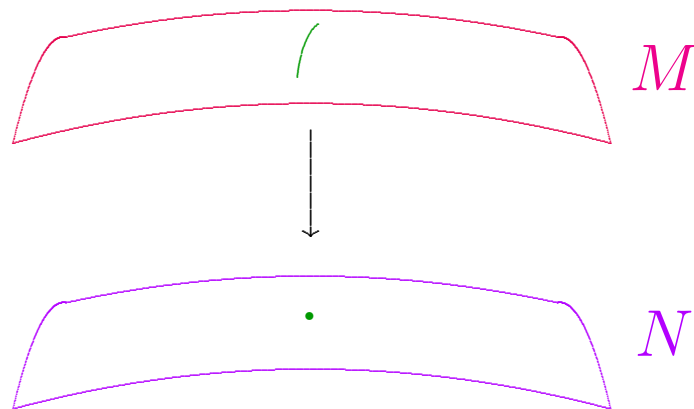


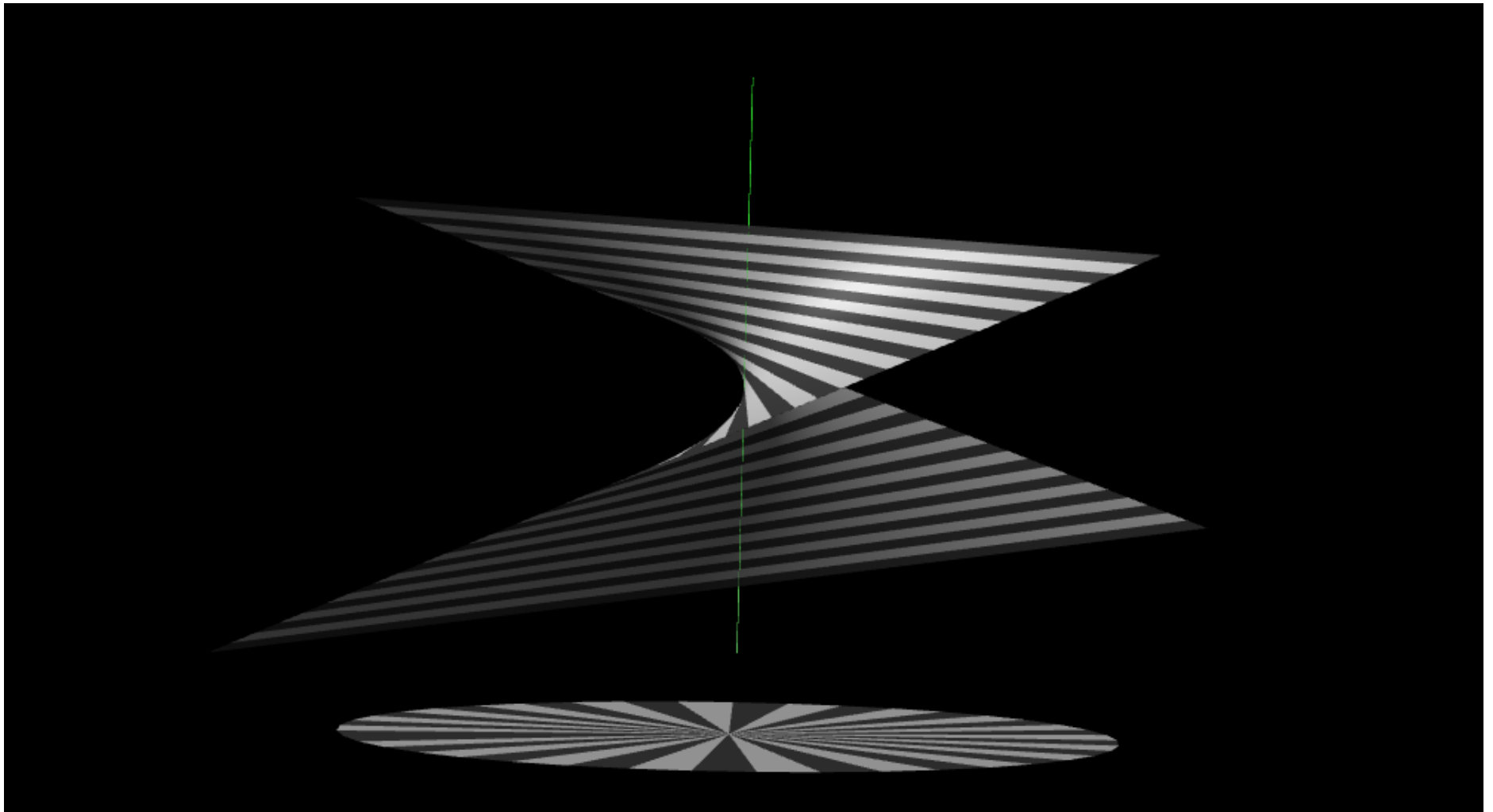
Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.



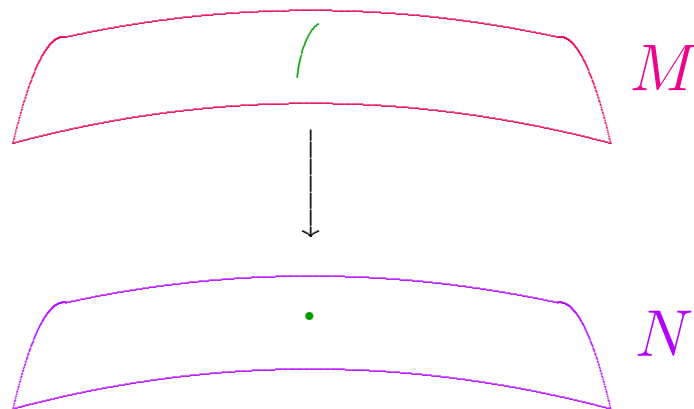


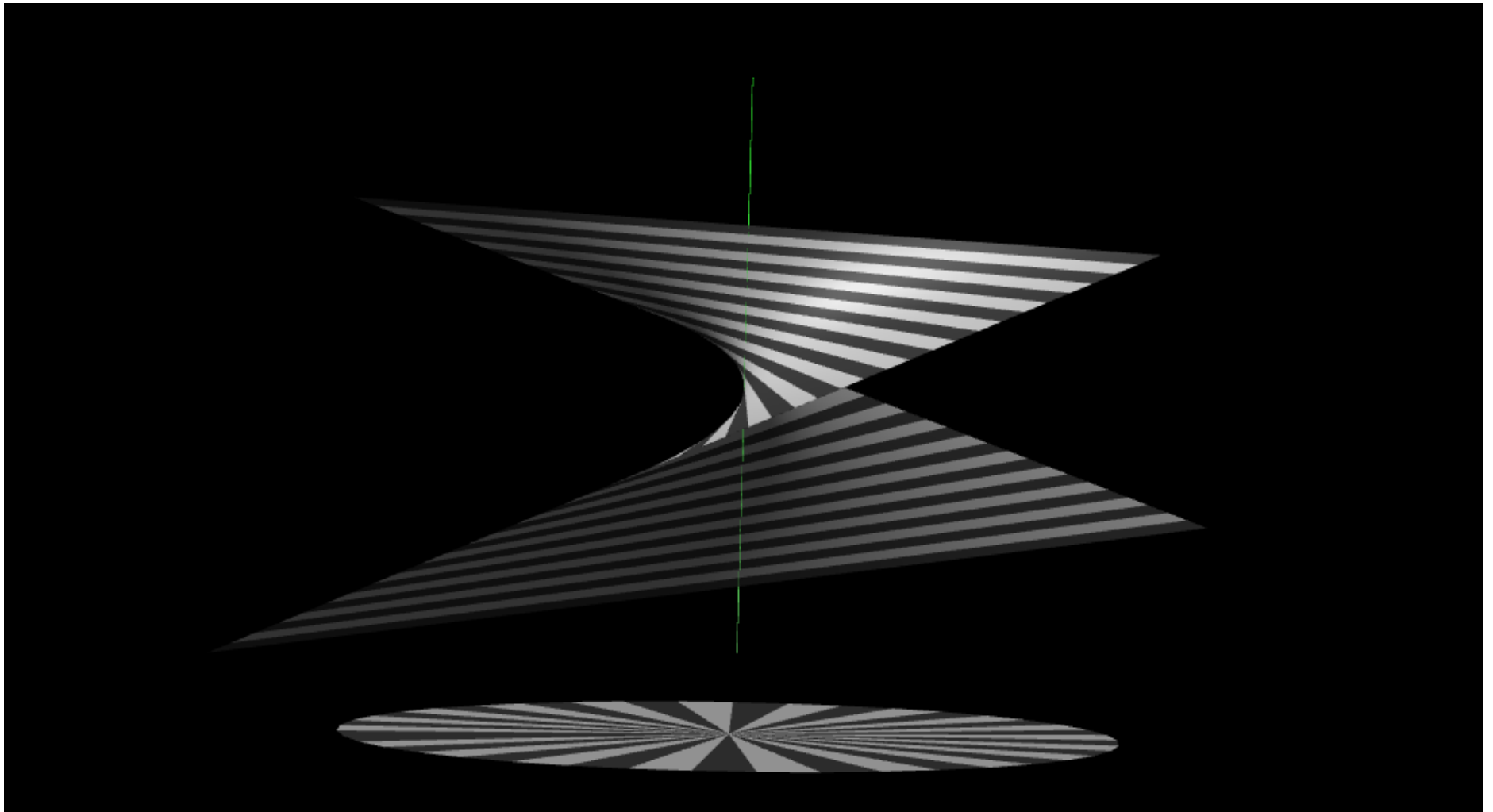
Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.



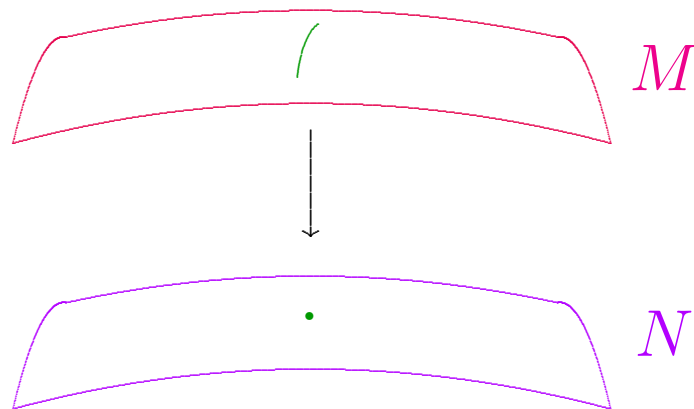


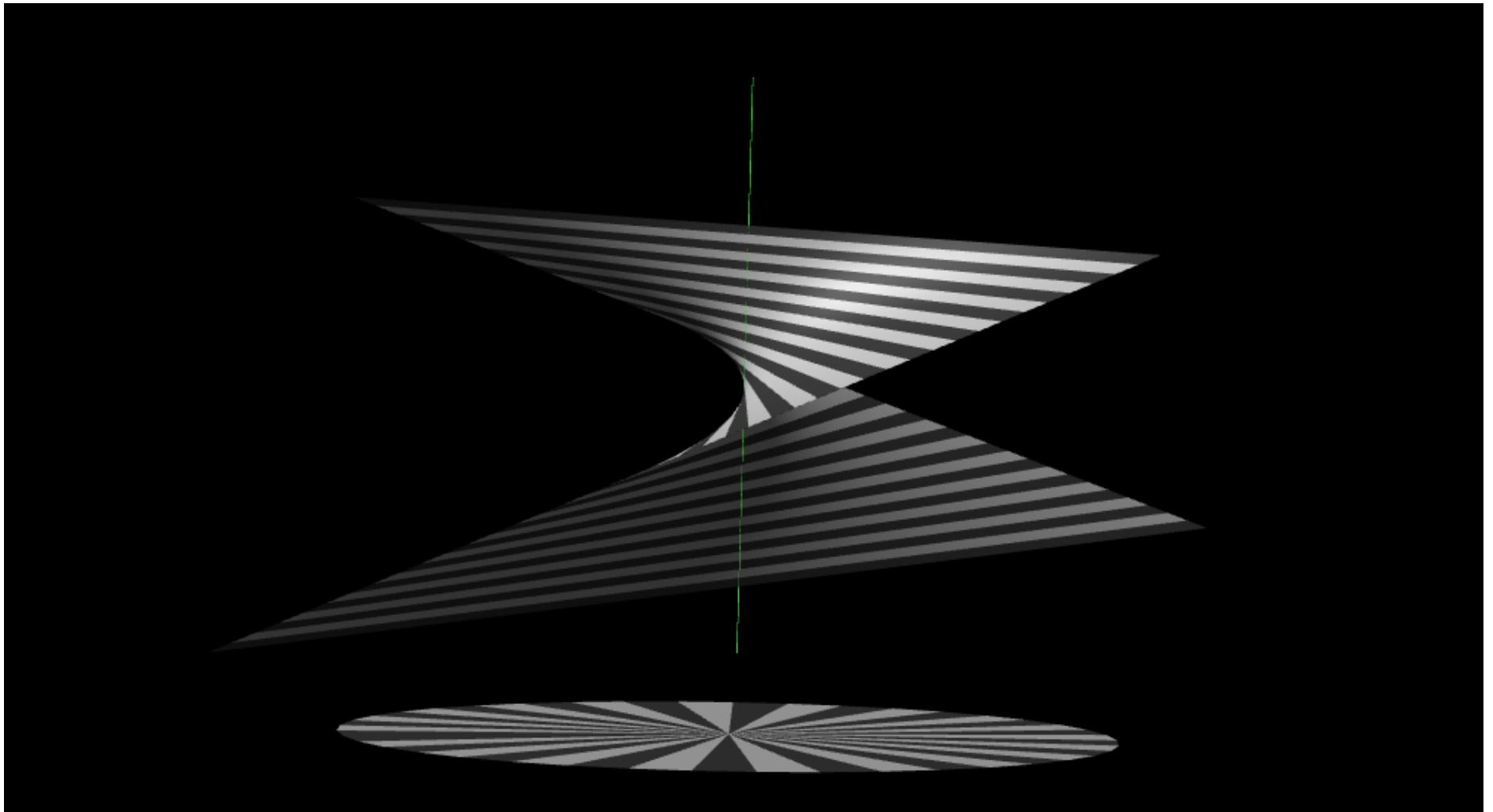
Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.



