Anti-Self-Dual 4-Manifolds,

Quasi-Fuchsian Groups, &

Almost-Kähler Geometry

Claude LeBrun
Stony Brook University

Workshop in Complex Differential Geometry
Vanderbilt University, 2 March, 2018

Most recent results joint with

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Christopher J. Bishop
Stony Brook University

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e-print: `arXiv:1708.03824 [math.DG]`

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To appear in Comm. An. Geom.

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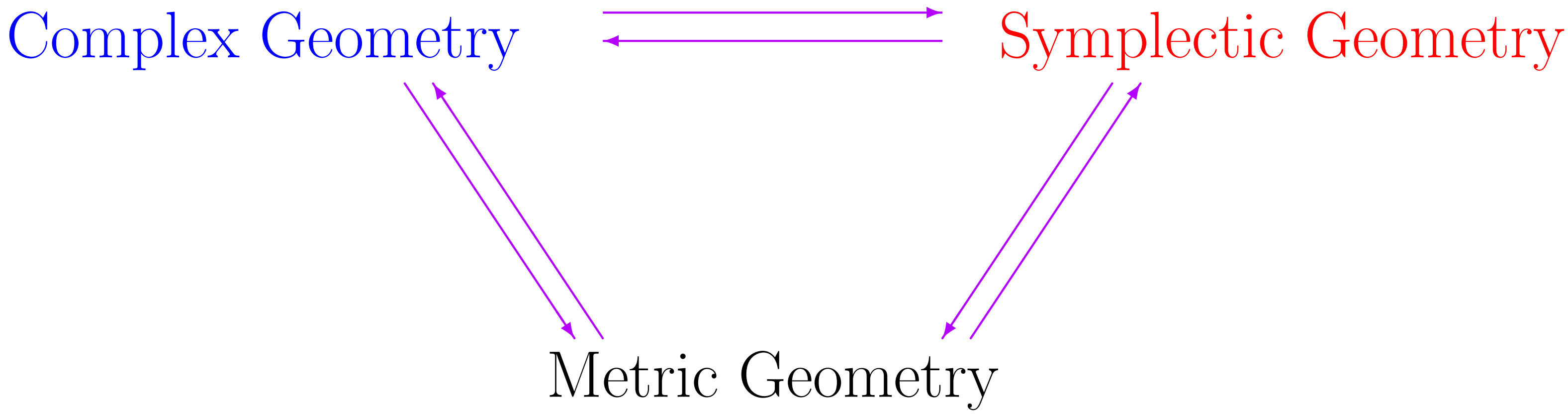

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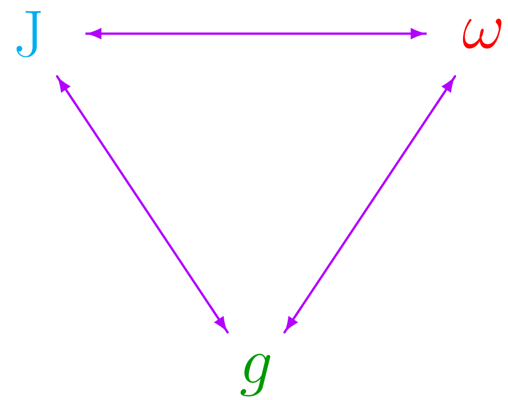

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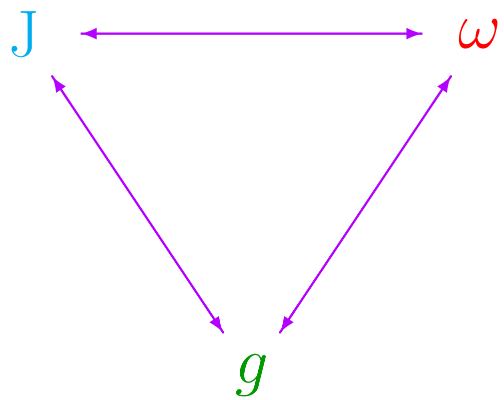

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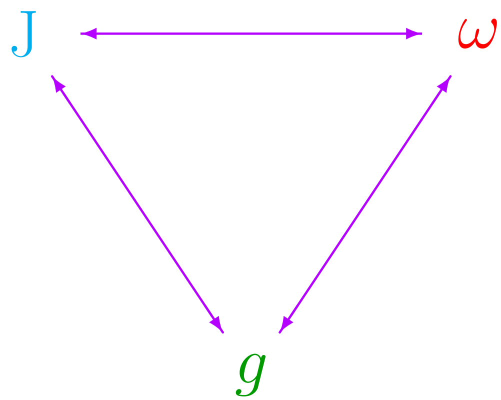

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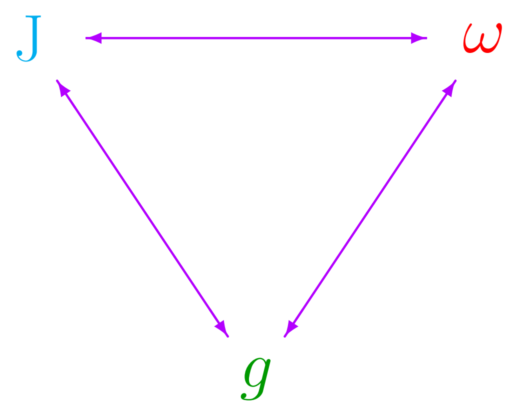

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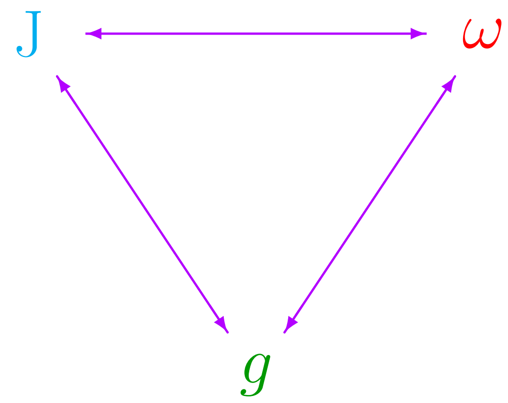

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Higher dimensions are demonstrably different.

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Imitates Kähler geometry in a non-Kähler setting.

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$$\text{Kähler} \iff J \text{ integrable.}$$

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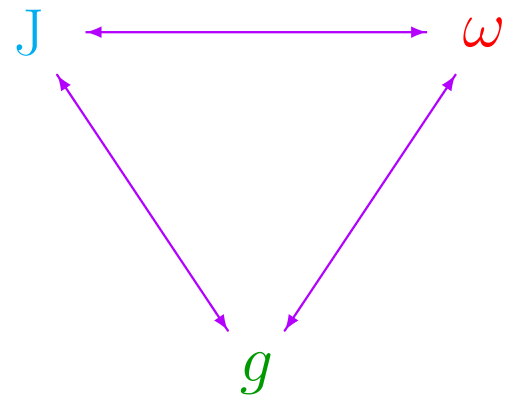

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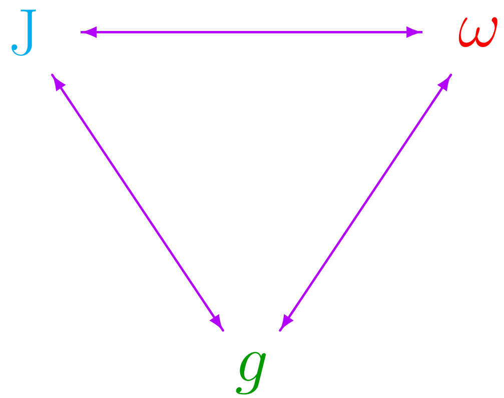

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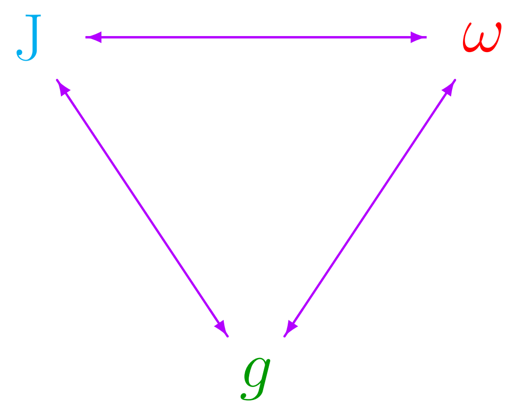

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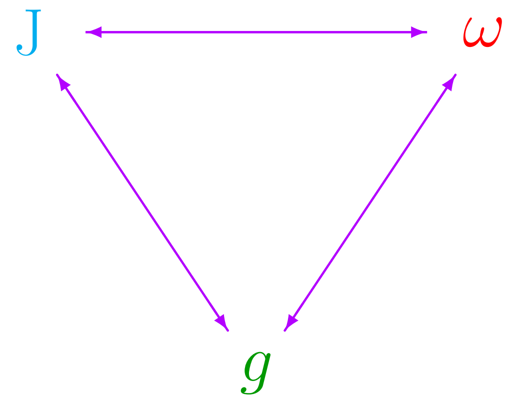
For example, can avoid explicitly mentioning J.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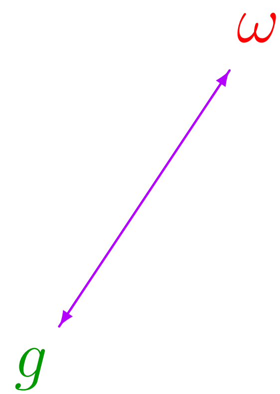
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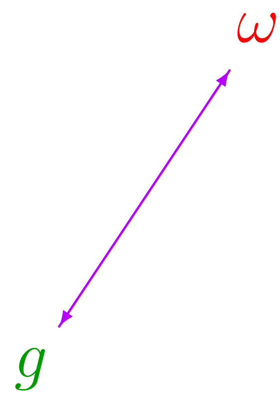
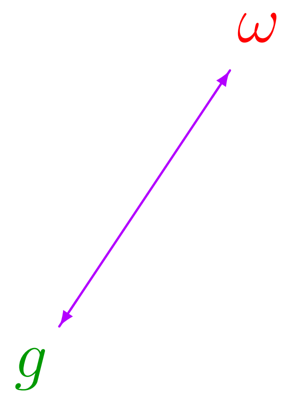
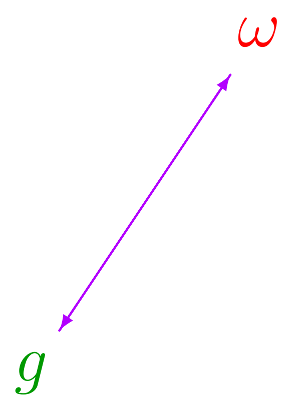


Lemma. *An oriented Riemannian manifold* (M^{2m}, g)



Lemma. *An oriented Riemannian manifold* (M^{2m}, g) *is almost-Kähler*



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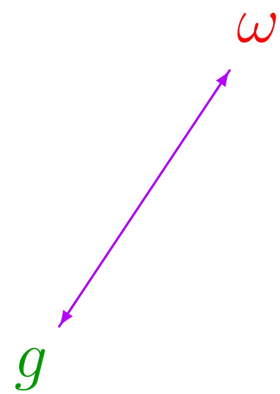

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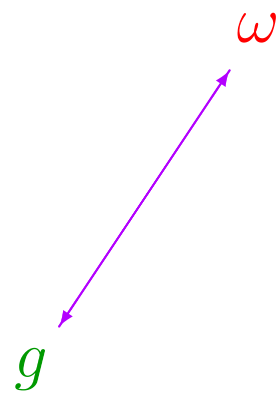

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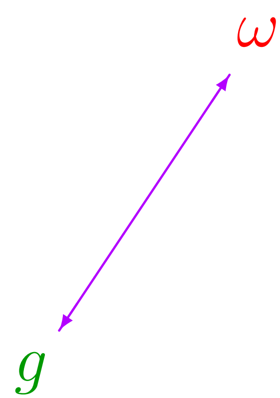

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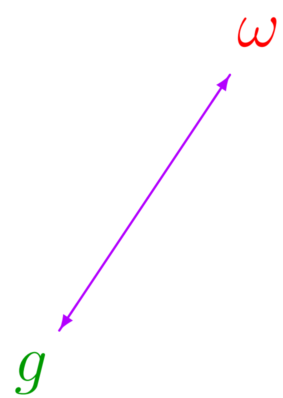

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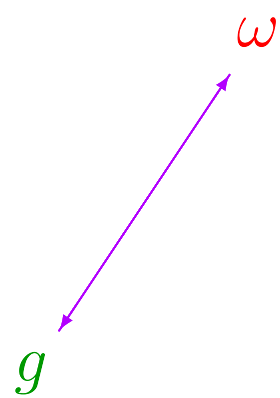

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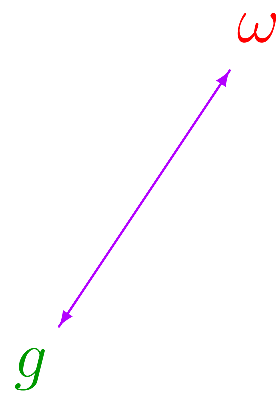
Simplifies dramatically when $m = 2$:



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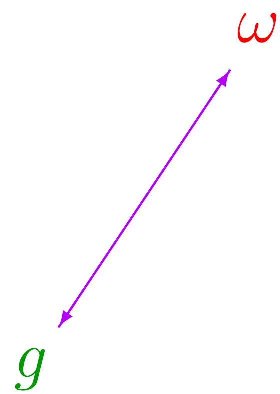

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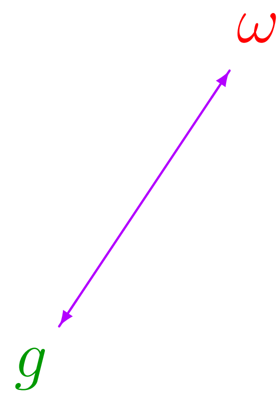