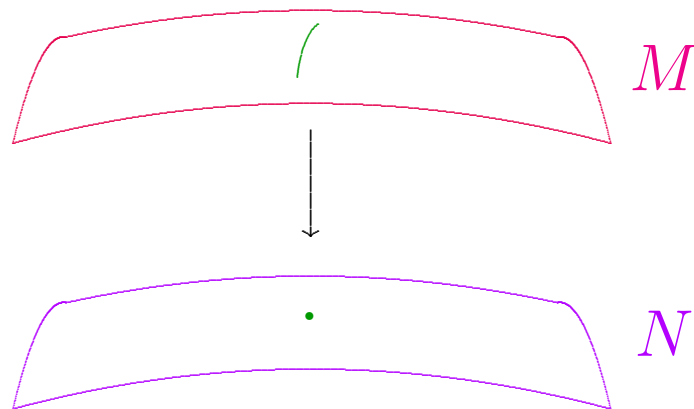


Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.

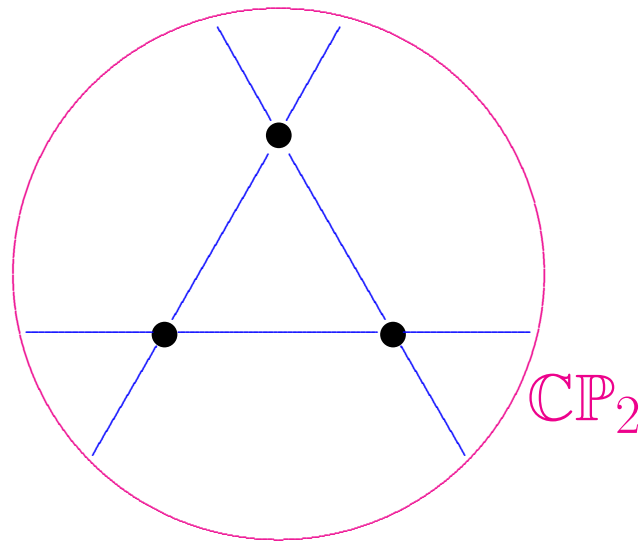


Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

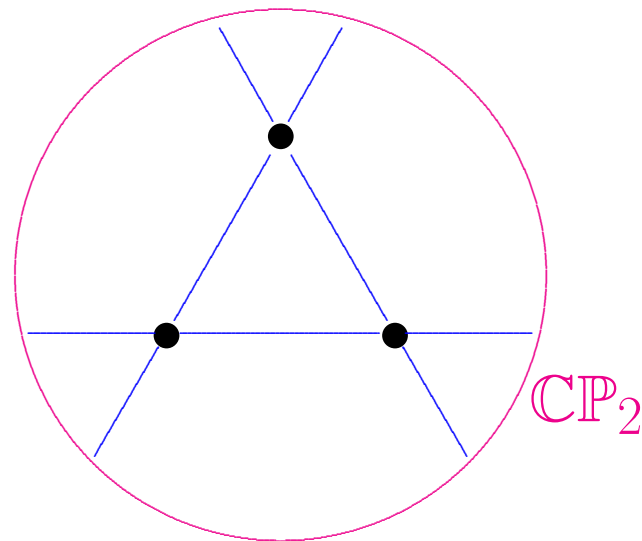


Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



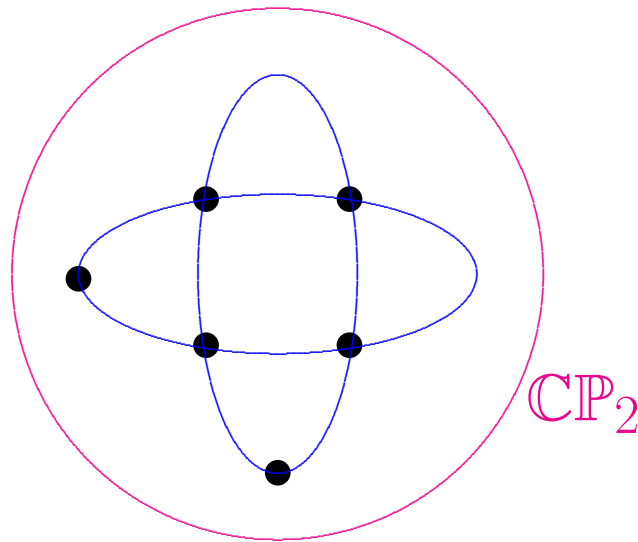
No 3 on a line,

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



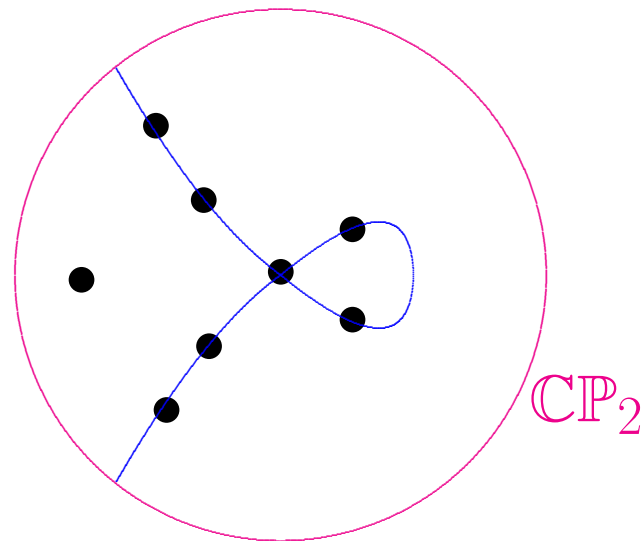
No 3 on a line, no 6 on conic,

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



No 3 on a line, no 6 on conic, no 8 on nodal cubic.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J -compatible*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J -compatible Hermitian,*

$$g = g(J\cdot, J\cdot)$$

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J -compatible Hermitian, Einstein metric,*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J -compatible Hermitian, Einstein metric, and this metric is unique*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J -compatible Hermitian, Einstein metric, and this metric is unique up to complex automorphisms and constant rescalings.*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J -compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J -compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

L '97: Such Einstein metrics are necessarily conformal to extremal Kähler metrics, in the sense of Calabi.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J -compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

L '97: Such Einstein metrics are necessarily conformal to extremal Kähler metrics, in the sense of Calabi. Most of them are actually Kähler-Einstein.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J -compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

L '97: Such Einstein metrics are necessarily conformal to extremal Kähler metrics, in the sense of Calabi. Most of them are actually Kähler-Einstein. There are exactly two exceptions.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J -compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J -compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

Existence: Page

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J -compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński,

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J -compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński, Siu,

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J -compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński, Siu, Tian-Yau,

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J -compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński, Siu, Tian-Yau, Tian,

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J -compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński, Siu, Tian-Yau, Tian,
Odaka-Spotti-Sun,

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J -compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński, Siu, Tian-Yau, Tian,
Odaka-Spotti-Sun, Chen-L-Weber.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J -compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

Uniqueness: Bando-Mabuchi '87

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J -compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

Uniqueness: Bando-Mabuchi '87, L '12.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J -compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

Gromov '81:

Gromov '81:

pre-compactness theorem for closed Riem' n -manifolds

Gromov '81:

pre-compactness theorem for closed Riem' n -manifolds
with bounds on sectional curvature, volume, diam.

Gromov '81:

pre-compactness theorem for closed Riem' n -manifolds
with bounds on sectional curvature, volume, diam.

Anderson '89, Bando-Kasue-Nakajima '89:

Gromov '81:

pre-compactness theorem for closed Riem' n -manifolds
with bounds on sectional curvature, volume, diam.

Anderson '89, Bando-Kasue-Nakajima '89:

replaced sectional curvature with Ricci control.

Gromov '81:

pre-compactness theorem for closed Riemann n -manifolds
with bounds on sectional curvature, volume, diam.

Anderson '89, Bando-Kasue-Nakajima '89:

replaced sectional curvature with Ricci control.

Best results were for Einstein (M^4, g) .

Gromov '81:

pre-compactness theorem for closed Riemann n -manifolds
with bounds on sectional curvature, volume, diam.

Anderson '89, Bando-Kasue-Nakajima '89:

replaced sectional curvature with Ricci control.

Best results were for Einstein (M^4, g) .

Theorem.

Gromov '81:

pre-compactness theorem for closed Riemann n -manifolds
with bounds on sectional curvature, volume, diam.

Anderson '89, Bando-Kasue-Nakajima '89:

replaced sectional curvature with Ricci control.

Best results were for Einstein (M^4, g) .

Theorem. *Let $\{(M_j, g_j)\}$*

Gromov '81:

pre-compactness theorem for closed Riem'n n -manifolds
with bounds on sectional curvature, volume, diam.

Anderson '89, Bando-Kasue-Nakajima '89:

replaced sectional curvature with Ricci control.

Best results were for Einstein (M^4, g) .

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt
connected Einstein 4-manifolds*

Gromov '81:

pre-compactness theorem for closed Riem'n n -manifolds
with bounds on sectional curvature, volume, diam.

Anderson '89, Bando-Kasue-Nakajima '89:

replaced sectional curvature with Ricci control.

Best results were for Einstein (M^4, g) .

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt
connected Einstein 4-manifolds with fixed Ein-
stein constant $\lambda > 0$,*

Gromov '81:

pre-compactness theorem for closed Riem'n n -manifolds
with bounds on sectional curvature, volume, diam.

Anderson '89, Bando-Kasue-Nakajima '89:

replaced sectional curvature with Ricci control.

Best results were for Einstein (M^4, g) .

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt
connected Einstein 4-manifolds with fixed Ein-
stein constant $\lambda > 0$, fixed Euler characteris-
tic χ ,*

Gromov '81:

pre-compactness theorem for closed Riem'n n -manifolds
with bounds on sectional curvature, volume, diam.

Anderson '89, Bando-Kasue-Nakajima '89:

replaced sectional curvature with Ricci control.

Best results were for Einstein (M^4, g) .

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $Vol(M_j, g_j)$ bounded away from zero.*

Gromov '81:

pre-compactness theorem for closed Riem'n n -manifolds
with bounds on sectional curvature, volume, diam.

Anderson '89, Bando-Kasue-Nakajima '89:

replaced sectional curvature with Ricci control.

Best results were for Einstein (M^4, g) .

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $Vol(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i})*

Gromov '81:

pre-compactness theorem for closed Riem'n n -manifolds
with bounds on sectional curvature, volume, diam.

Anderson '89, Bando-Kasue-Nakajima '89:

replaced sectional curvature with Ricci control.

Best results were for Einstein (M^4, g) .

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense*

Gromov '81:

pre-compactness theorem for closed Riemann n -manifolds with bounds on sectional curvature, volume, diam.

Anderson '89, Bando-Kasue-Nakajima '89:

replaced sectional curvature with Ricci control.

Best results were for Einstein (M^4, g) .

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞)*

Gromov '81:

pre-compactness theorem for closed Riemann n -manifolds with bounds on sectional curvature, volume, diam.

Anderson '89, Bando-Kasue-Nakajima '89:

replaced sectional curvature with Ricci control.

Best results were for Einstein (M^4, g) .

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞) with the same Einstein constant λ .*

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞) with the same Einstein constant λ .*

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞) with the same Einstein constant λ .*

- Mutually diffeomorphic $M_{j_i} \approx M^4$;

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞) with the same Einstein constant λ .*

- Mutually diffeomorphic $M_{j_i} \approx M^4$;
- X has only isolated singularities $\approx \mathbb{R}^4/\Gamma_\ell$;

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞) with the same Einstein constant λ .*

- Mutually diffeomorphic $M_{j_i} \approx M^4$;
- X has only isolated singularities $\approx \mathbb{R}^4/\Gamma_\ell$;
- Singular set $\mathfrak{S} \subset X$ is finite;

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞) with the same Einstein constant λ .*

- Mutually diffeomorphic $M_{j_i} \approx M^4$;
- X has only isolated singularities $\approx \mathbb{R}^4/\Gamma_\ell$;
- Singular set $\mathfrak{S} \subset X$ is finite;
- $(X - \mathfrak{S}) \approx$ open set $U \subset M$;

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞) with the same Einstein constant λ .*

- Mutually diffeomorphic $M_{j_i} \approx M^4$;
- X has only isolated singularities $\approx \mathbb{R}^4/\Gamma_\ell$;
- Singular set $\mathfrak{S} \subset X$ is finite;
- $(X - \mathfrak{S}) \approx$ open set $U \subset M$;
- Can arrange $(U, g_{j_i}) \rightarrow (X - \mathfrak{S}, g_\infty)$ smoothly;

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞) with the same Einstein constant λ .*

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞) with the same Einstein constant λ .*

- Each component of $M - U$ has diameter $\rightarrow 0$;

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞) with the same Einstein constant λ .*

- Each component of $M - U$ has diameter $\rightarrow 0$;
- Curvature blows up on each such component;

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞) with the same Einstein constant λ .*

- Each component of $M - U$ has diameter $\rightarrow 0$;
- Curvature blows up on each such component;
- But can rescale to keep bounded, and then take pointed Gromov-Hausdorff limits;

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞) with the same Einstein constant λ .*

- Each component of $M - U$ has diameter $\rightarrow 0$;
- Curvature blows up on each such component;
- But can rescale to keep bounded, and then take pointed Gromov-Hausdorff limits;
- These limits are Ricci-flat ALE orbifolds;

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞) with the same Einstein constant λ .*

- Each component of $M - U$ has diameter $\rightarrow 0$;
- Curvature blows up on each such component;
- But can rescale to keep bounded, and then take pointed Gromov-Hausdorff limits;
- These limits are Ricci-flat ALE orbifolds;

$$\sim \mathbb{R}^4 / \Gamma_\ell \text{ at infinity.}$$

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞) with the same Einstein constant λ .*

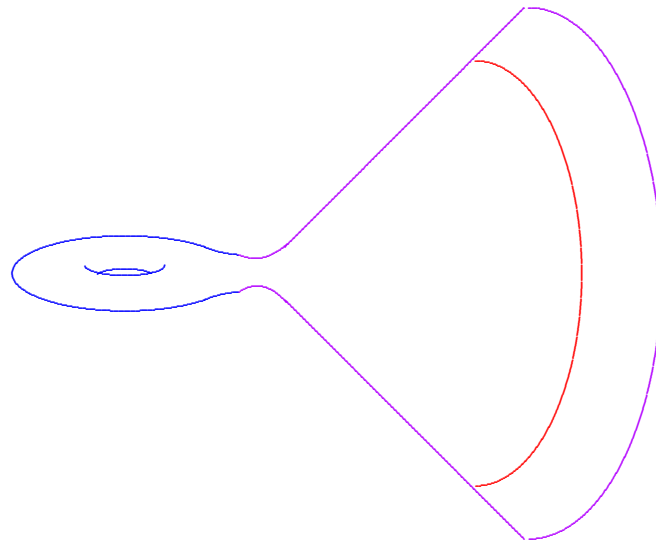
- Each component of $M - U$ has diameter $\rightarrow 0$;
- Curvature blows up on each such component;
- But can rescale to keep bounded, and then take pointed Gromov-Hausdorff limits;
- These limits are Ricci-flat ALE orbifolds;

$$g_{jk} = \delta_{jk} + O(|x|^{-4})$$

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞) with the same Einstein constant λ .*

- Each component of $M - U$ has diameter $\rightarrow 0$;
 - Curvature blows up on each such component;
 - But can rescale to keep bounded, and then take pointed Gromov-Hausdorff limits;
 - These limits are Ricci-flat ALE orbifolds;
- “Bubbling off gravitational instantons...”

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞) with the same Einstein constant λ .*



“Bubbling off gravitational instantons...”

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞) with the same Einstein constant λ .*

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞) with the same Einstein constant λ .*

- These “gravitational instantons” can have orbifold singularities;

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞) with the same Einstein constant λ .*

- These “gravitational instantons” can have orbifold singularities;
- But zooming in further then reveals a “deeper bubble” gravitational instanton;

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞) with the same Einstein constant λ .*

- These “gravitational instantons” can have orbifold singularities;
- But zooming in further then reveals a “deeper bubble” gravitational instanton;
- Terminates after finitely many steps:

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞) with the same Einstein constant λ .*

- These “gravitational instantons” can have orbifold singularities;
- But zooming in further then reveals a “deeper bubble” gravitational instanton;
- Terminates after finitely many steps:
- “Deepest bubbles” are smooth manifolds.

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞) with the same Einstein constant λ .*

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞) with the same Einstein constant λ .*

- One can reconstruct M from X and bubbles:

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞) with the same Einstein constant λ .*

- One can reconstruct M from X and bubbles:
- Cut out neighborhood each orbifold singularity;

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞) with the same Einstein constant λ .*

- One can reconstruct M from X and bubbles;
- Cut out neighborhood each orbifold singularity;
- Glue in corresponding gravitational instanton;

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞) with the same Einstein constant λ .*

- One can reconstruct M from X and bubbles;
- Cut out neighborhood each orbifold singularity;
- Glue in corresponding gravitational instanton;
- Iterate to eliminate all orbifold singularities.

Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ , and $\text{Vol}(M_j, g_j)$ bounded away from zero. Then there exists a subsequence (M_{j_i}, g_{j_i}) which converges in the Gromov-Hausdorff sense to cpt connected 4-dimensional Einstein orbifold (X, g_∞) with the same Einstein constant λ .*

We've now seen that certain compact Einstein orbifolds with $\lambda > 0$ are Gromov-Hausdorff limits of smooth compact Einstein 4-manifolds.

We've now seen that certain compact Einstein orbifolds with $\lambda > 0$ are Gromov-Hausdorff limits of smooth compact Einstein 4-manifolds.

Question. Does every compact Einstein orbifold with $\lambda > 0$ arise this way?

We've now seen that certain compact Einstein orbifolds with $\lambda > 0$ are Gromov-Hausdorff limits of smooth compact Einstein 4-manifolds.

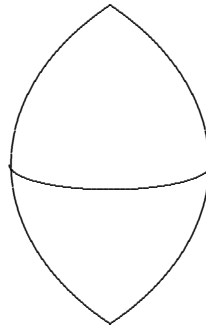
Question. Does every compact Einstein orbifold with $\lambda > 0$ arise this way?

No! Ozuch '22: $S^4/\mathbb{Z}_2$ is not such a limit!

We've now seen that certain compact Einstein orbifolds with $\lambda > 0$ are Gromov-Hausdorff limits of smooth compact Einstein 4-manifolds.

Question. Does every compact Einstein orbifold with $\lambda > 0$ arise this way?

No! Ozuch '22: $S^4/\mathbb{Z}_2$ is not such a limit!



$$S^4/\mathbb{Z}_2$$

We've now seen that certain compact Einstein orbifolds with $\lambda > 0$ are Gromov-Hausdorff limits of smooth compact Einstein 4-manifolds.

We've now seen that certain compact Einstein orbifolds with $\lambda > 0$ are Gromov-Hausdorff limits of smooth compact Einstein 4-manifolds.

Question. What about Kähler-Einstein orbifolds?

We've now seen that certain compact Einstein orbifolds with $\lambda > 0$ are Gromov-Hausdorff limits of smooth compact Einstein 4-manifolds.

Question. What about Kähler-Einstein orbifolds?

Odaka-Spotti-Sun '16: Classified the $\lambda > 0$ K-E orbifolds that are limits of smooth Kähler-Einstein manifolds.

We've now seen that certain compact Einstein orbifolds with $\lambda > 0$ are Gromov-Hausdorff limits of smooth compact Einstein 4-manifolds.

Question. What about Kähler-Einstein orbifolds?

Odaka-Spotti-Sun '16: Classified the $\lambda > 0$ K-E orbifolds that are limits of smooth Kähler-Einstein manifolds.

Most K-E orbifolds simply aren't such limits!

We've now seen that certain compact Einstein orbifolds with $\lambda > 0$ are Gromov-Hausdorff limits of smooth compact Einstein 4-manifolds.

Question. What about Kähler-Einstein orbifolds?

Odaka-Spotti-Sun '16: Classified the $\lambda > 0$ K-E orbifolds that are limits of smooth Kähler-Einstein manifolds.

Most K-E orbifolds simply aren't such limits!

But could these K-E orbifolds sometimes be limits of sequences of general Einstein manifolds?

We've now seen that certain compact Einstein orbifolds with $\lambda > 0$ are Gromov-Hausdorff limits of smooth compact Einstein 4-manifolds.

Question. What about Kähler-Einstein orbifolds?

Odaka-Spotti-Sun '16: Classified the $\lambda > 0$ K-E orbifolds that are limits of smooth Kähler-Einstein manifolds.

Most K-E orbifolds simply aren't such limits!

But could these K-E orbifolds sometimes be limits of sequences of general Einstein manifolds?

Goal: Show that this doesn't change anything!

Technical Hitch!

Technical Hitch! Whatever you may have been told,
ALE Ricci-flat 4-manifolds have not been classified!

Technical Hitch! Whatever you may have been told,
ALE Ricci-flat 4-manifolds have not been classified!

All known examples all have $W_+ = 0 \dots$

Technical Hitch! Whatever you may have been told,
ALE Ricci-flat 4-manifolds have not been classified!

All known examples all have $W_+ = 0 \dots$

Technical Hitch! Whatever you may have been told,
ALE Ricci-flat 4-manifolds have not been classified!

All known examples all have $W_+ = 0 \dots$

at least when correctly oriented!

Technical Hitch! Whatever you may have been told,
ALE Ricci-flat 4-manifolds have not been classified!

All known examples all have $W_+ = 0 \dots$

at least when correctly oriented!

Longstanding folk-conjecture: There are no others!

Technical Hitch! Whatever you may have been told,
ALE Ricci-flat 4-manifolds have not been classified!

All known examples all have $W_+ = 0 \dots$

at least when correctly oriented!

Longstanding folk-conjecture: There are no others!

We avoid this question by means of a definition!

Definition. Suppose that $(M, g_j) \rightarrow (X, g_\infty)$ in the Gromov-Hausdorff sense,

Definition. Suppose that $(M, g_j) \rightarrow (X, g_\infty)$ in the Gromov-Hausdorff sense, where the g_j

Definition. Suppose that $(M, g_j) \rightarrow (X, g_\infty)$ in the Gromov-Hausdorff sense, where the g_j are Einstein metrics of fixed $\lambda > 0$

Definition. Suppose that $(M, g_j) \rightarrow (X, g_\infty)$ in the Gromov-Hausdorff sense, where the g_j are Einstein metrics of fixed $\lambda > 0$ on a connected compact oriented M^4 ,

Definition. Suppose that $(M, g_j) \rightarrow (X, g_\infty)$ in the Gromov-Hausdorff sense, where the g_j are Einstein metrics of fixed $\lambda > 0$ on a connected compact oriented M^4 , and where X^4 is a compact orbifold with only isolated singularities.

Definition. Suppose that $(M, g_j) \rightarrow (X, g_\infty)$ in the Gromov-Hausdorff sense, where the g_j are Einstein metrics of fixed $\lambda > 0$ on a connected compact oriented M^4 , and where X^4 is a compact orbifold with only isolated singularities. We will then say that (M, g_j) is an **admissible sequence**,

Definition. Suppose that $(M, g_j) \rightarrow (X, g_\infty)$ in the Gromov-Hausdorff sense, where the g_j are Einstein metrics of fixed $\lambda > 0$ on a connected compact oriented M^4 , and where X^4 is a compact orbifold with only isolated singularities. We will then say that (M, g_j) is an **admissible sequence**, and that (X, g_∞) is an **admissible orbifold limit**,

Definition. Suppose that $(M, g_j) \rightarrow (X, g_\infty)$ in the Gromov-Hausdorff sense, where the g_j are Einstein metrics of fixed $\lambda > 0$ on a connected compact oriented M^4 , and where X^4 is a compact orbifold with only isolated singularities. We will then say that (M, g_j) is an **admissible sequence**, and that (X, g_∞) is an **admissible orbifold limit**, if every Ricci-flat ALE spaces (Y_i, h_i)

Definition. Suppose that $(M, g_j) \rightarrow (X, g_\infty)$ in the Gromov-Hausdorff sense, where the g_j are Einstein metrics of fixed $\lambda > 0$ on a connected compact oriented M^4 , and where X^4 is a compact orbifold with only isolated singularities. We will then say that (M, g_j) is an **admissible sequence**, and that (X, g_∞) is an **admissible orbifold limit**, if every Ricci-flat ALE spaces (Y_i, h_i) that bubbles off from the (M, g_j) satisfies $W_+ \equiv 0$

Definition. Suppose that $(M, g_j) \rightarrow (X, g_\infty)$ in the Gromov-Hausdorff sense, where the g_j are Einstein metrics of fixed $\lambda > 0$ on a connected compact oriented M^4 , and where X^4 is a compact orbifold with only isolated singularities. We will then say that (M, g_j) is an **admissible sequence**, and that (X, g_∞) is an **admissible orbifold limit**, if every Ricci-flat ALE spaces (Y_i, h_i) that bubbles off from the (M, g_j) satisfies $W_+ \equiv 0$ with respect to the induced orientation.