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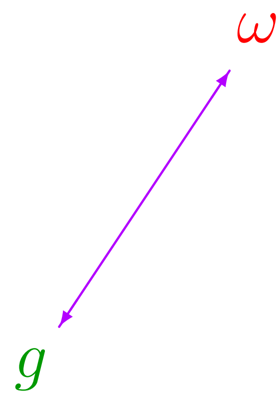

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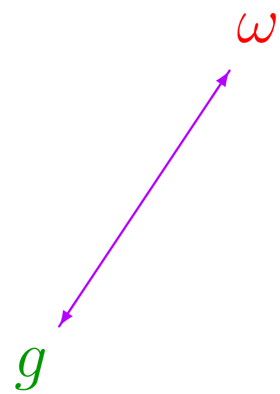

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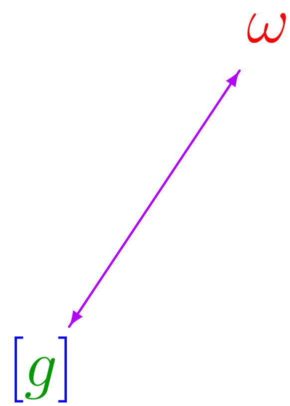
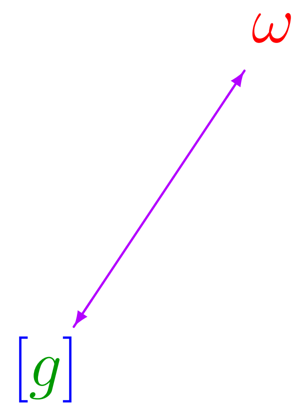


Proposition. *A conformal class* $[g]$ *on a smooth compact oriented* 4*-manifold* M *is represented by an almost-Kähler metric* g *iff it carries a self-dual* harmonic 2*-form* ω *that is* $\neq 0$ *everywhere.*



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Moreover, the set of conformal classes $[g]$ *on* M *that carry such a harmonic form* ω *is open in the* C^2 *topology.*

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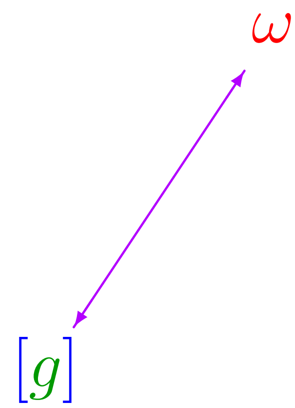

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“Conformal classes of symplectic type”

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In particular, the numbers

$$b_\pm(M) = \dim \mathcal{H}_g^\pm$$

are independent of g, and so are invariants of M.

$b_{\pm}(M)$?

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Diagonalize:

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$$
\left[\begin{array}{cc}
\underbrace{\begin{array}{ccc} +1 & & \\ & \ddots & \\ & & +1 \end{array}}_{b_+(M)} & \\
& b_-(M)\left\{\begin{array}{ccc} -1 & & \\ & \ddots & \\ & & -1 \end{array}\right.
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$$\begin{array}{ccc} H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) & \longrightarrow & \mathbb{R} \\ (\,[\varphi]\,,\,[\psi]\,) & \longmapsto & \int_M \varphi \wedge \psi \end{array}$$

Diagonalize:

$$\left[\begin{array}{cc} \underbrace{\begin{matrix} +1 & & \\ & \ddots & \\ & & +1 \end{matrix}}_{b_+(M)} & \\ & b_-(M)\left\{\begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix}\right. \end{array}\right].$$

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"Signature" of M.

Signature defined in terms of intersection pairing

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(Thom-Hirzebruch Signature Formula)

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Has major consequences in conformal geometry.

On oriented (M^4, g),

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

Riemann curvature of g

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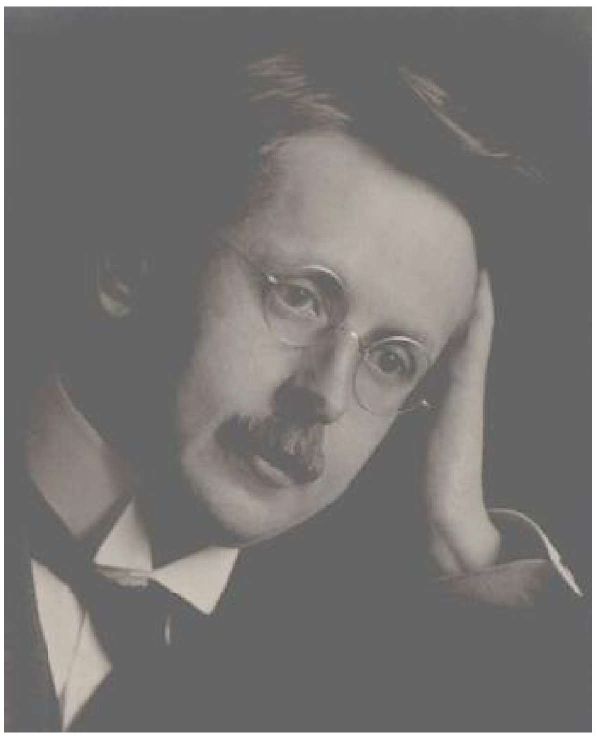

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Basic problems: For given smooth compact M^4,

- What is $\inf \mathscr{W}$?
- Do there exist minimizers?

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But we've already noted that

$$12\pi^2\tau(M) = \int_M \left(|W_+|^2 - |W_-|^2\right) d\mu_g$$

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So Weyl functional is essentially equivalent to

$$[g] \longmapsto \int_M |W_+|^2 d\mu_g$$

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(ASD)

Twistor picture of anti-self-duality condition:

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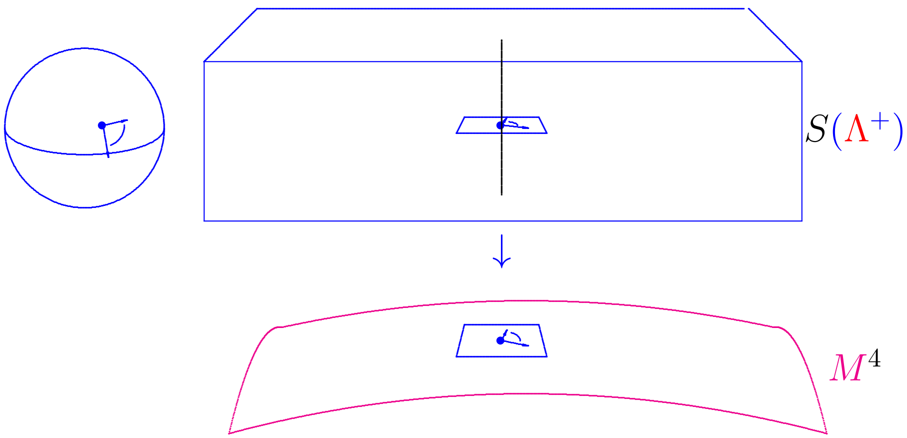

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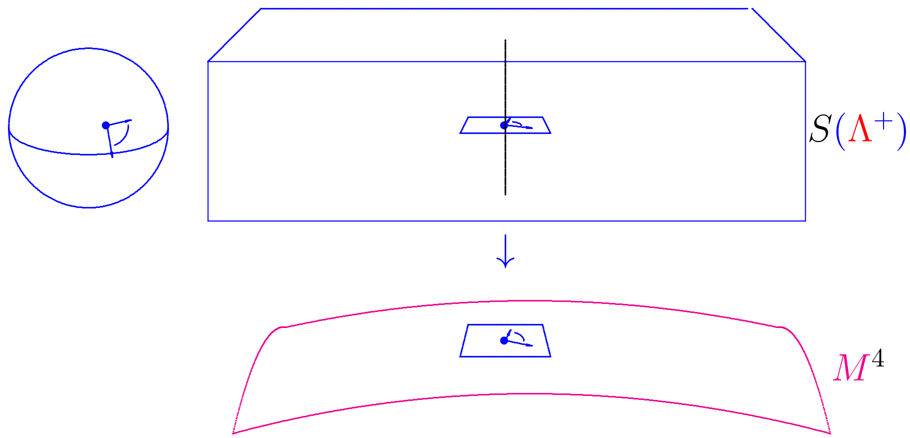

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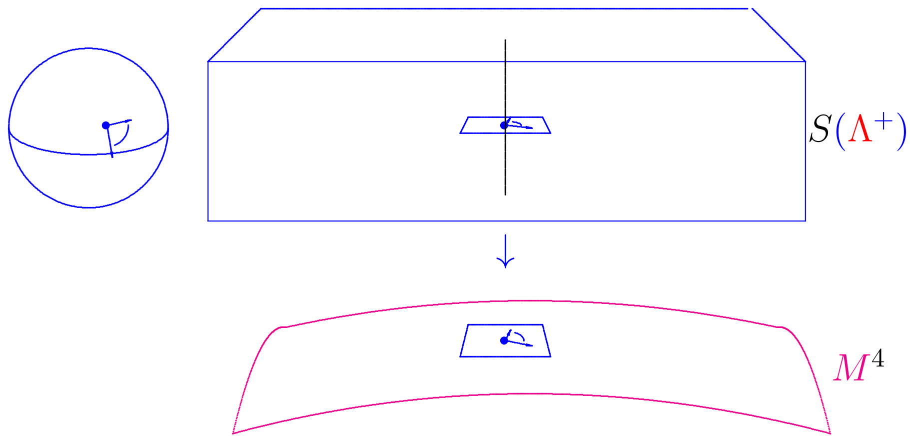

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Reconceptualizes earlier work by Penrose.

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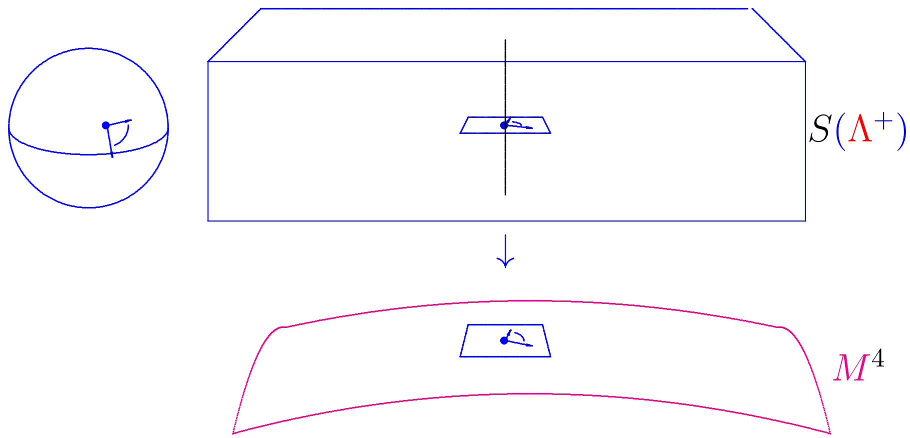

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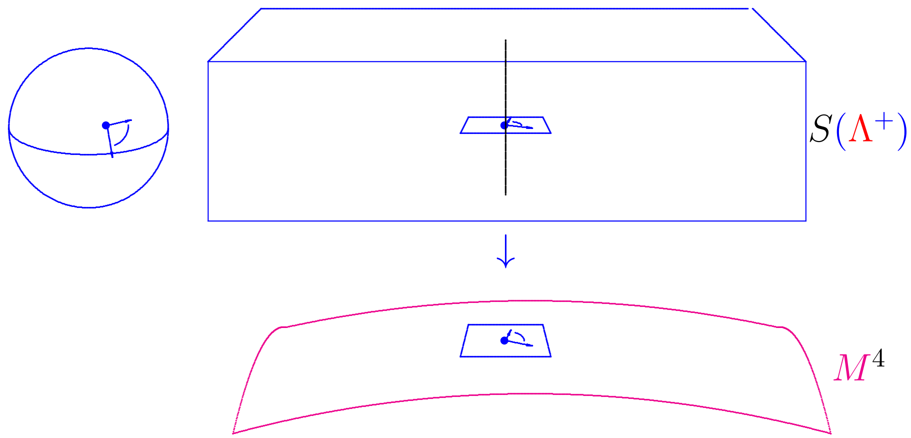

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Motivates study of ASD metrics,
and yields methods for constructing them.

So ASD metrics are linked to complex geometry. . .

A different link with complex geometry:

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If (M^4, g, J) is a Kähler surface,

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Scalar-flat Kähler surfaces:

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Results proved about SFK in '90s foreshadowed
many more recent results about general case.

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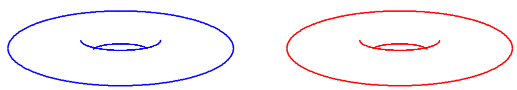
Connected sum #:



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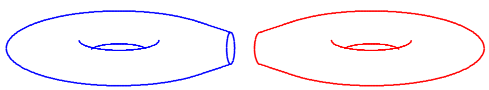

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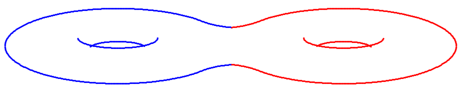

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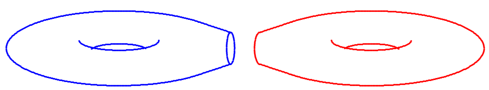

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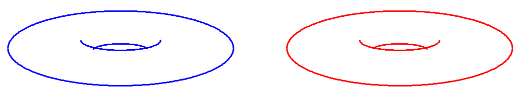

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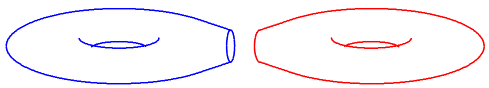

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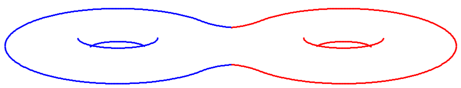

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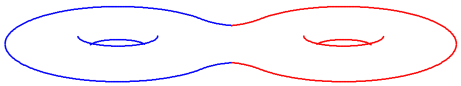

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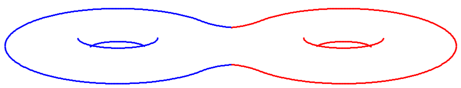

Blowing Up: $M \rightsquigarrow M \# \overline{\mathbb{CP}}_2$

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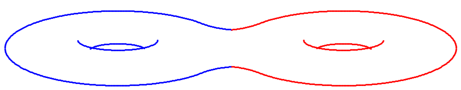

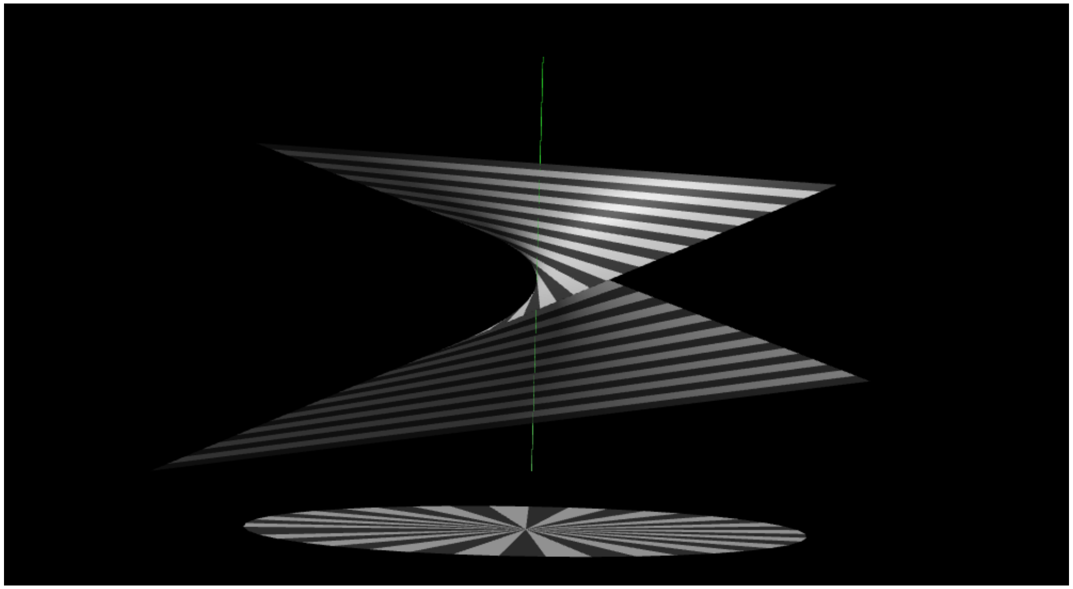

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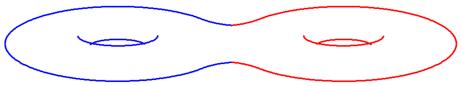

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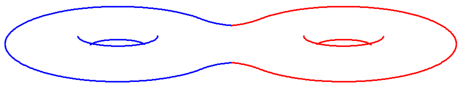

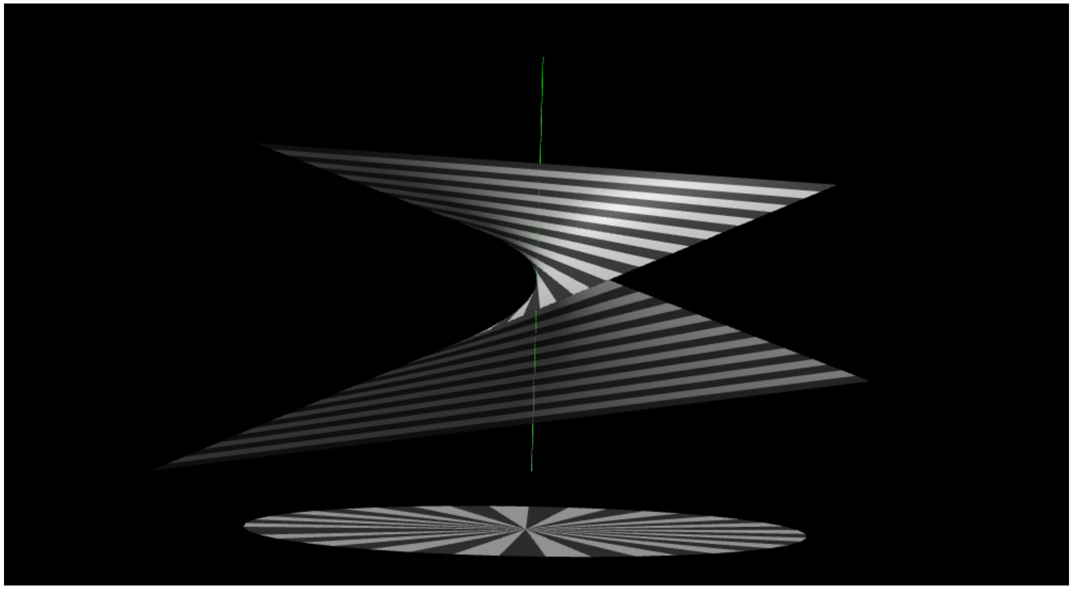

A different link with complex geometry:

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Classification up to diffeomorphism:

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  - $(T^2 \times S^2) \# k \overline{\mathbb{CP}}_2$, $k \geq 1$
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Classification up to diffeomorphism:

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Theorem (L '15). *True for conformal classes of symplectic type.*

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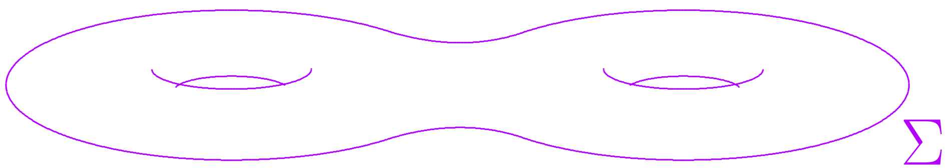

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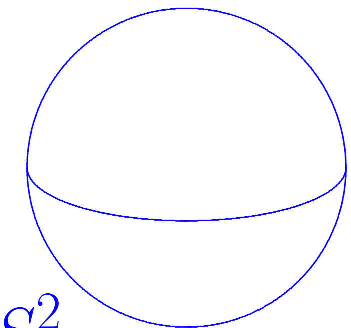
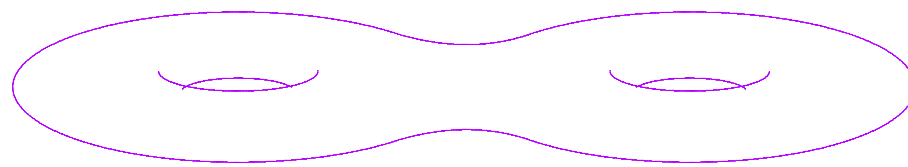


S^2



Σ

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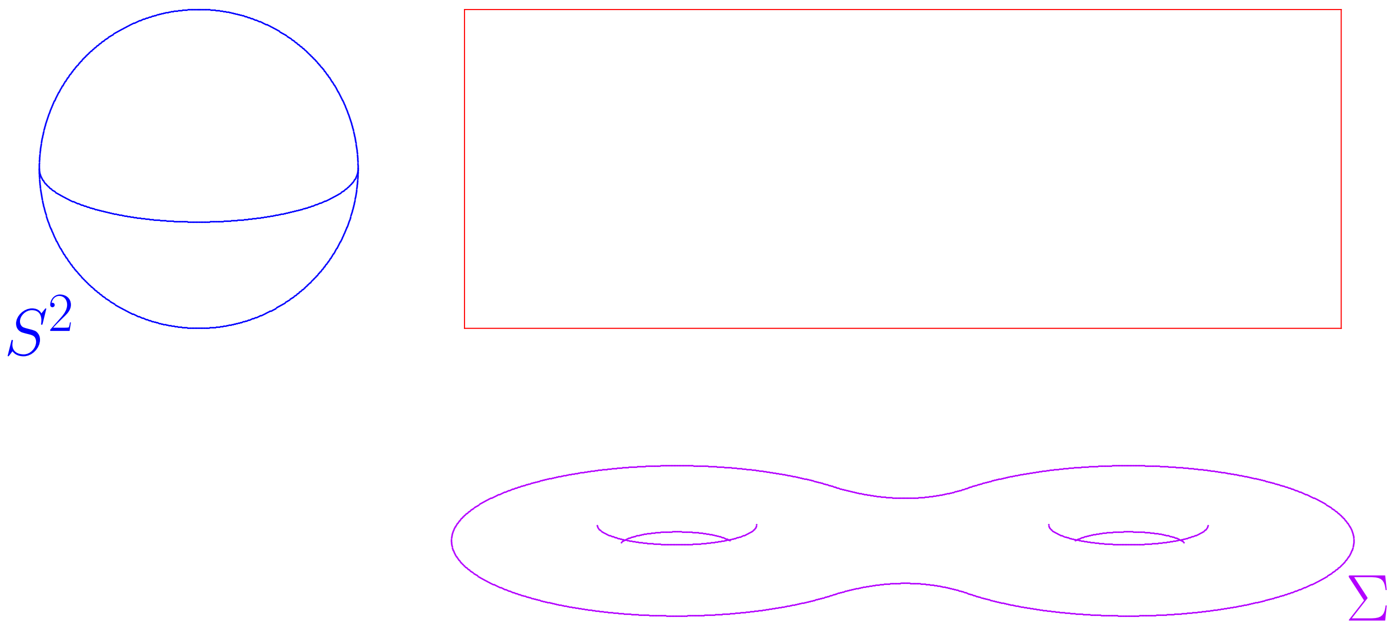

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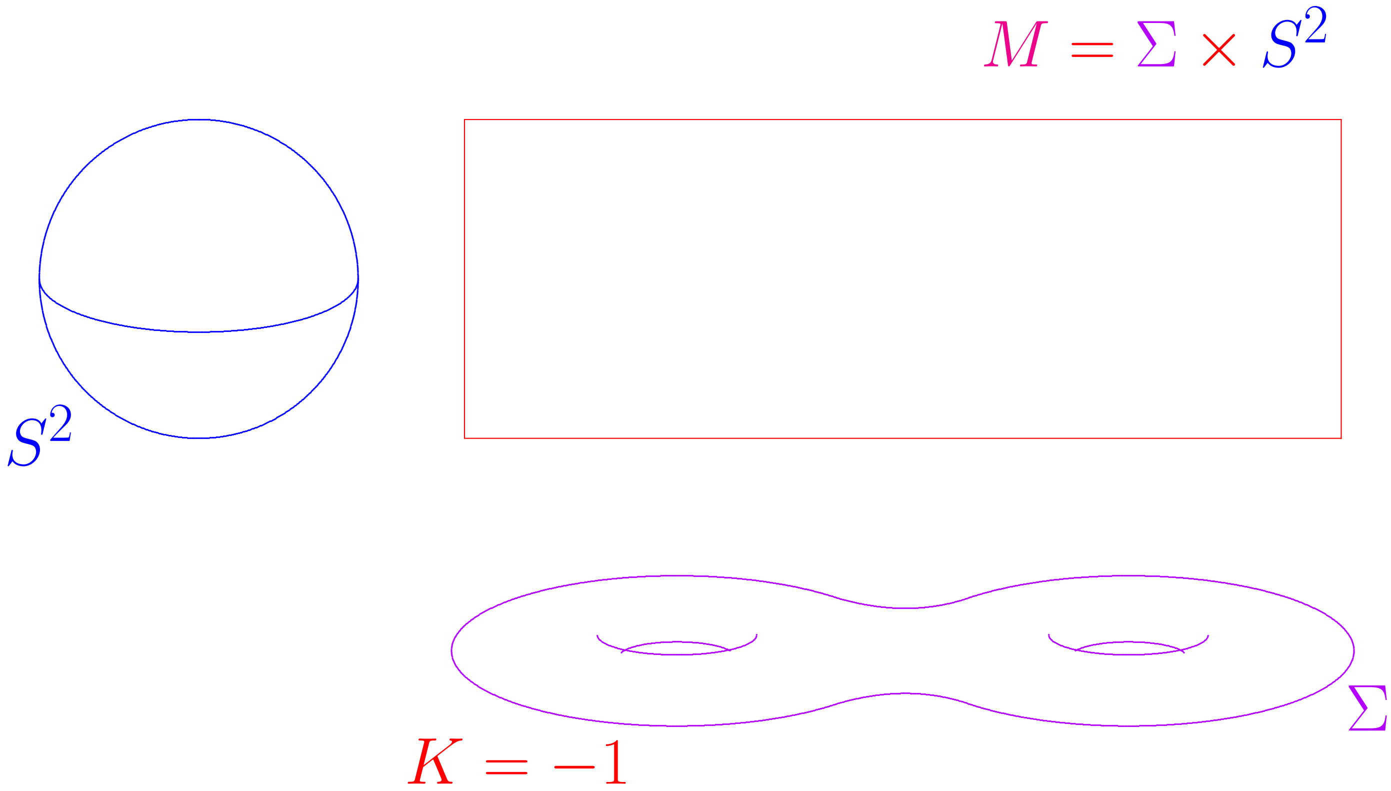