Proposition. *Any admissible orbifold limit (X, g_∞)*

Proposition. *Any admissible orbifold limit (X, g_∞) only has singularities of type T.*

Proposition. *Any admissible orbifold limit (X, g_∞) only has singularities of type T.*

Definition. *An isolated orbifold singularity of an oriented orbifold X^4*

Proposition. *Any admissible orbifold limit (X, g_∞) only has singularities of type T.*

Definition. *An isolated orbifold singularity of an oriented orbifold X^4 will be said to be of type T*

Proposition. *Any admissible orbifold limit (X, g_∞) only has singularities of type T .*

Definition. *An isolated orbifold singularity of an oriented orbifold X^4 will be said to be of type T if it has an open neighborhood modeled on $\mathbb{R}^4/\Gamma$*

Proposition. *Any admissible orbifold limit (X, g_∞) only has singularities of type T .*

Definition. *An isolated orbifold singularity of an oriented orbifold X^4 will be said to be of type T if it has an open neighborhood modeled on $\mathbb{R}^4/\Gamma$ for a finite subgroup $\Gamma < SO(4)$*

Proposition. *Any admissible orbifold limit (X, g_∞) only has singularities of type T .*

Definition. *An isolated orbifold singularity of an oriented orbifold X^4 will be said to be of type T if it has an open neighborhood modeled on $\mathbb{R}^4/\Gamma$ for a finite subgroup $\Gamma < SO(4)$ for which $\mathbb{R}^4/\Gamma$ is also*

Proposition. *Any admissible orbifold limit (X, g_∞) only has singularities of type T .*

Definition. *An isolated orbifold singularity of an oriented orbifold X^4 will be said to be of type T if it has an open neighborhood modeled on $\mathbb{R}^4/\Gamma$ for a finite subgroup $\Gamma < SO(4)$ for which $\mathbb{R}^4/\Gamma$ is also the tangent cone at infinity*

Proposition. *Any admissible orbifold limit (X, g_∞) only has singularities of type T .*

Definition. *An isolated orbifold singularity of an oriented orbifold X^4 will be said to be of type T if it has an open neighborhood modeled on $\mathbb{R}^4/\Gamma$ for a finite subgroup $\Gamma < SO(4)$ for which $\mathbb{R}^4/\Gamma$ is also the tangent cone at infinity of an anti-self-dual Ricci-flat ALE manifold (Y^4, h) .*

Proposition. *Any admissible orbifold limit (X, g_∞) only has singularities of type T .*

Definition. *An isolated orbifold singularity of an oriented orbifold X^4 will be said to be of type T if it has an open neighborhood modeled on $\mathbb{R}^4/\Gamma$ for a finite subgroup $\Gamma < SO(4)$ for which $\mathbb{R}^4/\Gamma$ is also the tangent cone at infinity of an anti-self-dual Ricci-flat ALE manifold (Y^4, h) .*

Subtle point: Gravitational instantons that bubble off could be orbifolds rather than manifolds!

Proposition. *Any admissible orbifold limit (X, g_∞) only has singularities of type T .*

Definition. *An isolated orbifold singularity of an oriented orbifold X^4 will be said to be of type T if it has an open neighborhood modeled on $\mathbb{R}^4/\Gamma$ for a finite subgroup $\Gamma < SO(4)$ for which $\mathbb{R}^4/\Gamma$ is also the tangent cone at infinity of an anti-self-dual Ricci-flat ALE manifold (Y^4, h) .*

Subtle point: Gravitational instantons that bubble off could be orbifolds rather than manifolds! But this has no effect on their tangent cones at infinity.

Proposition. *Any admissible orbifold limit (X, g_∞) only has singularities of type T .*

Definition. *An isolated orbifold singularity of an oriented orbifold X^4 will be said to be of type T if it has an open neighborhood modeled on $\mathbb{R}^4/\Gamma$ for a finite subgroup $\Gamma < SO(4)$ for which $\mathbb{R}^4/\Gamma$ is also the tangent cone at infinity of an anti-self-dual Ricci-flat ALE manifold (Y^4, h) .*

Proposition. *Any admissible orbifold limit (X, g_∞) only has singularities of type T .*

Definition. *An isolated orbifold singularity of an oriented orbifold X^4 will be said to be of type T if it has an open neighborhood modeled on $\mathbb{R}^4/\Gamma$ for a finite subgroup $\Gamma < SO(4)$ for which $\mathbb{R}^4/\Gamma$ is also the tangent cone at infinity of an anti-self-dual Ricci-flat ALE manifold (Y^4, h) .*

We can now recast this definition in a much more concrete and effective form

Proposition. *Any admissible orbifold limit (X, g_∞) only has singularities of type T .*

Definition. *An isolated orbifold singularity of an oriented orbifold X^4 will be said to be of type T if it has an open neighborhood modeled on $\mathbb{R}^4/\Gamma$ for a finite subgroup $\Gamma < SO(4)$ for which $\mathbb{R}^4/\Gamma$ is also the tangent cone at infinity of an anti-self-dual Ricci-flat ALE manifold (Y^4, h) .*

We can now recast this definition in a much more concrete and effective form by citing classification results of **Ioana Şuvaina** and **Evan Wright**.

Proposition. *An oriented singularity $\mathbb{R}^4/\Gamma$*

Proposition. *An oriented singularity $\mathbb{R}^4/\Gamma$ is of type T*

Proposition. *An oriented singularity $\mathbb{R}^4/\Gamma$ is of type T iff $\Gamma < SO(4)$ is conjugate to*

Proposition. *An oriented singularity $\mathbb{R}^4/\Gamma$ is of type T iff $\Gamma < SO(4)$ is conjugate to a subgroup of $U(2)$*

Proposition. *An oriented singularity $\mathbb{R}^4/\Gamma$ is of type T iff $\Gamma < SO(4)$ is conjugate to a subgroup of $U(2)$ of one of the two following types:*

Proposition. *An oriented singularity $\mathbb{R}^4/\Gamma$ is of type T iff $\Gamma < SO(4)$ is conjugate to a subgroup of $U(2)$ of one of the two following types:*

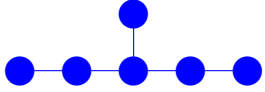
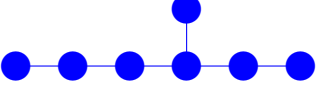
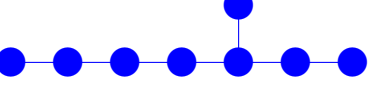
- *Finite subgroups $\Gamma < SU(2)$.*

Proposition. *An oriented singularity $\mathbb{R}^4/\Gamma$ is of type T iff $\Gamma < SO(4)$ is conjugate to a subgroup of $U(2)$ of one of the two following types:*

- *Finite subgroups $\Gamma < SU(2)$. These are classified by the simply-laced Dynkin diagrams:*

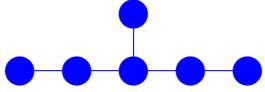
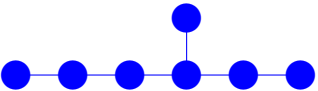
Proposition. An oriented singularity $\mathbb{R}^4/\Gamma$ is of type $\mathbf{T}$ iff $\Gamma < SO(4)$ is conjugate to a subgroup of $U(2)$ of one of the two following types:

- Finite subgroups $\Gamma < SU(2)$. These are classified by the *simply-laced Dynkin diagrams*:

<i>Dynkin Diagram</i>	$\Gamma < SU(2)$	<i>Classical Figure</i>
A_k 	<i>Cyclic</i>	
D_k 	<i>Binary Dihedral</i>	
E_6 	<i>Binary Tetrahedral</i>	
E_7 	<i>Binary Octahedral</i>	
E_8 	<i>Binary Icosahedral</i>	

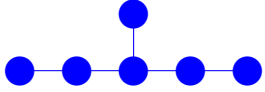
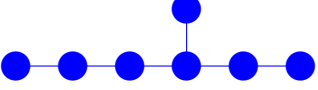
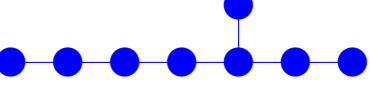
Proposition. *An oriented singularity $\mathbb{R}^4/\Gamma$ is of type T iff $\Gamma < SO(4)$ is conjugate to a subgroup of $U(2)$ of one of the two following types:*

- *Finite subgroups $\Gamma < SU(2)$. These are classified by the simply-laced Dynkin diagrams:*

<i>Dynkin Diagram</i>	$\Gamma < SU(2)$	<i>Classical Figure</i>
A_k 	$\mathbb{Z}_{k+1}$	
D_k 	Dih_{k-2}^*	
E_6 	<i>Binary Tetrahedral</i>	
E_7 	<i>Binary Octahedral</i>	
E_8 	<i>Binary Icosahedral</i>	

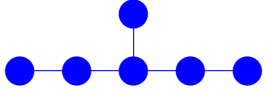
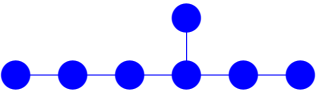
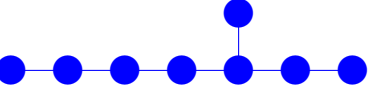
Proposition. An oriented singularity $\mathbb{R}^4/\Gamma$ is of type $\mathbf{T}$ iff $\Gamma < SO(4)$ is conjugate to a subgroup of $U(2)$ of one of the two following types:

- Finite subgroups $\Gamma < SU(2)$. These are classified by the *simply-laced Dynkin diagrams*:

<i>Dynkin Diagram</i>	$\Gamma < SU(2)$	<i>Classical Figure</i>
A_k 	<i>Cyclic</i>	
D_k 	<i>Binary Dihedral</i>	
E_6 	<i>Binary Tetrahedral</i>	
E_7 	<i>Binary Octahedral</i>	
E_8 	<i>Binary Icosahedral</i>	

Proposition. *An oriented singularity $\mathbb{R}^4/\Gamma$ is of type T iff $\Gamma < SO(4)$ is conjugate to a subgroup of $U(2)$ of one of the two following types:*

- *Finite subgroups $\Gamma < SU(2)$. These are classified by the simply-laced Dynkin diagrams:*

<i>Dynkin Diagram</i>	$\Gamma < SU(2)$	<i>Classical Figure</i>
A_k 	<i>Cyclic</i>	
D_k 	<i>Binary Dihedral</i>	
E_6 	<i>Binary Tetrahedral</i>	
E_7 	<i>Binary Octahedral</i>	
E_8 	<i>Binary Icosahedral</i>	

The corresponding gravitational instantons are exactly the hyper-Kähler ALE manifolds.

Proposition. *An oriented singularity $\mathbb{R}^4/\Gamma$ is of type T iff $\Gamma < SO(4)$ is conjugate to a subgroup of $U(2)$ of one of the two following types:*

- *Finite subgroups $\Gamma < SU(2)$. These are classified by the simply-laced Dynkin diagrams.*

Proposition. *An oriented singularity $\mathbb{R}^4/\Gamma$ is of type T iff $\Gamma < SO(4)$ is conjugate to a subgroup of $U(2)$ of one of the two following types:*

- *Finite subgroups $\Gamma < SU(2)$. These are classified by the simply-laced Dynkin diagrams.*
- *Cyclic groups $\mathbb{Z}_{\ell m^2} < U(2)$,*

Proposition. *An oriented singularity $\mathbb{R}^4/\Gamma$ is of type T iff $\Gamma < SO(4)$ is conjugate to a subgroup of $U(2)$ of one of the two following types:*

- *Finite subgroups $\Gamma < SU(2)$. These are classified by the simply-laced Dynkin diagrams.*
- *Cyclic groups $\mathbb{Z}_{\ell_{m^2}} < U(2)$, $m \geq 2$,*

Proposition. *An oriented singularity $\mathbb{R}^4/\Gamma$ is of type T iff $\Gamma < SO(4)$ is conjugate to a subgroup of $U(2)$ of one of the two following types:*

- *Finite subgroups $\Gamma < SU(2)$. These are classified by the simply-laced Dynkin diagrams.*
- *Cyclic groups $\mathbb{Z}_{\ell_{m^2}} < U(2)$, $m \geq 2$, gen'd by*

Proposition. *An oriented singularity $\mathbb{R}^4/\Gamma$ is of type T iff $\Gamma < SO(4)$ is conjugate to a subgroup of $U(2)$ of one of the two following types:*

- *Finite subgroups $\Gamma < SU(2)$. These are classified by the simply-laced Dynkin diagrams.*
- *Cyclic groups $\mathbb{Z}_{\ell_{m^2}} < U(2)$, $m \geq 2$, gen'd by*

Proposition. *An oriented singularity $\mathbb{R}^4/\Gamma$ is of type T iff $\Gamma < SO(4)$ is conjugate to a subgroup of $U(2)$ of one of the two following types:*

- *Finite subgroups $\Gamma < SU(2)$. These are classified by the *simply-laced Dynkin diagrams*.*
- *Cyclic groups $\mathbb{Z}_{\ell m^2} < U(2)$, $m \geq 2$, gen'd by*

$$\begin{bmatrix} \zeta & \\ & \zeta^{\ell mn-1} \end{bmatrix}$$

Proposition. *An oriented singularity $\mathbb{R}^4/\Gamma$ is of type T iff $\Gamma < SO(4)$ is conjugate to a subgroup of $U(2)$ of one of the two following types:*

- *Finite subgroups $\Gamma < SU(2)$. These are classified by the *simply-laced Dynkin diagrams*.*
- *Cyclic groups $\mathbb{Z}_{\ell m^2} < U(2)$, $m \geq 2$, gen'd by*

$$\begin{bmatrix} \zeta & \\ & \zeta^{\ell m n - 1} \end{bmatrix}$$

for $\zeta = e^{2\pi i/\ell m^2}$,

Proposition. *An oriented singularity $\mathbb{R}^4/\Gamma$ is of type T iff $\Gamma < SO(4)$ is conjugate to a subgroup of $U(2)$ of one of the two following types:*

- *Finite subgroups $\Gamma < SU(2)$. These are classified by the *simply-laced Dynkin diagrams*.*
- *Cyclic groups $\mathbb{Z}_{\ell m^2} < U(2)$, $m \geq 2$, gen'd by*

$$\begin{bmatrix} \zeta & \\ & \zeta^{\ell mn-1} \end{bmatrix}$$

for $\zeta = e^{2\pi i/\ell m^2}$, with $n < m$, $\gcd(m, n) = 1$.