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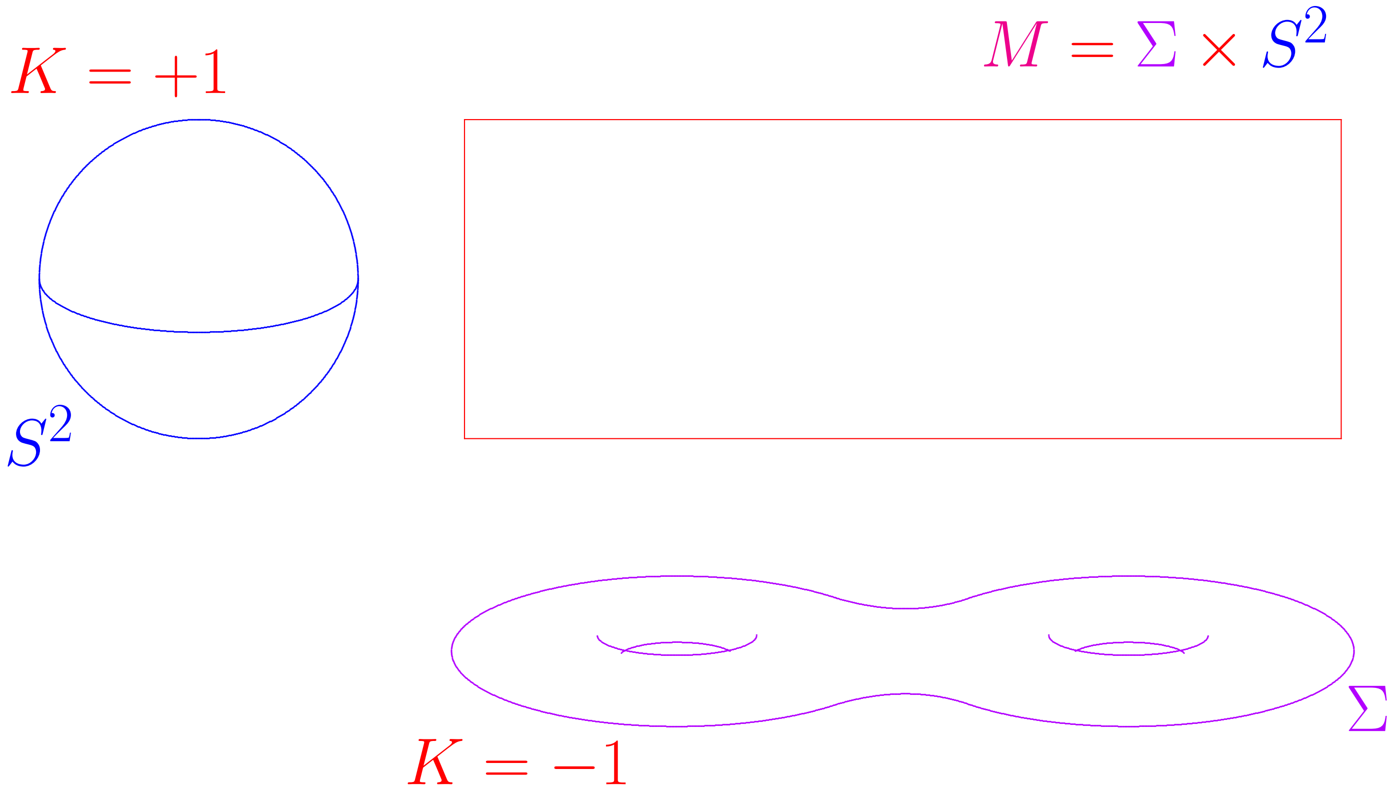

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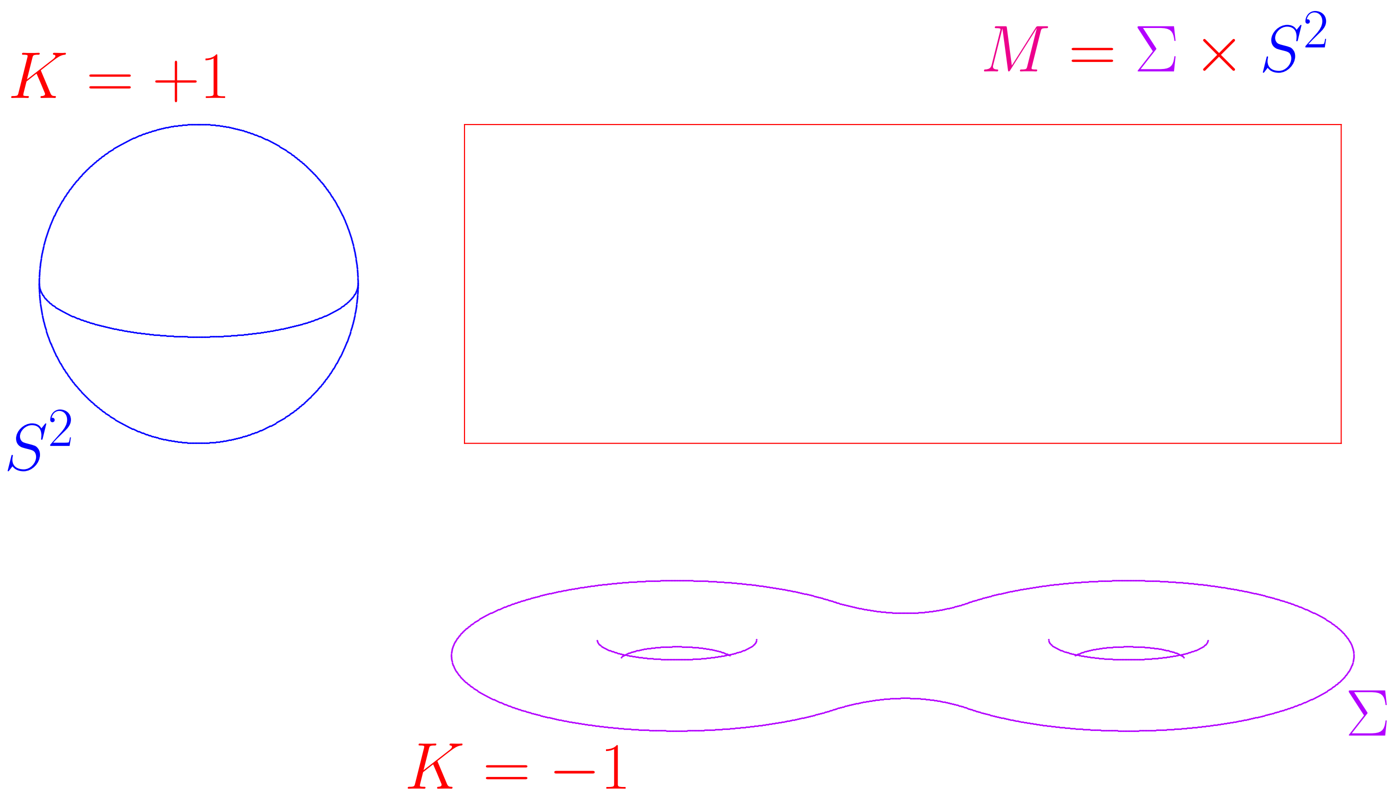

Product is scalar-flat

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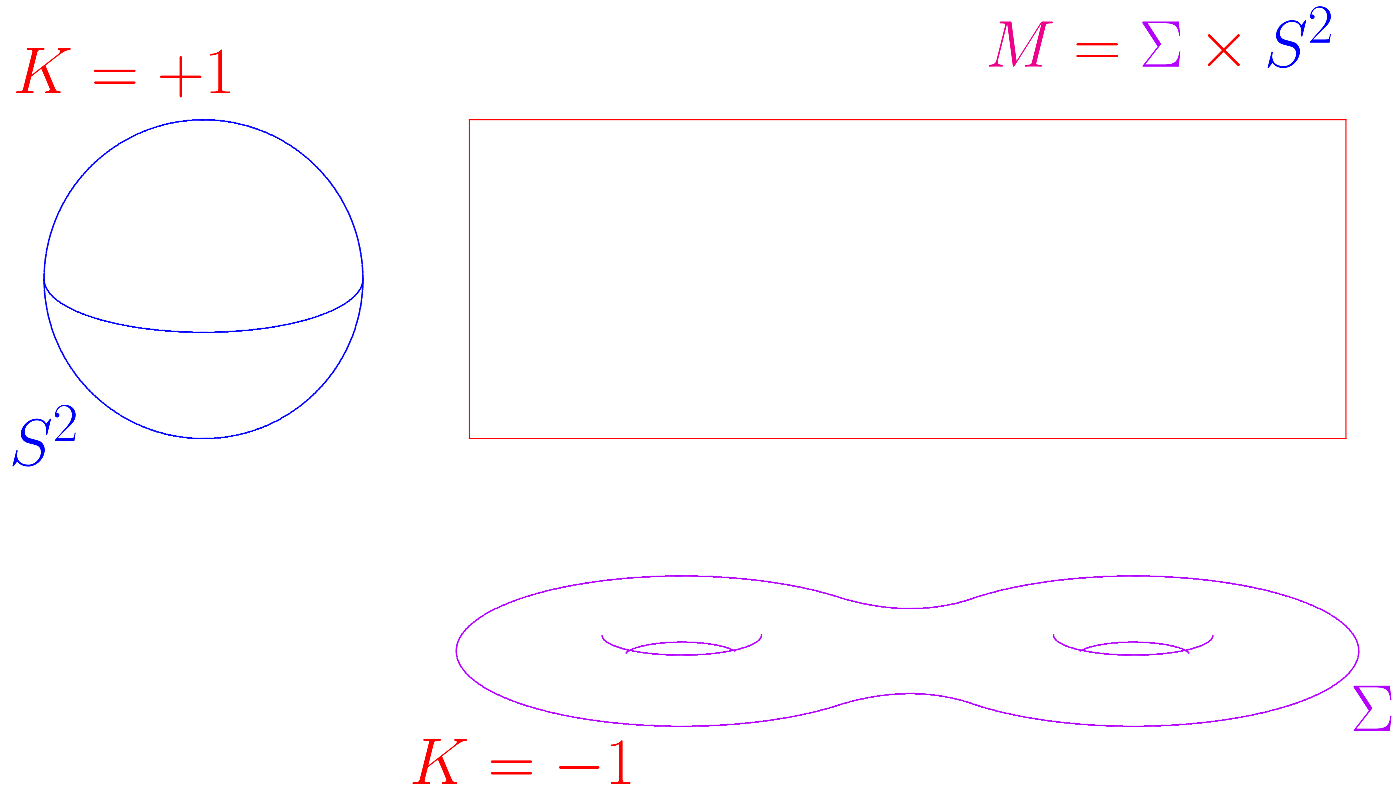

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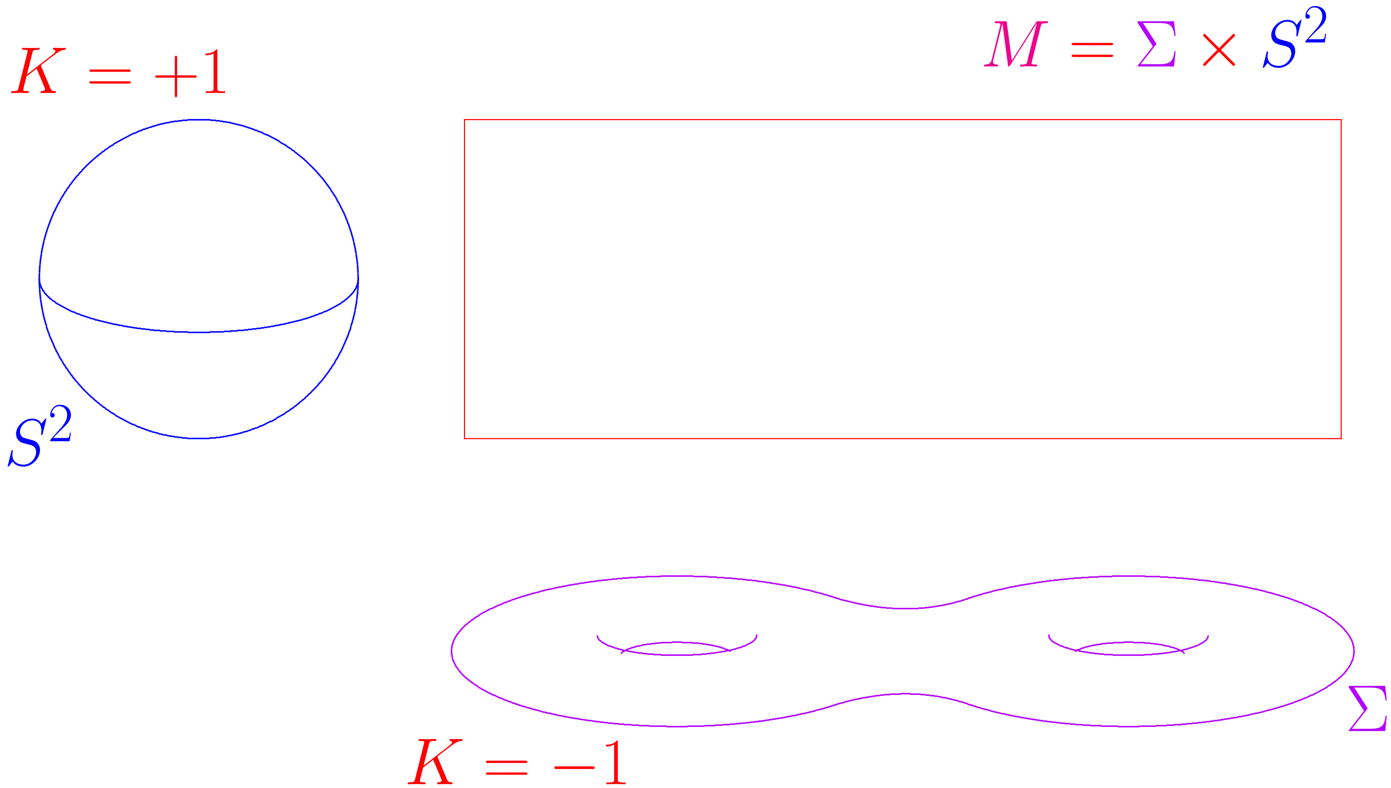

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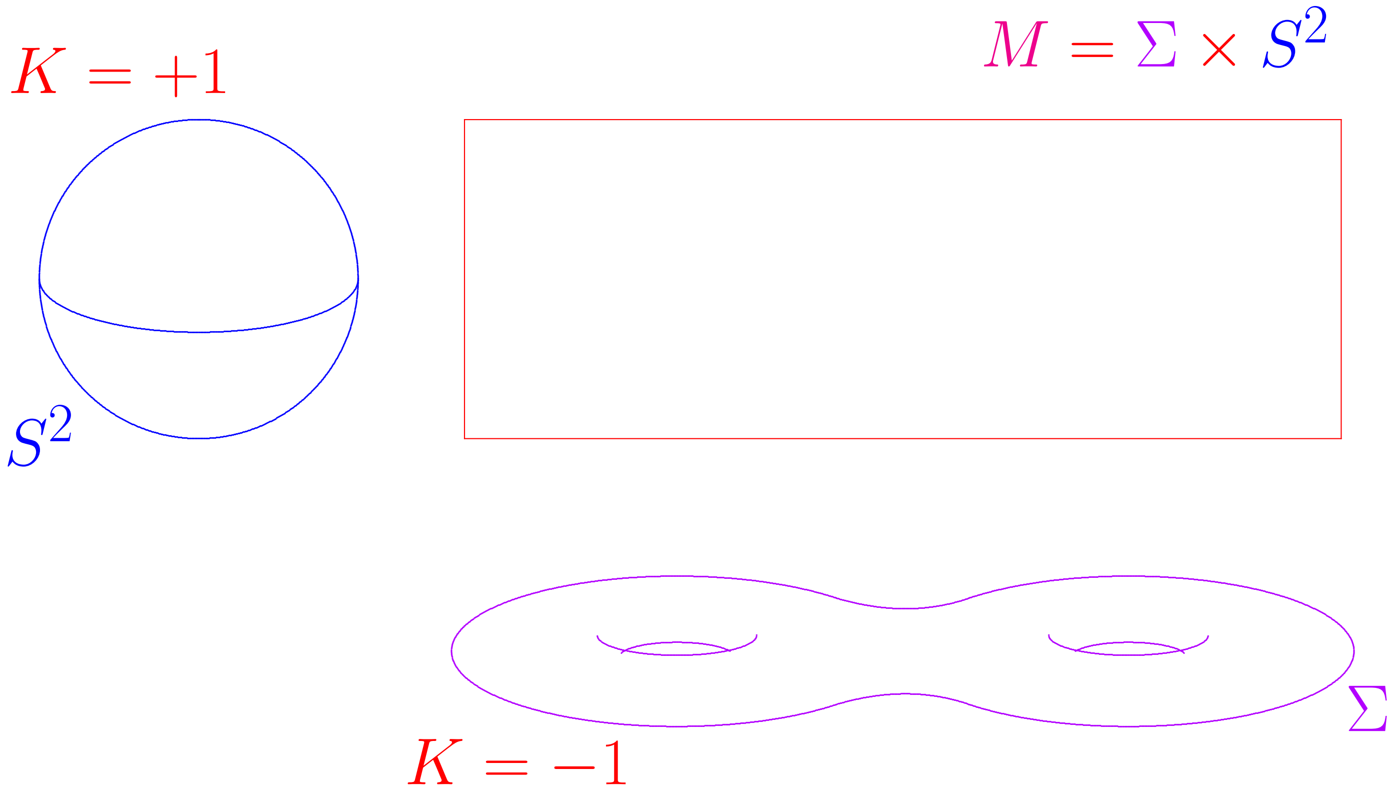

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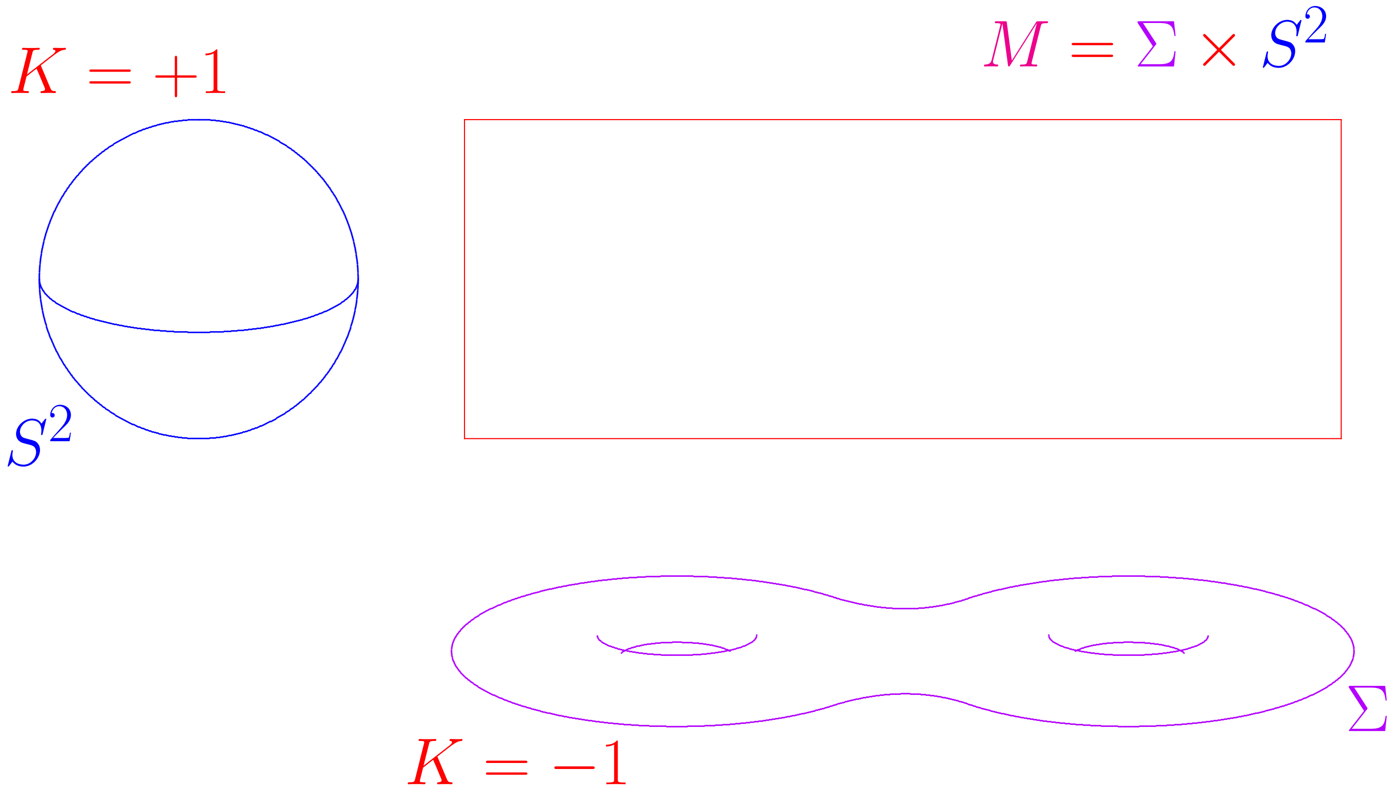

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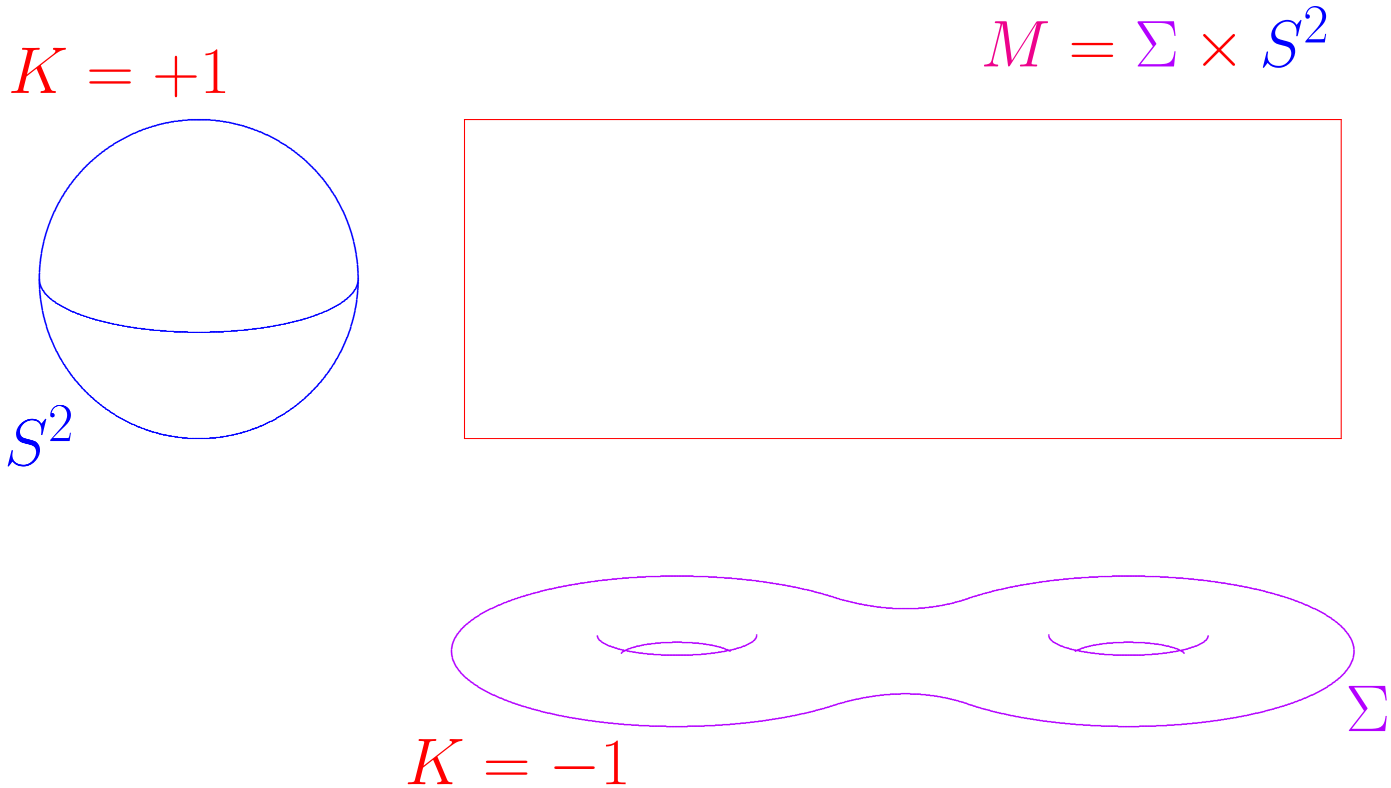

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Locally conformally flat!

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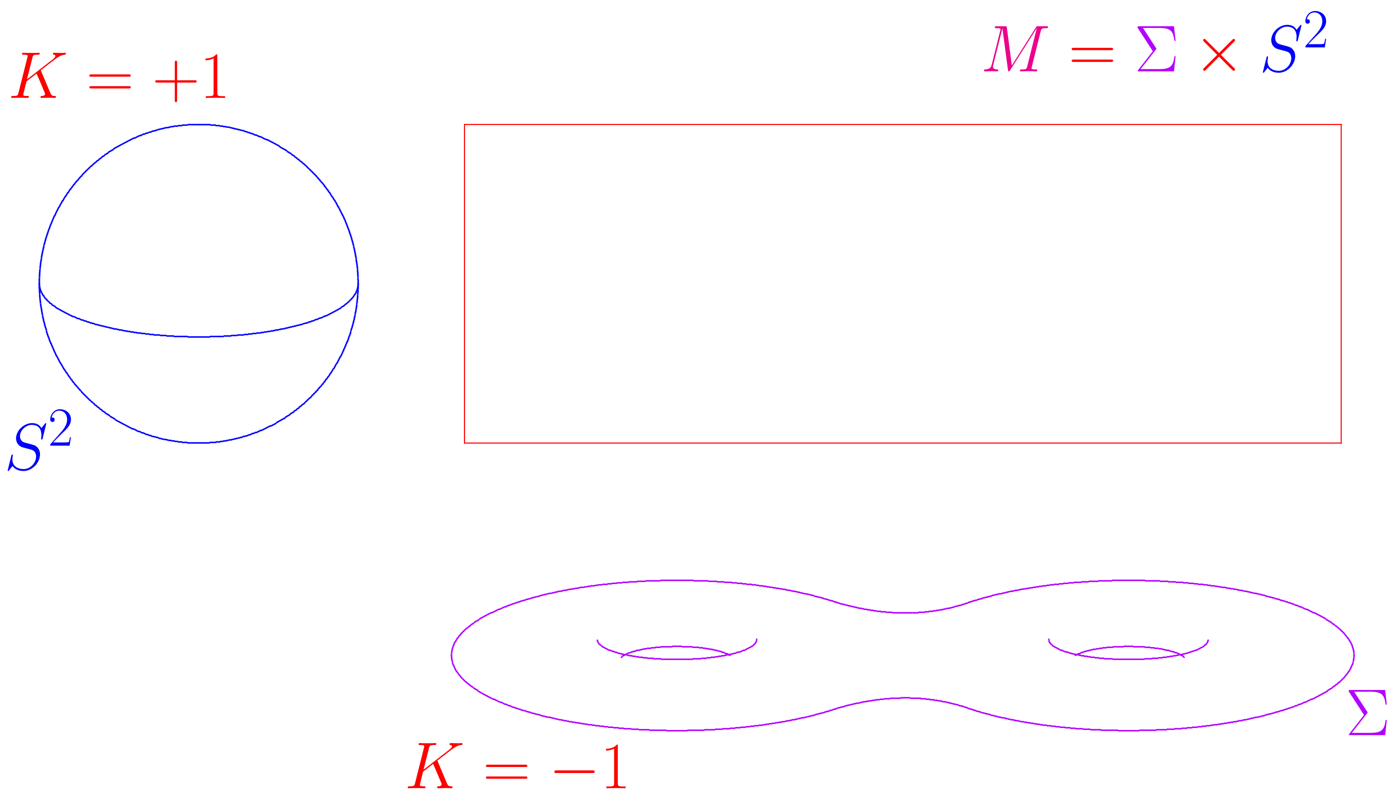