Proposition. *An oriented singularity $\mathbb{R}^4/\Gamma$ is of type T iff $\Gamma < SO(4)$ is conjugate to a subgroup of $U(2)$ of one of the two following types:*

- *Finite subgroups $\Gamma < SU(2)$. These are classified by the *simply-laced Dynkin diagrams*.*
- *Cyclic groups $\mathbb{Z}_{\ell m^2} < U(2)$, $m \geq 2$, gen'd by*

$$\begin{bmatrix} \zeta & \\ & \zeta^{\ell mn-1} \end{bmatrix}$$

for $\zeta = e^{2\pi i/\ell m^2}$, with $n < m$, $\gcd(m, n) = 1$.

Standard shorthand: $\frac{1}{\ell m^2}(1, \ell mn - 1)$.

Proposition. *An oriented singularity $\mathbb{R}^4/\Gamma$ is of type T iff $\Gamma < SO(4)$ is conjugate to a subgroup of $U(2)$ of one of the two following types:*

- *Finite subgroups $\Gamma < SU(2)$. These are classified by the *simply-laced Dynkin diagrams*.*
- *Cyclic groups $\mathbb{Z}_{\ell m^2} < U(2)$, $m \geq 2$, gen'd by*

$$\begin{bmatrix} \zeta & \\ & \zeta^{\ell mn-1} \end{bmatrix}$$

for $\zeta = e^{2\pi i/\ell m^2}$, with $n < m$, $\gcd(m, n) = 1$.

Proposition. *An oriented singularity $\mathbb{R}^4/\Gamma$ is of type T iff $\Gamma < SO(4)$ is conjugate to a subgroup of $U(2)$ of one of the two following types:*

- *Finite subgroups $\Gamma < SU(2)$. These are classified by the simply-laced Dynkin diagrams.*
- *Cyclic groups $\mathbb{Z}_{\ell m^2} < U(2)$, $m \geq 2$, gen'd by*

$$\begin{bmatrix} \zeta & \\ & \zeta^{\ell mn-1} \end{bmatrix}$$

for $\zeta = e^{2\pi i/\ell m^2}$, with $n < m$, $\gcd(m, n) = 1$.

This second possibility arises in connection with gravitational instantons that are Ricci-flat Kähler, but not simply-connected.

Proposition. *An oriented singularity $\mathbb{R}^4/\Gamma$ is of type T iff $\Gamma < SO(4)$ is conjugate to a subgroup of $U(2)$ of one of the two following types:*

- *Finite subgroups $\Gamma < SU(2)$. These are classified by the simply-laced Dynkin diagrams.*
- *Cyclic groups $\mathbb{Z}_{\ell m^2} < U(2)$, $m \geq 2$, gen'd by*

$$\begin{bmatrix} \zeta & \\ & \zeta^{\ell mn-1} \end{bmatrix}$$

for $\zeta = e^{2\pi i/\ell m^2}$, with $n < m$, $\gcd(m, n) = 1$.

This second possibility arises in connection with gravitational instantons that are Ricci-flat Kähler, but not simply-connected. These are all free $\mathbb{Z}_m$ -quotients of $A_{\ell m-1}$ hyper-Kähler instantons.

Proposition. *An oriented singularity $\mathbb{R}^4/\Gamma$ is of type T iff $\Gamma < SO(4)$ is conjugate to a subgroup of $U(2)$ of one of the two following types:*

- *Finite subgroups $\Gamma < SU(2)$. These are classified by the *simply-laced Dynkin diagrams*.*
- *Cyclic groups $\mathbb{Z}_{\ell m^2} < U(2)$, $m \geq 2$, gen'd by*

$$\begin{bmatrix} \zeta & \\ & \zeta^{\ell mn-1} \end{bmatrix}$$

for $\zeta = e^{2\pi i/\ell m^2}$, with $n < m$, $\gcd(m, n) = 1$.

Proposition. *An oriented singularity $\mathbb{R}^4/\Gamma$ is of type T iff $\Gamma < SO(4)$ is conjugate to a subgroup of $U(2)$ of one of the two following types:*

- *Finite subgroups $\Gamma < SU(2)$. These are classified by the simply-laced Dynkin diagrams.*
- *Cyclic groups $\mathbb{Z}_{\ell m^2} < U(2)$, $m \geq 2$, gen'd by*

$$\begin{bmatrix} \zeta & \\ & \zeta^{\ell mn-1} \end{bmatrix}$$

for $\zeta = e^{2\pi i/\ell m^2}$, with $n < m$, $\gcd(m, n) = 1$.

The corresponding singularities $\mathbb{C}^2/\Gamma$ are exactly the ones that algebraic geometers term singularities of class T.

Proposition. *An oriented singularity $\mathbb{R}^4/\Gamma$ is of type **T** iff $\Gamma < SO(4)$ is conjugate to a subgroup of $U(2)$ of one of the two following types:*

- *Finite subgroups $\Gamma < SU(2)$. These are classified by the *simply-laced Dynkin diagrams*.*
- *Cyclic groups $\mathbb{Z}_{\ell m^2} < U(2)$, $m \geq 2$, gen'd by*

$$\begin{bmatrix} \zeta & \\ & \zeta^{\ell mn-1} \end{bmatrix}$$

for $\zeta = e^{2\pi i/\ell m^2}$, with $n < m$, $\gcd(m, n) = 1$.

The corresponding singularities $\mathbb{C}^2/\Gamma$ are exactly the ones that algebraic geometers term singularities of **class T**. For them, the defining property is that they have $\mathbb{Q}$ -Gorenstein local smoothings...

Example. $\mathbb{CP}_2/\mathbb{Z}_p$, where the $\mathbb{Z}_p$ action is generated by

Example. $\mathbb{CP}_2/\mathbb{Z}_p$, where the $\mathbb{Z}_p$ action is generated by

$$\begin{bmatrix} 1 & & \\ & e^{2\pi i/p} & \\ & & e^{-2\pi i/p} \end{bmatrix}.$$

Example. $\mathbb{CP}_2/\mathbb{Z}_p$, where the $\mathbb{Z}_p$ action is generated by

$$\begin{bmatrix} 1 & & \\ & e^{2\pi i/p} & \\ & & e^{-2\pi i/p} \end{bmatrix}.$$

If $p \geq 5$ is a prime, this is K-E orbifold with only isolated singularities which cannot be a limit of admissible type.

Example. $\mathbb{CP}_2/\mathbb{Z}_p$, where the $\mathbb{Z}_p$ action is generated by

$$\begin{bmatrix} 1 & & \\ & e^{2\pi i/p} & \\ & & e^{-2\pi i/p} \end{bmatrix}.$$

If $p \geq 5$ is a prime, this is K-E orbifold with only isolated singularities which cannot be a limit of admissible type. In fact, even forbidden if we allow reverse-oriented instantons!

Example. $\mathbb{CP}_2/\mathbb{Z}_p$, where the $\mathbb{Z}_p$ action is generated by

$$\begin{bmatrix} 1 & & \\ & e^{2\pi i/p} & \\ & & e^{-2\pi i/p} \end{bmatrix}.$$

If $p \geq 5$ is a prime, this is $K-E$ orbifold with only isolated singularities which cannot be a limit of admissible type. In fact, even forbidden if we allow reverse-oriented instantons!

Proposition. *Either there are infinitely many topological types of compact $K-E$ 4-dimensional orbifolds with only isolated singularities that cannot arise as Gromov-Hausdorff limits of smooth Einstein manifolds,*

Example. $\mathbb{CP}_2/\mathbb{Z}_p$, where the $\mathbb{Z}_p$ action is generated by

$$\begin{bmatrix} 1 & & \\ & e^{2\pi i/p} & \\ & & e^{-2\pi i/p} \end{bmatrix}.$$

If $p \geq 5$ is a prime, this is **K-E** orbifold with only isolated singularities which cannot be a limit of admissible type. In fact, even forbidden if we allow reverse-oriented instantons!

Proposition. *Either there are infinitely many topological types of compact **K-E** 4-dimensional orbifolds with only isolated singularities that cannot arise as Gromov-Hausdorff limits of smooth Einstein manifolds, or else there are infinitely many Ricci-flat ALE 4-manifolds that remain to be discovered.*

Example. $\mathbb{CP}_2/\mathbb{Z}_p$, where the $\mathbb{Z}_p$ action is generated by

$$\begin{bmatrix} 1 & & \\ & e^{2\pi i/p} & \\ & & e^{-2\pi i/p} \end{bmatrix}.$$

If $p \geq 5$ is a prime, this is K-E orbifold with only isolated singularities which cannot be a limit of admissible type. In fact, even forbidden if we allow reverse-oriented instantons!

Example. $\mathbb{CP}_2/\mathbb{Z}_p$, where the $\mathbb{Z}_p$ action is generated by

$$\begin{bmatrix} 1 & & \\ & e^{2\pi i/p} & \\ & & e^{-2\pi i/p} \end{bmatrix}.$$

If $p \geq 5$ is a prime, this is K-E orbifold with only isolated singularities which cannot be a limit of admissible type. In fact, even forbidden if we allow reverse-oriented instantons!

By contrast, if $p = 3$, then it is actually a limit of K-E metrics on $\mathbb{CP}_2 \# 6\overline{\mathbb{CP}}_2$,

Example. $\mathbb{CP}_2/\mathbb{Z}_p$, where the $\mathbb{Z}_p$ action is generated by

$$\begin{bmatrix} 1 & & \\ & e^{2\pi i/p} & \\ & & e^{-2\pi i/p} \end{bmatrix}.$$

If $p \geq 5$ is a prime, this is K-E orbifold with only isolated singularities which cannot be a limit of admissible type. In fact, even forbidden if we allow reverse-oriented instantons!

By contrast, if $p = 3$, then it is actually a limit of K-E metrics on $\mathbb{CP}_2 \# 6\overline{\mathbb{CP}_2}$, where three A_2 instantons bubble off in the limit.

Example. $\mathbb{CP}_2/\mathbb{Z}_p$, where the $\mathbb{Z}_p$ action is generated by

$$\begin{bmatrix} 1 & & \\ & e^{2\pi i/p} & \\ & & e^{-2\pi i/p} \end{bmatrix}.$$

If $p \geq 5$ is a prime, this is K-E orbifold with only isolated singularities which cannot be a limit of admissible type. In fact, even forbidden if we allow reverse-oriented instantons!

By contrast, if $p = 3$, then it is actually a limit of K-E metrics on $\mathbb{CP}_2 \# 6\overline{\mathbb{CP}_2}$, where three A_2 instantons bubble off in the limit.

These examples form part of a systematic picture...

Odaka-Spotti-Sun:

Odaka-Spotti-Sun:

Classified Gromov-Hausdorff limits of Del Pezzos.

Odaka-Spotti-Sun:

Classified Gromov-Hausdorff limits of Del Pezzos.

Of course, one only gets non-trivial limits for

Odaka-Spotti-Sun:

Classified Gromov-Hausdorff limits of Del Pezzos.

Of course, one only gets non-trivial limits for

$$\mathbb{CP}_2 \#^k \overline{\mathbb{CP}_2},$$

Odaka-Spotti-Sun:

Classified Gromov-Hausdorff limits of Del Pezzos.

Of course, one only gets non-trivial limits for

$$\mathbb{CP}_2 \#^k \overline{\mathbb{CP}_2}, \quad 5 \leq k \leq 8,$$

Odaka-Spotti-Sun:

Classified Gromov-Hausdorff limits of Del Pezzos.

Of course, one only gets non-trivial limits for

$$\mathbb{CP}_2 \#^k \overline{\mathbb{CP}_2}, \quad 5 \leq k \leq 8,$$

because the others are rigid.