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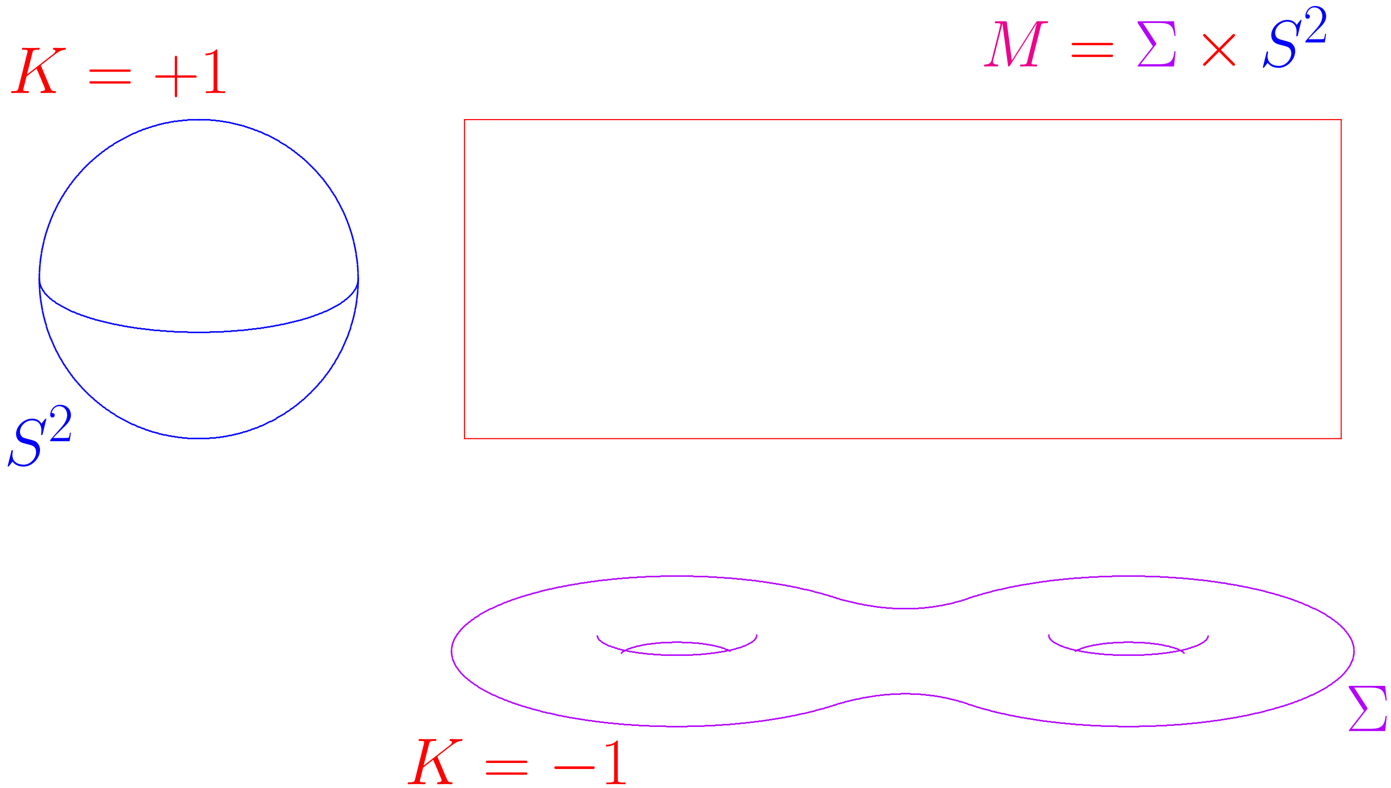

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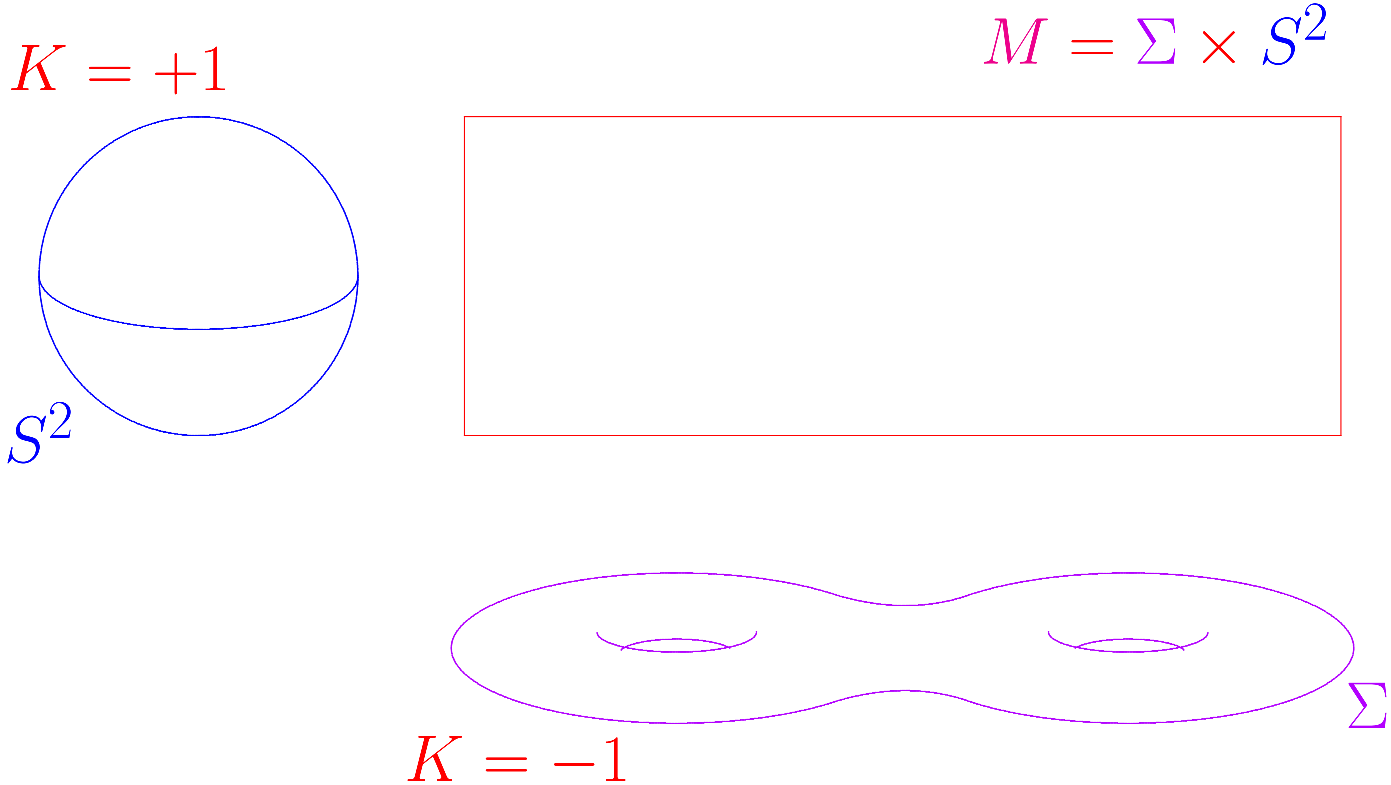

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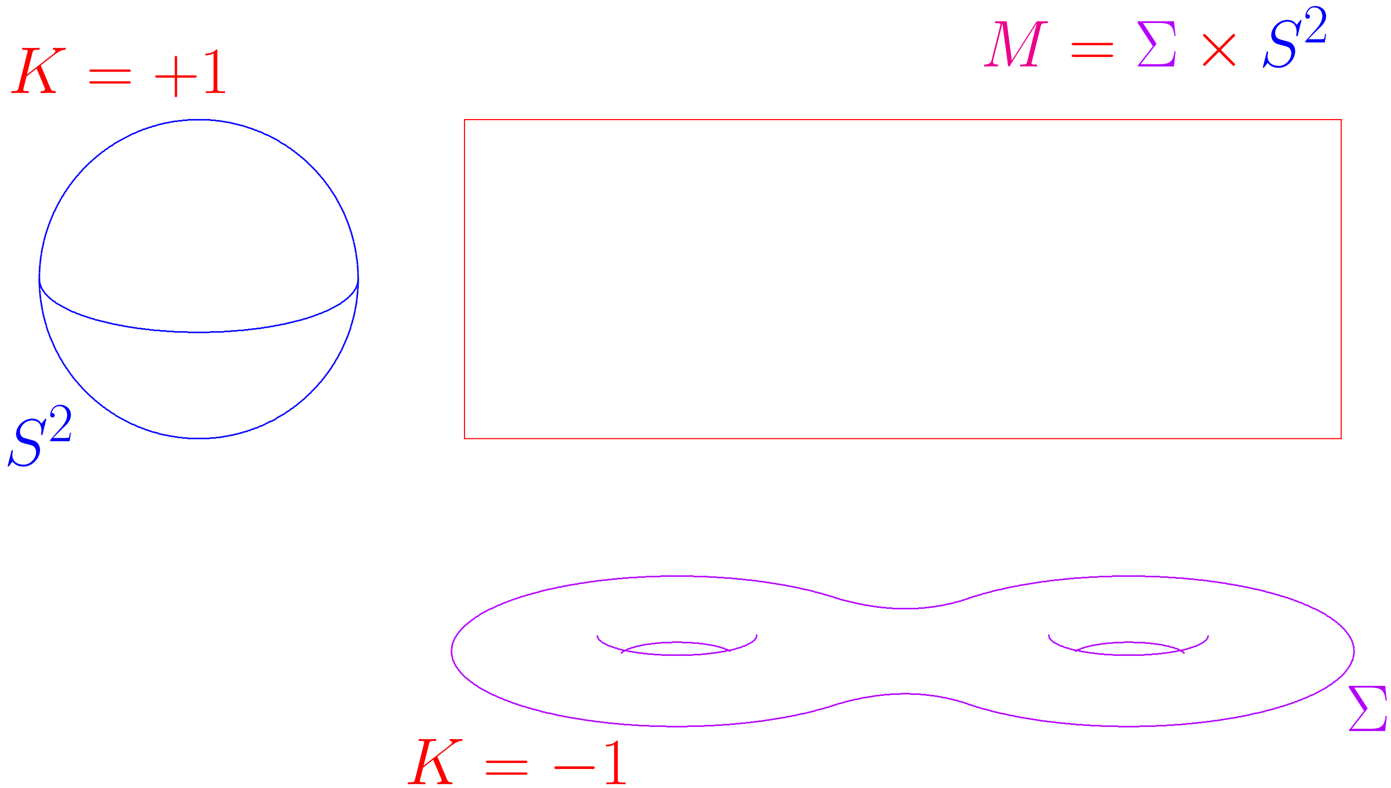

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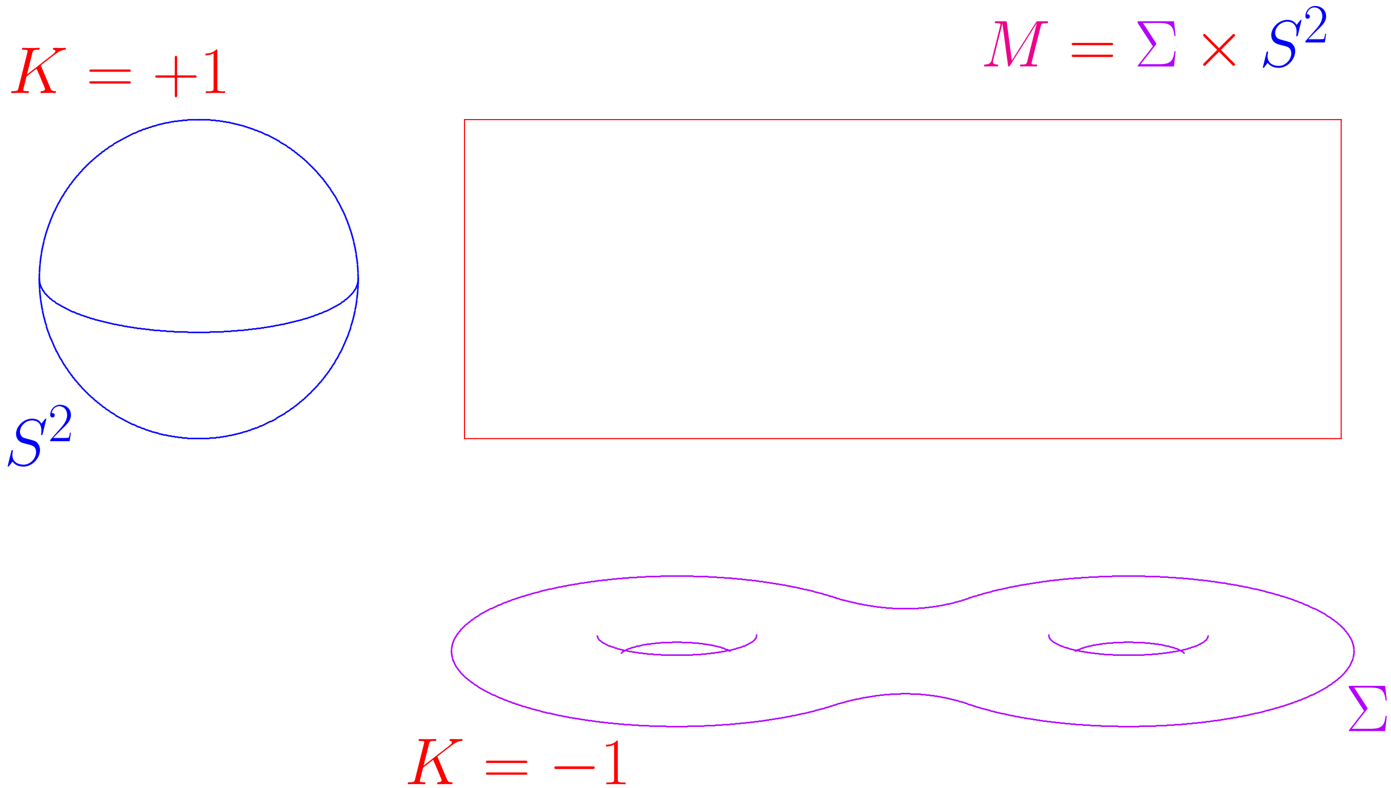