Odaka-Spotti-Sun:

Classified Gromov-Hausdorff limits of Del Pezzos.

Of course, one only gets non-trivial limits for

$$\mathbb{CP}_2 \#^k \overline{\mathbb{CP}_2}, \quad 5 \leq k \leq 8.$$

Odaka-Spotti-Sun:

Classified Gromov-Hausdorff limits of Del Pezzos.

Of course, one only gets non-trivial limits for

$$\mathbb{CP}_2 \#^k \overline{\mathbb{CP}_2}, \quad 5 \leq k \leq 8.$$

- $\mathbb{CP}_2 \#^5 \overline{\mathbb{CP}_2}$: only A_1 singularities arise.

Odaka-Spotti-Sun:

Classified Gromov-Hausdorff limits of Del Pezzos.

Of course, one only gets non-trivial limits for

$$\mathbb{CP}_2 \#^k \overline{\mathbb{CP}_2}, \quad 5 \leq k \leq 8.$$

- $\mathbb{CP}_2 \#^5 \overline{\mathbb{CP}_2}$: only A_1 singularities arise.
- $\mathbb{CP}_2 \#^6 \overline{\mathbb{CP}_2}$: only A_1 or A_2 singularities arise.

Odaka-Spotti-Sun:

Classified Gromov-Hausdorff limits of Del Pezzos.

Of course, one only gets non-trivial limits for

$$\mathbb{CP}_2 \#^k \overline{\mathbb{CP}_2}, \quad 5 \leq k \leq 8.$$

- $\mathbb{CP}_2 \#^5 \overline{\mathbb{CP}_2}$: only A_1 singularities arise.
- $\mathbb{CP}_2 \#^6 \overline{\mathbb{CP}_2}$: only A_1 or A_2 singularities arise.
- $\mathbb{CP}_2 \#^7 \overline{\mathbb{CP}_2}$: only $A_1, \dots, A_4$ or $\frac{1}{4}(1, 1)$ arise.

Odaka-Spotti-Sun:

Classified Gromov-Hausdorff limits of Del Pezzos.

Of course, one only gets non-trivial limits for

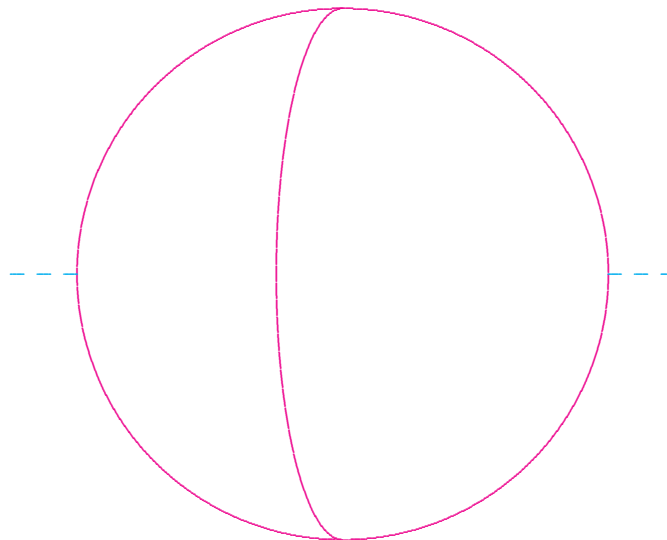
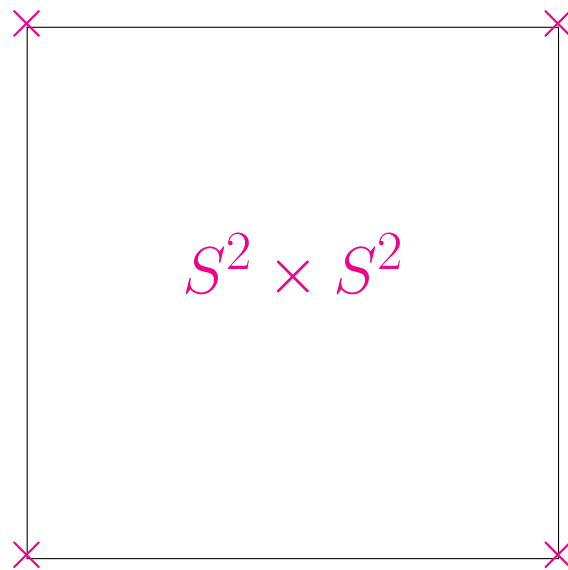
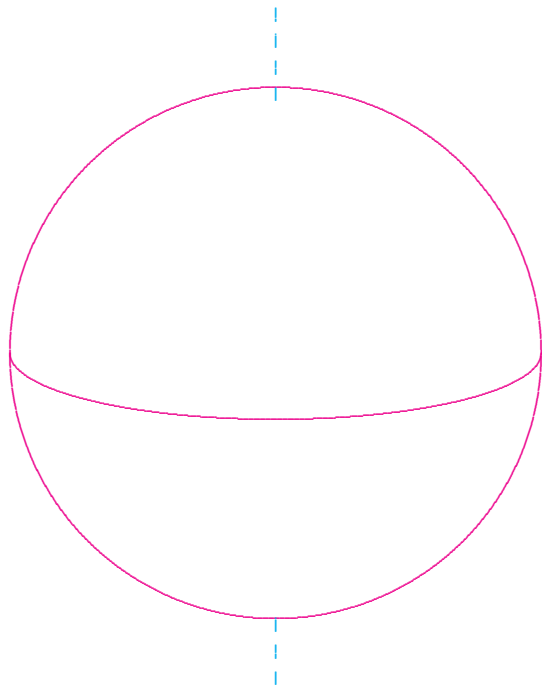
$$\mathbb{CP}_2 \#^k \overline{\mathbb{CP}}_2, \quad 5 \leq k \leq 8.$$

- $\mathbb{CP}_2 \#^5 \overline{\mathbb{CP}}_2$: only A_1 singularities arise.
- $\mathbb{CP}_2 \#^6 \overline{\mathbb{CP}}_2$: only A_1 or A_2 singularities arise.
- $\mathbb{CP}_2 \#^7 \overline{\mathbb{CP}}_2$: only $A_1, \dots, A_4$ or $\frac{1}{4}(1, 1)$ arise.
- $\mathbb{CP}_2 \#^8 \overline{\mathbb{CP}}_2$: only (!) $A_1, \dots, A_{10}, D_4, \frac{1}{4}(1, 1), \frac{1}{8}(1, 3),$ or $\frac{1}{9}(1, 2)$ singularities arise.

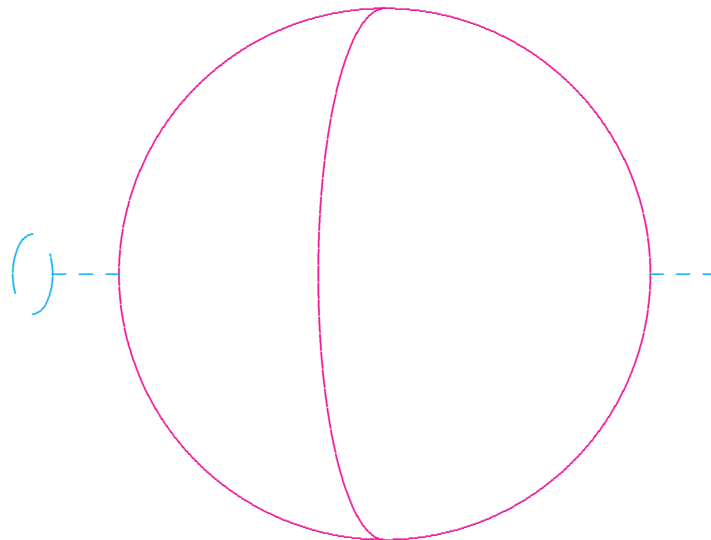
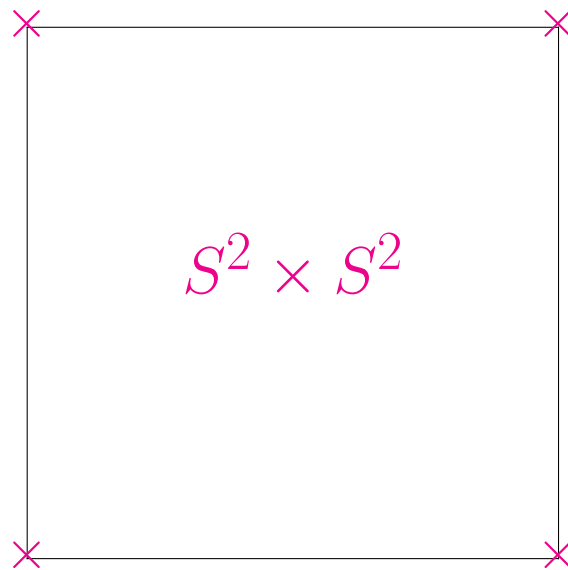
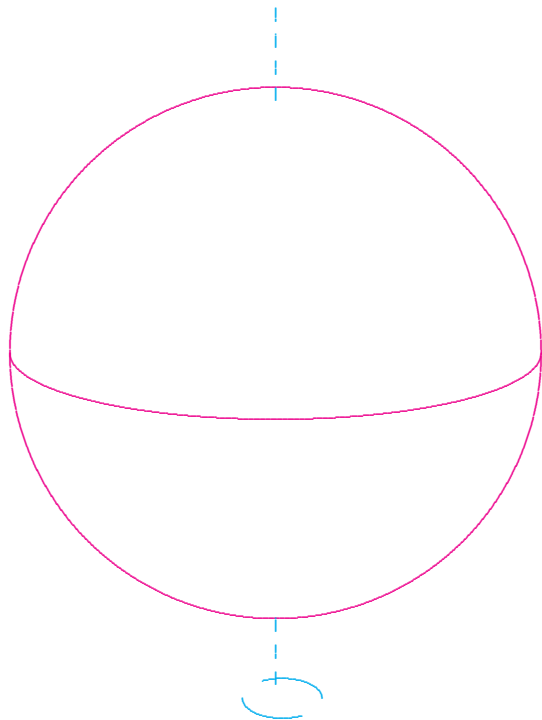
Example.

Example. $(S^2 \times S^2)/\mathbb{Z}_2$

Example. $(S^2 \times S^2)/\mathbb{Z}_2$



Example. $(S^2 \times S^2)/\mathbb{Z}_2$



Example. $(S^2 \times S^2)/\mathbb{Z}_2$

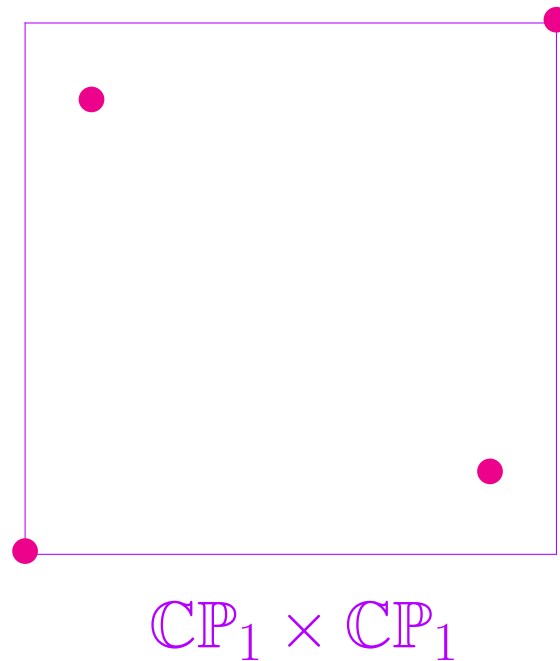
is limit of sequence of smooth Kähler-Einstein on

$$\mathbb{CP}_2 \# 5 \overline{\mathbb{CP}}_2 = (\mathbb{CP}_1 \times \mathbb{CP}_1) \# 4 \overline{\mathbb{CP}}_2$$

Example. $(S^2 \times S^2)/\mathbb{Z}_2$

is limit of sequence of smooth Kähler-Einstein on

$$\mathbb{CP}_2 \# 5\overline{\mathbb{CP}}_2 = (\mathbb{CP}_1 \times \mathbb{CP}_1) \# 4\overline{\mathbb{CP}}_2$$



Example. $(S^2 \times S^2)/\mathbb{Z}_2$

is limit of sequence of smooth Kähler-Einstein on

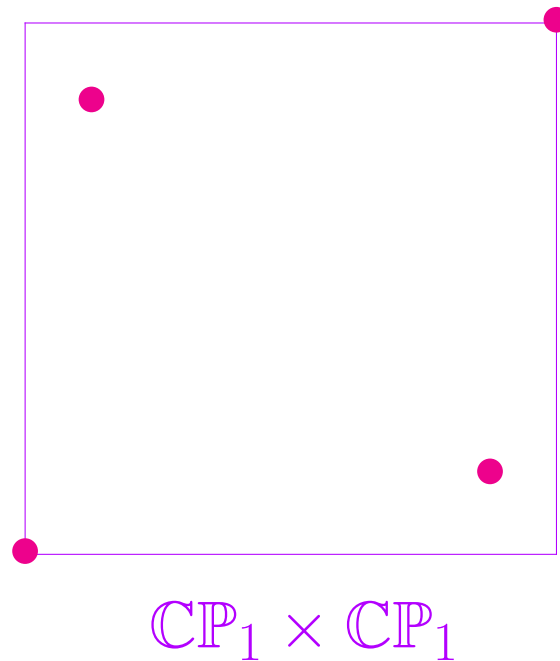
$$\mathbb{CP}_2 \# 5\overline{\mathbb{CP}}_2 = (\mathbb{CP}_1 \times \mathbb{CP}_1) \# 4\overline{\mathbb{CP}}_2$$