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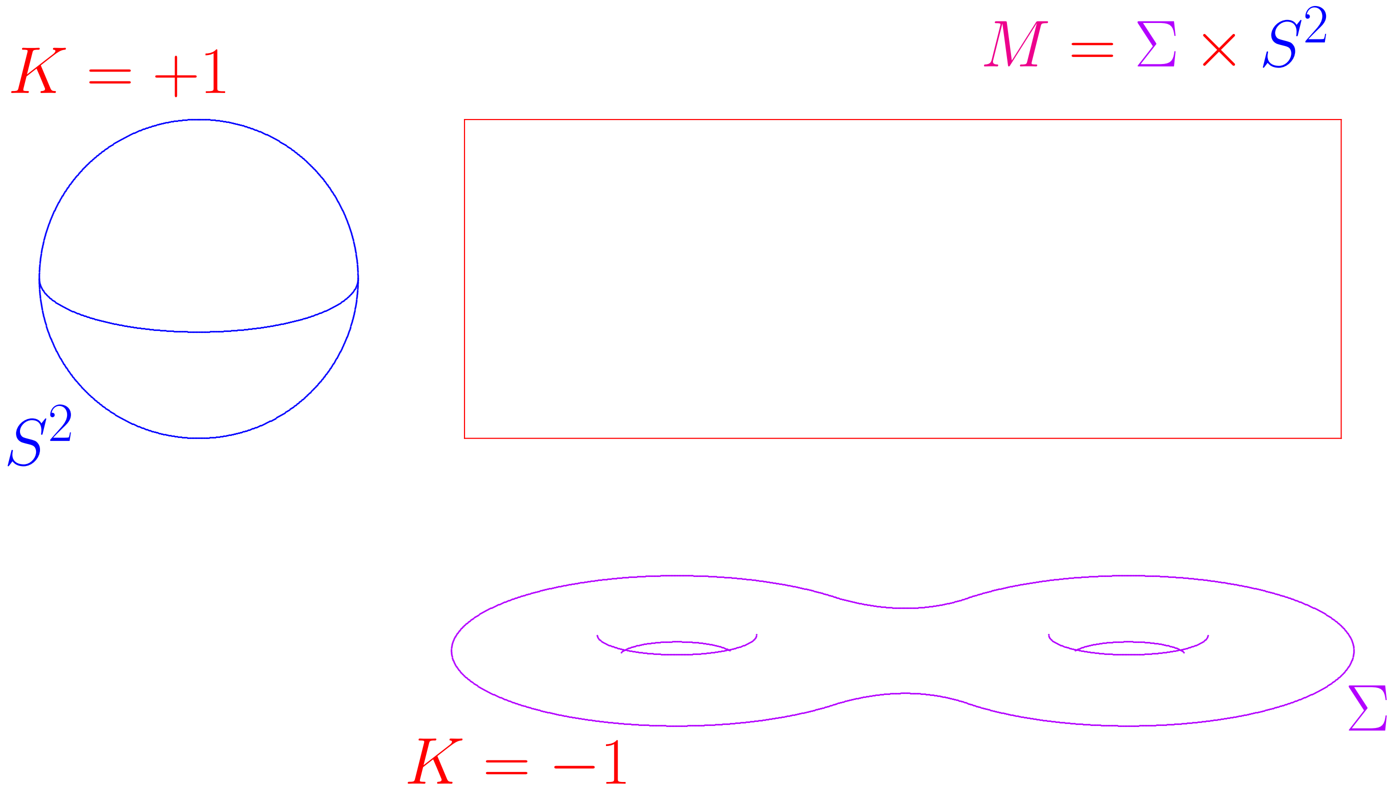

Scalar-flat Kähler deformations: $12(\mathfrak{g} - 1)$ moduli

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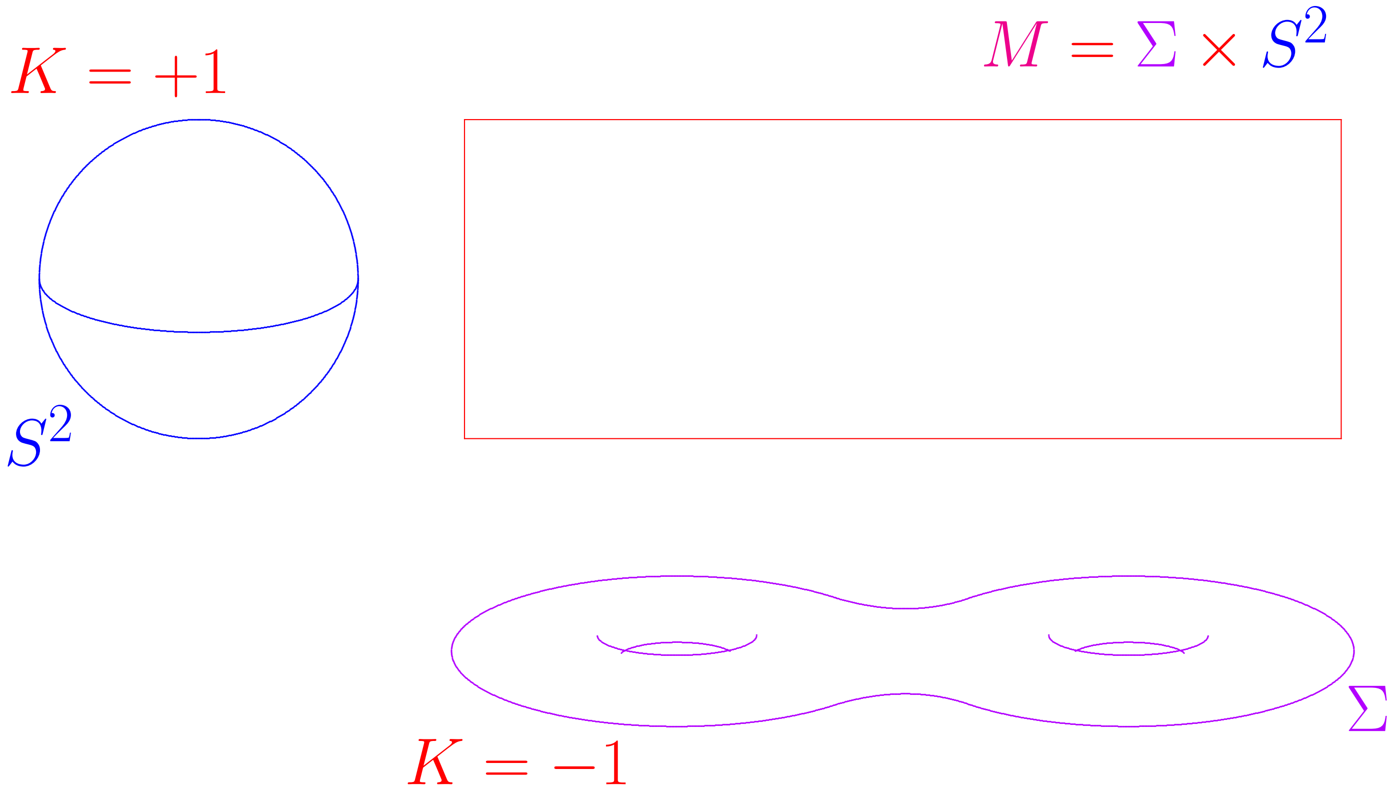

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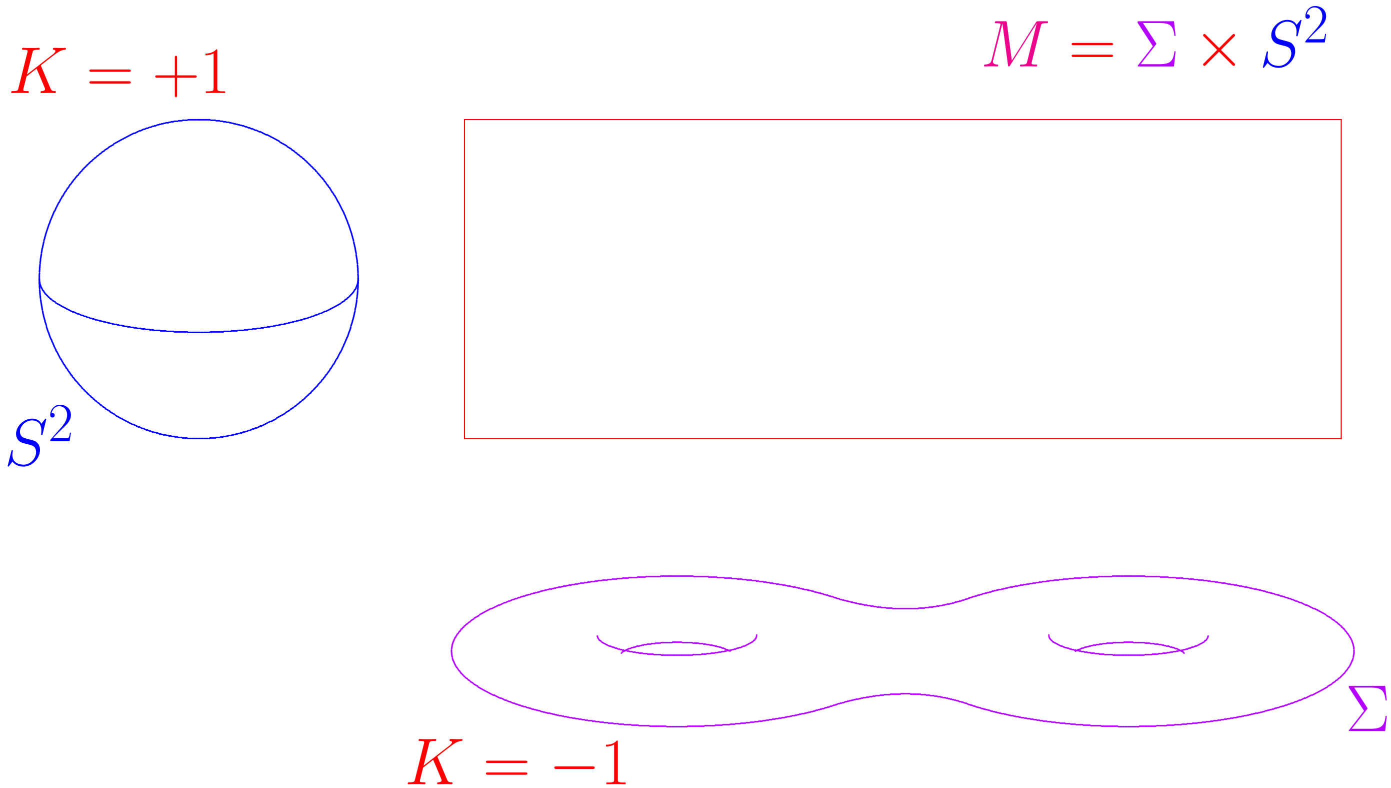

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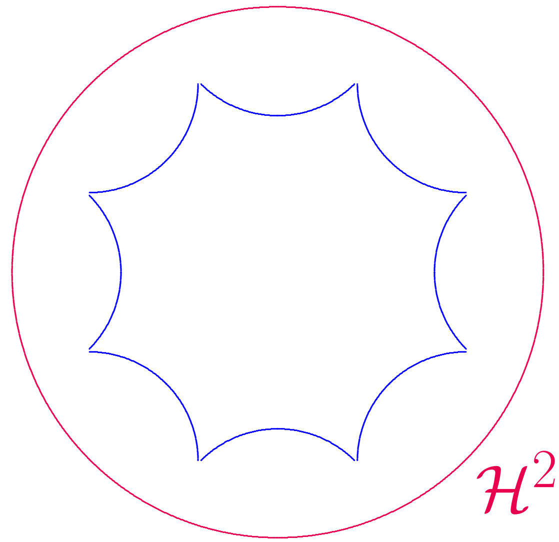
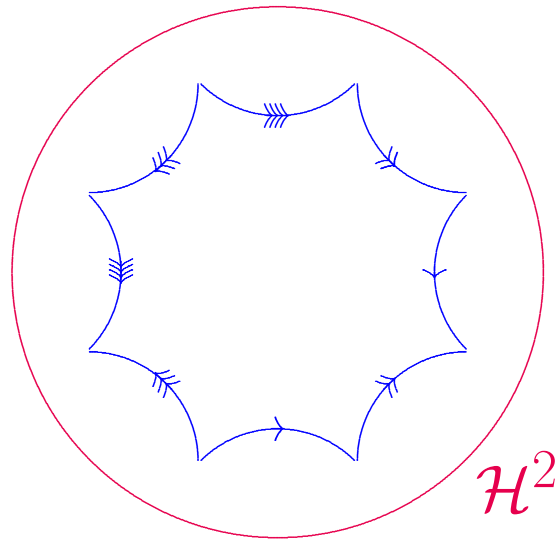
We begin by revisiting hyperbolic metrics on Σ.