Example. $(S^2 \times S^2)/\mathbb{Z}_2$

is limit of sequence of smooth Kähler-Einstein on

$$\mathbb{CP}_2 \# 5\overline{\mathbb{CP}}_2 = (\mathbb{CP}_1 \times \mathbb{CP}_1) \# 4\overline{\mathbb{CP}}_2$$



Example. $(S^2 \times S^2)/\mathbb{Z}_2$

is limit of sequence of smooth Kähler-Einstein on

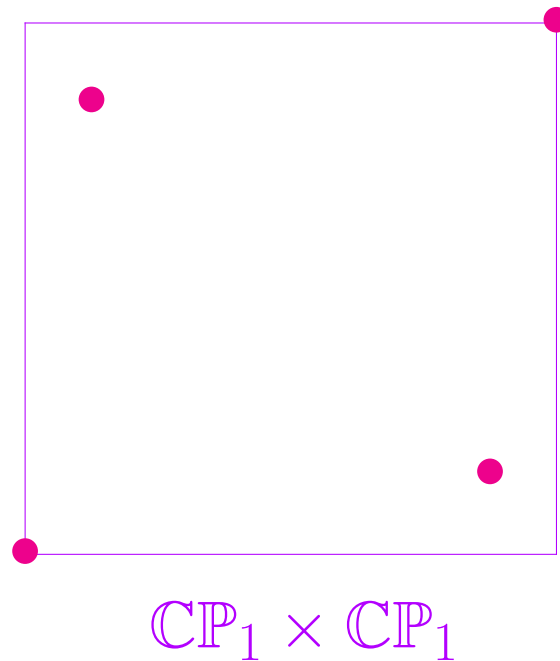
$$\mathbb{CP}_2 \# 5\overline{\mathbb{CP}}_2 = (\mathbb{CP}_1 \times \mathbb{CP}_1) \# 4\overline{\mathbb{CP}}_2$$



Example. $(S^2 \times S^2)/\mathbb{Z}_2$

is limit of sequence of smooth Kähler-Einstein on

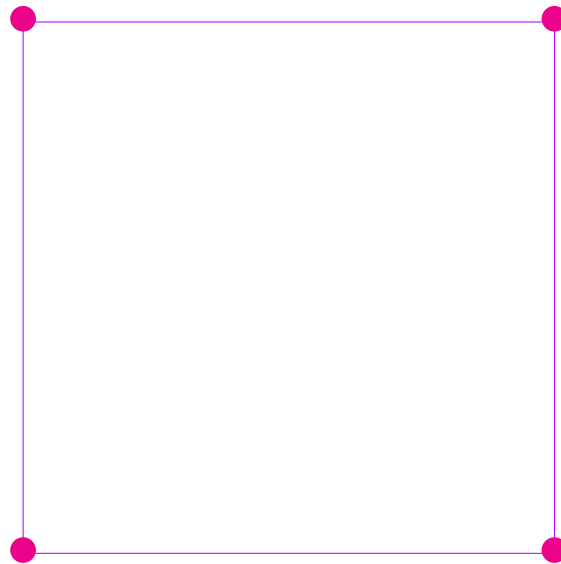
$$\mathbb{CP}_2 \# 5\overline{\mathbb{CP}}_2 = (\mathbb{CP}_1 \times \mathbb{CP}_1) \# 4\overline{\mathbb{CP}}_2$$



Example. $(S^2 \times S^2)/\mathbb{Z}_2$

is limit of sequence of smooth Kähler-Einstein on

$$\mathbb{CP}_2 \# 5\overline{\mathbb{CP}}_2 = (\mathbb{CP}_1 \times \mathbb{CP}_1) \# 4\overline{\mathbb{CP}}_2$$

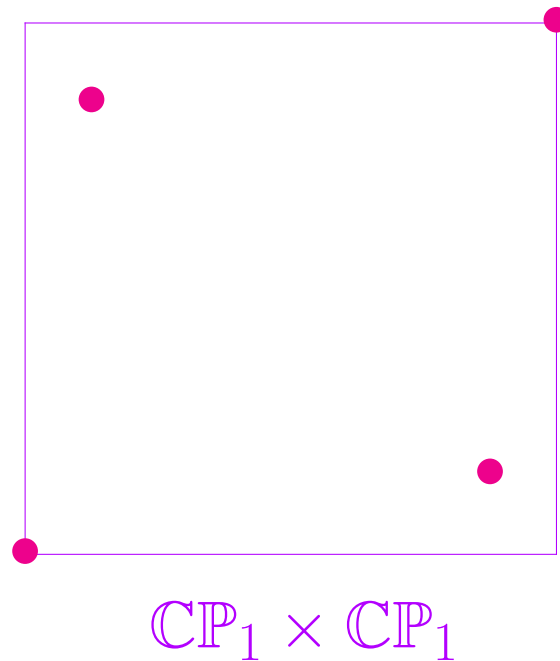


$$\mathbb{CP}_1 \times \mathbb{CP}_1$$

Example. $(S^2 \times S^2)/\mathbb{Z}_2$

is limit of sequence of smooth Kähler-Einstein on

$$\mathbb{CP}_2 \# 5\overline{\mathbb{CP}}_2 = (\mathbb{CP}_1 \times \mathbb{CP}_1) \# 4\overline{\mathbb{CP}}_2$$



Example. $(S^2 \times S^2)/\mathbb{Z}_2$

is limit of sequence of smooth Kähler-Einstein on

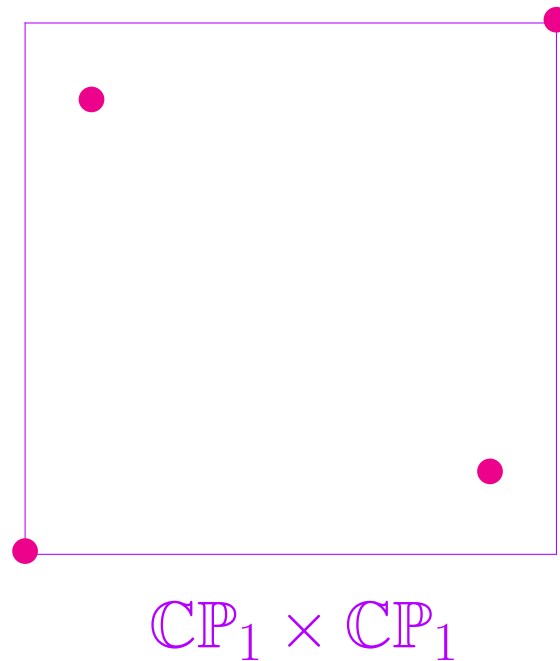
$$\mathbb{CP}_2 \# 5\overline{\mathbb{CP}}_2 = (\mathbb{CP}_1 \times \mathbb{CP}_1) \# 4\overline{\mathbb{CP}}_2$$



Example. $(S^2 \times S^2)/\mathbb{Z}_2$

is limit of sequence of smooth Kähler-Einstein on

$$\mathbb{CP}_2 \# 5\overline{\mathbb{CP}}_2 = (\mathbb{CP}_1 \times \mathbb{CP}_1) \# 4\overline{\mathbb{CP}}_2$$



Example. $(S^2 \times S^2)/\mathbb{Z}_2$

is limit of sequence of smooth Kähler-Einstein on

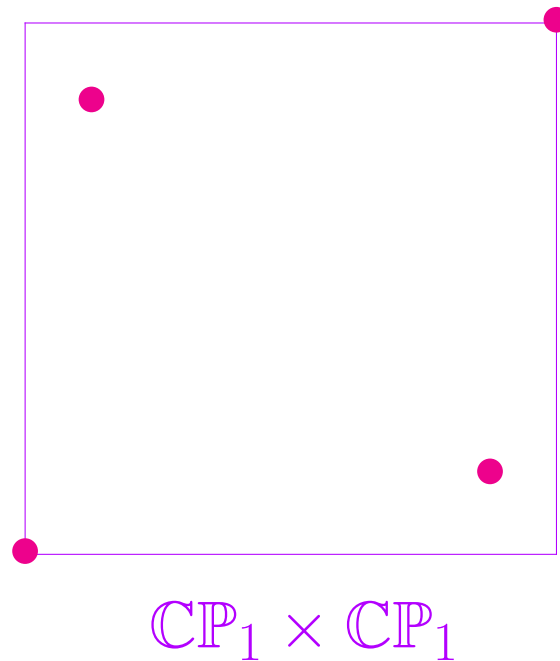
$$\mathbb{CP}_2 \# 5\overline{\mathbb{CP}}_2 = (\mathbb{CP}_1 \times \mathbb{CP}_1) \# 4\overline{\mathbb{CP}}_2$$



Example. $(S^2 \times S^2)/\mathbb{Z}_2$

is limit of sequence of smooth Kähler-Einstein on

$$\mathbb{CP}_2 \# 5\overline{\mathbb{CP}}_2 = (\mathbb{CP}_1 \times \mathbb{CP}_1) \# 4\overline{\mathbb{CP}}_2$$



We thus understand limits of $\lambda > 0$ K-E metrics.

We thus understand limits of $\lambda > 0$ K-E metrics.

But what about limits of general Einstein metrics?

Theorem A.

Theorem A. *Let (X, g_∞) be a compact *Kähler-Einstein orbifold**

Theorem A. *Let (X, g_∞) be a compact **Kähler-Einstein** orbifold of Einstein constant $\lambda > 0$,*

Theorem A. *Let (X, g_∞) be a compact **Kähler-Einstein** orbifold of Einstein constant $\lambda > 0$, and suppose that (M^4, g_j) is an **admissible** sequence*

Theorem A. *Let (X, g_∞) be a compact **Kähler-Einstein** orbifold of Einstein constant $\lambda > 0$, and suppose that (M^4, g_j) is an **admissible sequence** of compact oriented Einstein manifolds*

Theorem A. *Let (X, g_∞) be a compact **Kähler-Einstein** orbifold of Einstein constant $\lambda > 0$, and suppose that (M^4, g_j) is an **admissible sequence** of compact oriented Einstein manifolds that converges to (X, g_∞)*

Theorem A. *Let (X, g_∞) be a compact **Kähler-Einstein** orbifold of Einstein constant $\lambda > 0$, and suppose that (M^4, g_j) is an **admissible sequence** of compact oriented Einstein manifolds that converges to (X, g_∞) in the Gromov-Hausdorff sense.*

Theorem A. *Let (X, g_∞) be a compact **Kähler-Einstein** orbifold of Einstein constant $\lambda > 0$, and suppose that (M^4, g_j) is an **admissible sequence** of compact oriented Einstein manifolds that converges to (X, g_∞) in the Gromov-Hausdorff sense. Then, for all $j \gg 0$,*

Theorem A. *Let (X, g_∞) be a compact **Kähler-Einstein** orbifold of Einstein constant $\lambda > 0$, and suppose that (M^4, g_j) is an **admissible sequence** of compact oriented Einstein manifolds that converges to (X, g_∞) in the Gromov-Hausdorff sense. Then, for all $j \gg 0$, the (M^4, g_j) are actually all **Kähler-Einstein**.*

Theorem A. *Let (X, g_∞) be a compact **Kähler-Einstein** orbifold of Einstein constant $\lambda > 0$, and suppose that (M^4, g_j) is an **admissible sequence** of compact oriented Einstein manifolds that converges to (X, g_∞) in the Gromov-Hausdorff sense. Then, for all $j \gg 0$, the (M^4, g_j) are actually all **Kähler-Einstein**. In particular,*

Theorem A. *Let (X, g_∞) be a compact **Kähler-Einstein** orbifold of Einstein constant $\lambda > 0$, and suppose that (M^4, g_j) is an **admissible sequence** of compact oriented Einstein manifolds that converges to (X, g_∞) in the Gromov-Hausdorff sense. Then, for all $j \gg 0$, the (M^4, g_j) are actually all **Kähler-Einstein**. In particular, (X, g_∞) must therefore be*

Theorem A. *Let (X, g_∞) be a compact **Kähler-Einstein** orbifold of Einstein constant $\lambda > 0$, and suppose that (M^4, g_j) is an **admissible sequence** of compact oriented Einstein manifolds that converges to (X, g_∞) in the Gromov-Hausdorff sense. Then, for all $j \gg 0$, the (M^4, g_j) are actually all **Kähler-Einstein**. In particular, (X, g_∞) must therefore be one of the **Kähler-Einstein orbifolds***

Theorem A. *Let (X, g_∞) be a compact **Kähler-Einstein** orbifold of Einstein constant $\lambda > 0$, and suppose that (M^4, g_j) is an **admissible sequence** of compact oriented Einstein manifolds that converges to (X, g_∞) in the Gromov-Hausdorff sense. Then, for all $j \gg 0$, the (M^4, g_j) are actually all **Kähler-Einstein**. In particular, (X, g_∞) must therefore be one of the **Kähler-Einstein orbifolds** classified by **Odaka-Spotti-Sun**.*

Theorem B.

Theorem B. *Let (X, g_∞) be a compact *Einstein orbifold**

Theorem B. *Let (X, g_∞) be a compact Einstein orbifold which is Hermitian*

Theorem B. *Let (X, g_∞) be a compact *Einstein* orbifold which is *Hermitian* with respect to an integrable complex structure J on X .*

Theorem B. *Let (X, g_∞) be a compact *Einstein* orbifold which is *Hermitian* with respect to an integrable complex structure J on X .*

$$g_\infty(J\cdot, J\cdot) = g_\infty$$

Theorem B. *Let (X, g_∞) be a compact *Einstein* orbifold which is *Hermitian* with respect to an integrable complex structure J on X .*

Theorem B. *Let (X, g_∞) be a compact *Einstein* orbifold which is *Hermitian* with respect to an integrable complex structure J on X . Assume that X has at least one singular point,*

Theorem B. *Let (X, g_∞) be a compact *Einstein* orbifold which is *Hermitian* with respect to an integrable complex structure J on X . Assume that X has at least one singular point, and suppose that (X, g_∞) is the Gromov-Hausdorff limit*

Theorem B. *Let (X, g_∞) be a compact *Einstein* orbifold which is *Hermitian* with respect to an integrable complex structure J on X . Assume that X has at least one singular point, and suppose that (X, g_∞) is the Gromov-Hausdorff limit of an admissible sequence*

$$(M, g_j) \rightarrow (X, g_\infty)$$

Theorem B. *Let (X, g_∞) be a compact *Einstein* orbifold which is *Hermitian* with respect to an integrable complex structure J on X . Assume that X has at least one singular point, and suppose that (X, g_∞) is the Gromov-Hausdorff limit of an admissible sequence*

$$(M, g_j) \rightarrow (X, g_\infty)$$

of Einstein metrics of fixed $\lambda > 0$

Theorem B. *Let (X, g_∞) be a compact **Einstein** orbifold which is **Hermitian** with respect to an integrable complex structure **J** on X . Assume that X has at least one singular point, and suppose that (X, g_∞) is the Gromov-Hausdorff limit of an admissible sequence*

$$(\mathbf{M}, g_j) \rightarrow (X, g_\infty)$$

*of Einstein metrics of fixed $\lambda > 0$ on a connected compact smooth oriented 4-manifold **M** .*

Theorem B. *Let (X, g_∞) be a compact **Einstein** orbifold which is **Hermitian** with respect to an integrable complex structure **J** on X . Assume that X has at least one singular point, and suppose that (X, g_∞) is the Gromov-Hausdorff limit of an admissible sequence*

$$(\mathbf{M}, g_j) \rightarrow (X, g_\infty)$$

*of Einstein metrics of fixed $\lambda > 0$ on a connected compact smooth oriented 4-manifold **M** . Then (X, g_∞) is actually **Kähler-Einstein**,*

Theorem B. *Let (X, g_∞) be a compact **Einstein** orbifold which is **Hermitian** with respect to an integrable complex structure **J** on **X**. Assume that **X** has at least one singular point, and suppose that (X, g_∞) is the Gromov-Hausdorff limit of an admissible sequence*

$$(\mathbf{M}, g_j) \rightarrow (X, g_\infty)$$

*of Einstein metrics of fixed $\lambda > 0$ on a connected compact smooth oriented 4-manifold **M**. Then (X, g_∞) is actually **Kähler-Einstein**, and **Theorem A** therefore applies.*

Theorem B. *Let (X, g_∞) be a compact **Einstein** orbifold which is Hermitian with respect to an integrable complex structure J on X . Assume that X has at least one singular point, and suppose that (X, g_∞) is the Gromov-Hausdorff limit of an admissible sequence*

$$(\mathbf{M}, g_j) \rightarrow (X, g_\infty)$$

*of Einstein metrics of fixed $\lambda > 0$ on a connected compact smooth oriented 4-manifold $\mathbf{M}$. Then (X, g_∞) is actually **Kähler-Einstein**, and **Theorem A** therefore applies.*

By the Riemannian Goldberg-Sachs Theorem,

Theorem B. *Let (X, g_∞) be a compact **Einstein** orbifold which is Hermitian with respect to an integrable complex structure J on X . Assume that X has at least one singular point, and suppose that (X, g_∞) is the Gromov-Hausdorff limit of an admissible sequence*

$$(\mathbf{M}, g_j) \rightarrow (X, g_\infty)$$

*of Einstein metrics of fixed $\lambda > 0$ on a connected compact smooth oriented 4-manifold $\mathbf{M}$. Then (X, g_∞) is actually **Kähler-Einstein**, and **Theorem A** therefore applies.*

By the Riemannian Goldberg-Sachs Theorem, the Hermitian assumption is equivalent to assuming that the orbifold Einstein metric g_∞ is **conformally Kähler**.

Theorem B. *Let (X, g_∞) be a compact **Einstein** orbifold which is **Hermitian** with respect to an integrable complex structure **J** on **X**. Assume that **X** has at least one singular point, and suppose that (X, g_∞) is the Gromov-Hausdorff limit of an admissible sequence*

$$(\mathbf{M}, g_j) \rightarrow (X, g_\infty)$$

*of Einstein metrics of fixed $\lambda > 0$ on a connected compact smooth oriented 4-manifold **M**. Then (X, g_∞) is actually **Kähler-Einstein**, and **Theorem A** therefore applies.*

This assertion is peculiar to dimension 4.

Theorem B. *Let (X, g_∞) be a compact *Einstein* orbifold which is *Hermitian* with respect to an integrable complex structure J on X . Assume that X has at least one singular point, and suppose that (X, g_∞) is the Gromov-Hausdorff limit of an admissible sequence*

$$(M, g_j) \rightarrow (X, g_\infty)$$

*of Einstein metrics of fixed $\lambda > 0$ on a connected compact smooth oriented 4-manifold M . Then (X, g_∞) is actually *Kähler-Einstein*, and **Theorem A** therefore applies.*

This assertion is peculiar to dimension 4.
It is false in all higher dimensions!

Theorem B. *Let (X, g_∞) be a compact **Einstein** orbifold which is **Hermitian** with respect to an integrable complex structure **J** on **X**. Assume that **X** has at least one singular point, and suppose that (X, g_∞) is the Gromov-Hausdorff limit of an admissible sequence*

$$(\mathbf{M}, g_j) \rightarrow (X, g_\infty)$$

*of Einstein metrics of fixed $\lambda > 0$ on a connected compact smooth oriented 4-manifold **M**. Then (X, g_∞) is actually **Kähler-Einstein**, and **Theorem A** therefore applies.*

A key tool used in the proofs of the main theorems is based on a criterion discovered by [Peng Wu '21](#).

A key tool used in the proofs of the main theorems is based on a criterion discovered by Peng Wu '21.

A more transparent proof was then given in L '21.

Theorem (Wu '21, L '21).

Theorem (Wu '21, L '21). *Let (M, g) be a simply-connected compact oriented Einstein 4-manifold,*

Theorem (Wu '21, L '21). *Let (M, g) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \rightarrow \Lambda^+$$

Theorem (Wu '21, L '21). *Let (M, g) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \rightarrow \Lambda^+$$

satisfies

$$\det(W^+) > 0$$

Theorem (Wu '21, L '21). *Let (M, g) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \rightarrow \Lambda^+$$

satisfies

$$\det(W^+) > 0$$

at every point of M .

Theorem (Wu '21, L '21). *Let (M, g) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \rightarrow \Lambda^+$$

satisfies

$$\det(W^+) > 0$$

at every point of M . Then g is conformal

Theorem (Wu '21, L '21). *Let (M, g) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \rightarrow \Lambda^+$$

satisfies

$$\det(W^+) > 0$$

at every point of M . Then g is conformal to an orientation-compatible extremal Kähler metric h

Theorem (Wu '21, L '21). *Let (M, g) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \rightarrow \Lambda^+$$

satisfies

$$\det(W^+) > 0$$

at every point of M . Then g is conformal to an orientation-compatible extremal Kähler metric h with scalar curvature $s > 0$ on M .

Theorem (Wu '21, L '21). *Let (M, g) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \rightarrow \Lambda^+$$

satisfies

$$\det(W^+) > 0$$

at every point of M . Then g is conformal to an orientation-compatible extremal Kähler metric h with scalar curvature $s > 0$ on M .

In most cases, this implies that g is Kähler-Einstein.

Theorem (Wu '21, L '21). *Let (M, g) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \rightarrow \Lambda^+$$

satisfies

$$\det(W^+) > 0$$

at every point of M . Then g is conformal to an orientation-compatible extremal Kähler metric h with scalar curvature $s > 0$ on M .

In most cases, this implies that g is Kähler-Einstein. There are two exceptions,

Theorem (Wu '21, L '21). *Let (M, g) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \rightarrow \Lambda^+$$

satisfies

$$\det(W^+) > 0$$

at every point of M . Then g is conformal to an orientation-compatible extremal Kähler metric h with scalar curvature $s > 0$ on M .

In most cases, this implies that g is Kähler-Einstein. There are two exceptions, but these are both rigid,

Theorem (Wu '21, L '21). *Let (M, g) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \rightarrow \Lambda^+$$

satisfies

$$\det(W^+) > 0$$

at every point of M . Then g is conformal to an orientation-compatible extremal Kähler metric h with scalar curvature $s > 0$ on M .

In most cases, this implies that g is Kähler-Einstein. There are two exceptions, but these are both rigid, and thus never lead to non-trivial G-H limits.

Theorem (Wu '21, L '21). *Let (M, g) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \rightarrow \Lambda^+$$

satisfies

$$\det(W^+) > 0$$

at every point of M . Then g is conformal to an orientation-compatible extremal Kähler metric h with scalar curvature $s > 0$ on M .

In most cases, this implies that g is Kähler-Einstein. There are two exceptions, but these are both rigid, and thus never lead to non-trivial G-H limits. Conversely, $\lambda > 0$ K-E $\implies \det(W^+) > 0$.

Applying Peng Wu's criterion to prove Theorems A and B is technically very delicate.

Applying Peng Wu's criterion to prove Theorems A and B is technically very delicate.

Condition $\det(W_+) > 0$ is required everywhere!

Applying Peng Wu's criterion to prove Theorems A and B is technically very delicate.

Condition $\det(W_+) > 0$ is required everywhere!

Easy away from “gravitational instanton” regions.

Applying Peng Wu's criterion to prove Theorems A and B is technically very delicate.

Condition $\det(W_+) > 0$ is required everywhere!

Easy away from “gravitational instanton” regions.

Harder in the regions of singularity formation!

Applying Peng Wu's criterion to prove Theorems A and B is technically very delicate.

Condition $\det(W_+) > 0$ is required everywhere!

Easy away from “gravitational instanton” regions.

Harder in the regions of singularity formation!

But results of Biquard and Ozuch provide the key!

Applying Peng Wu's criterion to prove Theorems A and B is technically very delicate.

Condition $\det(W_+) > 0$ is required everywhere!

Easy away from “gravitational instanton” regions.

Harder in the regions of singularity formation!

But results of Biquard and Ozuch provide the key!

Curvature tensor behaves roughly like Kähler paradigm in regions of singularity formation.

Applying Peng Wu's criterion to prove Theorems A and B is technically very delicate.

Condition $\det(W_+) > 0$ is required everywhere!

Easy away from “gravitational instanton” regions.

Harder in the regions of singularity formation!

But results of Biquard and Ozuch provide the key!

Curvature tensor behaves roughly like Kähler paradigm in regions of singularity formation.

Technically hardest when curvature accumulates on many different length-scales, giving rise to a complicated bubble tree.

Basic strategy:

Basic strategy:

Weaken Einstein condition to Einstein-modulo obstructions.

Basic strategy:

Weaken Einstein condition to Einstein-modulo obstructions.

Construct universal family of such metrics that desingularize given Einstein orbifold.

Basic strategy:

Weaken Einstein condition to Einstein-modulo obstructions.

Construct universal family of such metrics that desingularize given Einstein orbifold.

Show that Wu's criterion holds for these metrics.

Basic strategy:

Weaken Einstein condition to Einstein-modulo obstructions.

Construct universal family of such metrics that desingularize given Einstein orbifold.

Show that Wu's criterion holds for these metrics.

Then show that the given Einstein metrics g_j belong to this universal family for all $j \gg 0$.

Basic strategy:

Weaken Einstein condition to Einstein-modulo obstructions.

Construct universal family of such metrics that desingularize given Einstein orbifold.

Show that Wu's criterion holds for these metrics.

Then show that the given Einstein metrics g_j belong to this universal family for all $j \gg 0$.

And hence that they are actually conformally Kähler!

Thanks for the Invitation!

Thanks for the Invitation!



It's a real pleasure to be here!

It's a real pleasure to be here!



It's a real pleasure to be here!

It's a real pleasure to be here!