$\mathcal{H}^2$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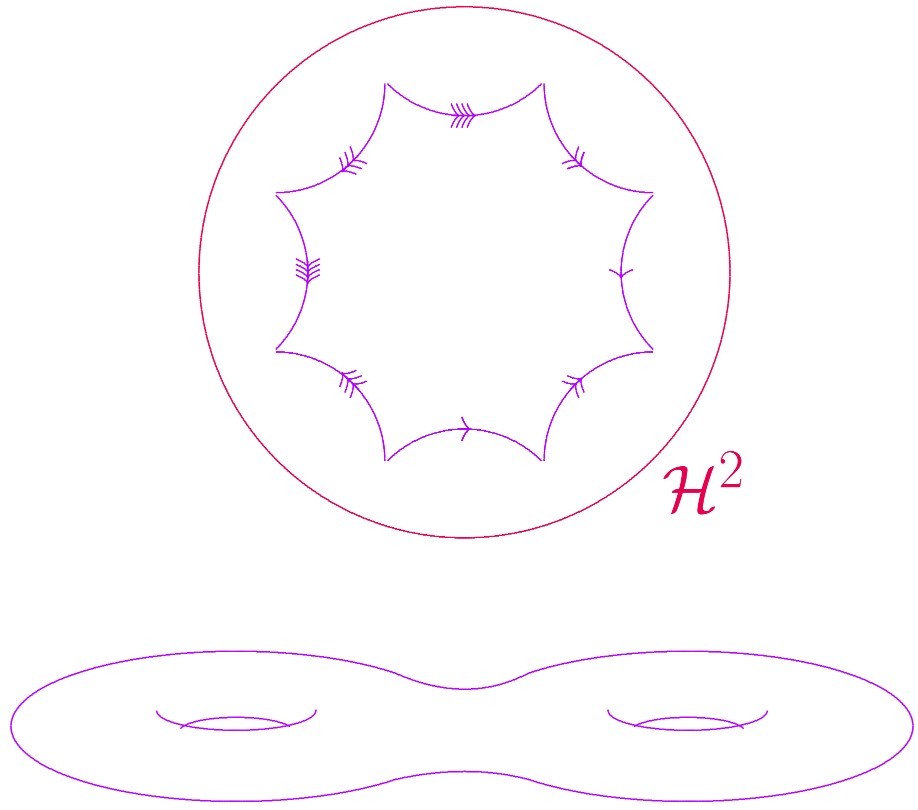
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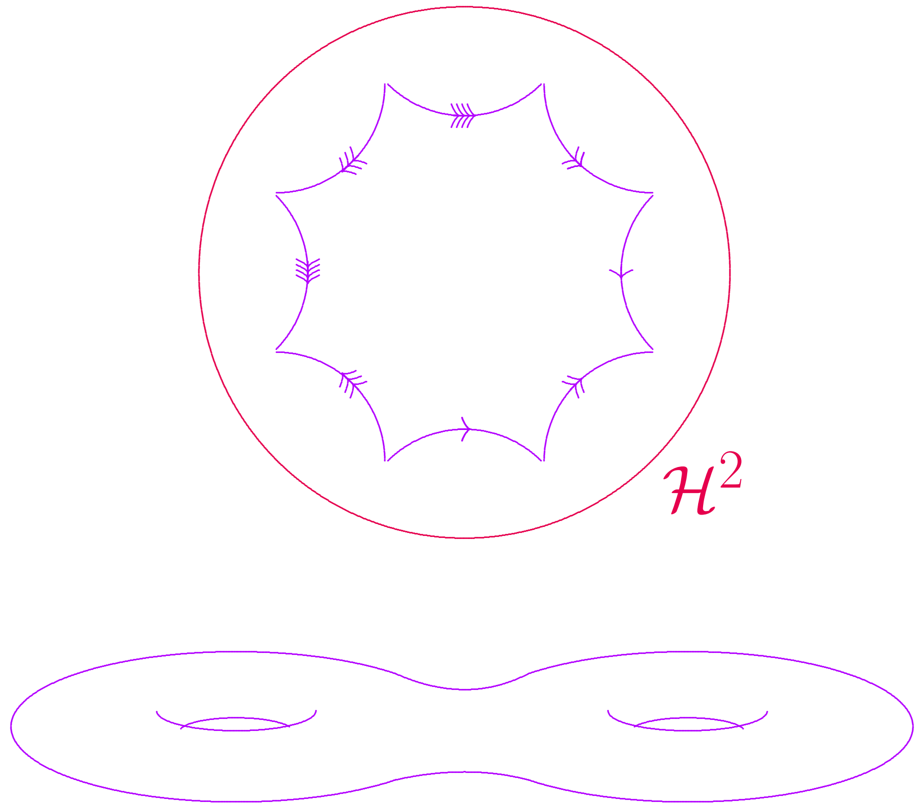

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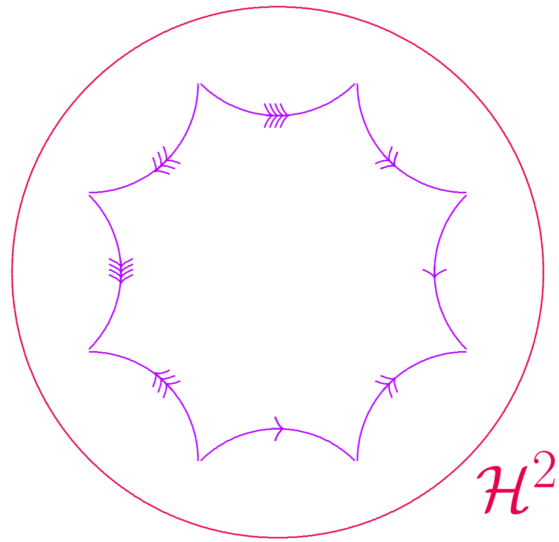

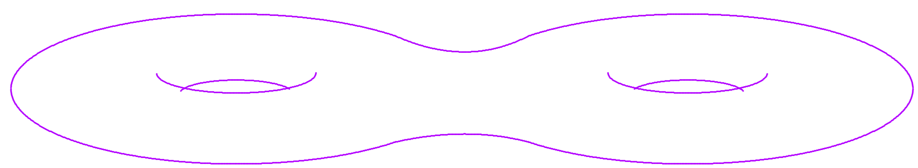

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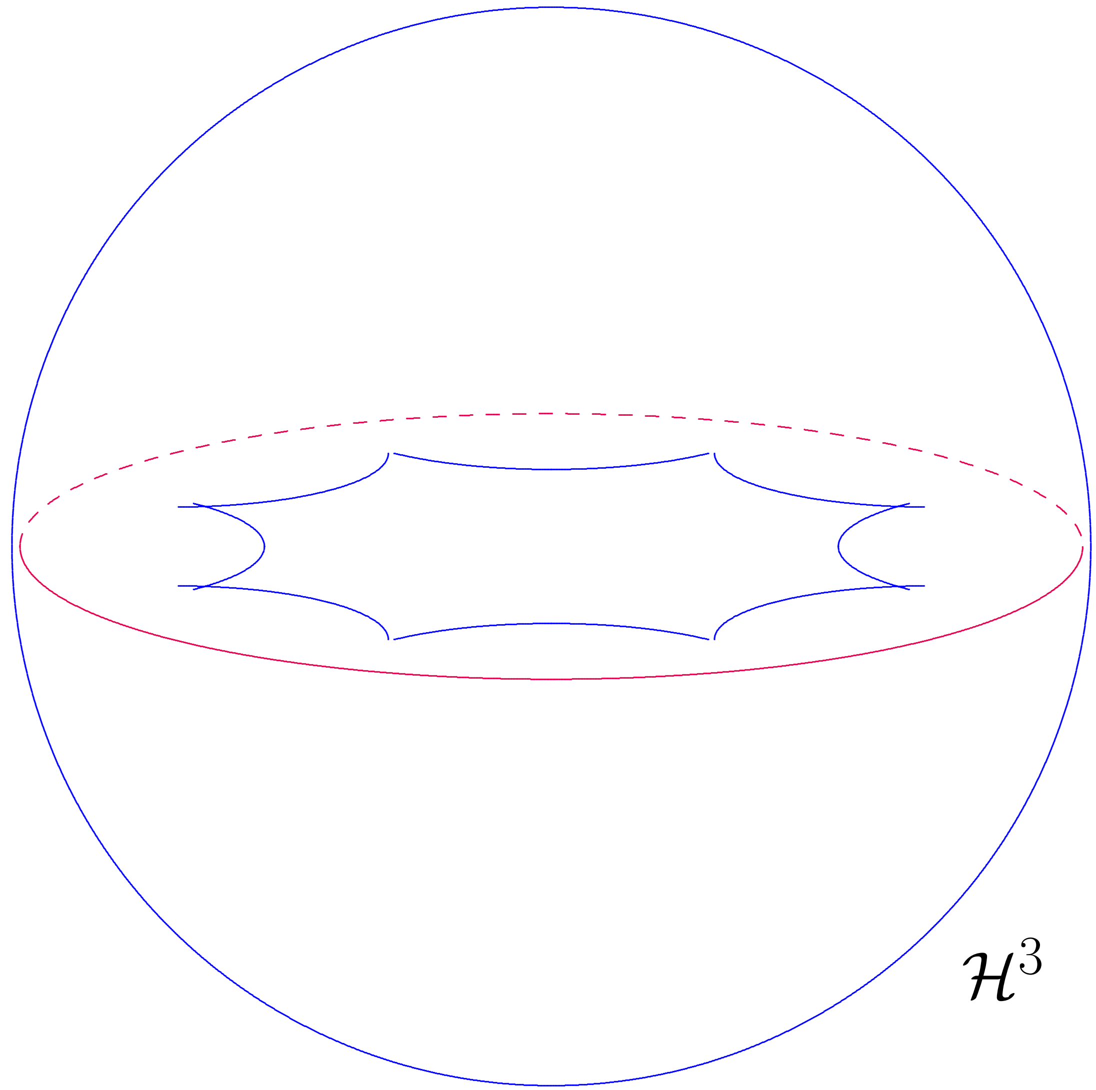
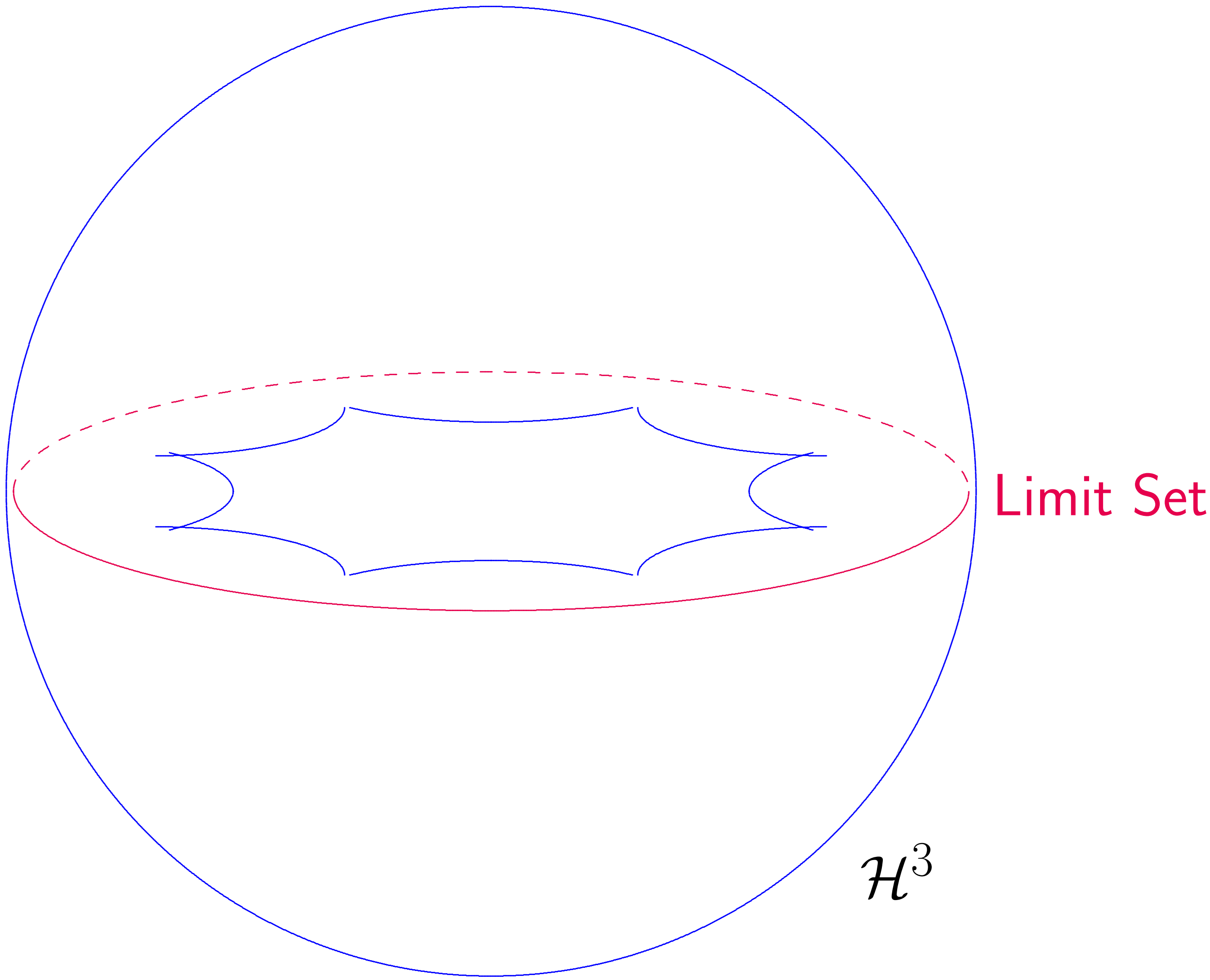
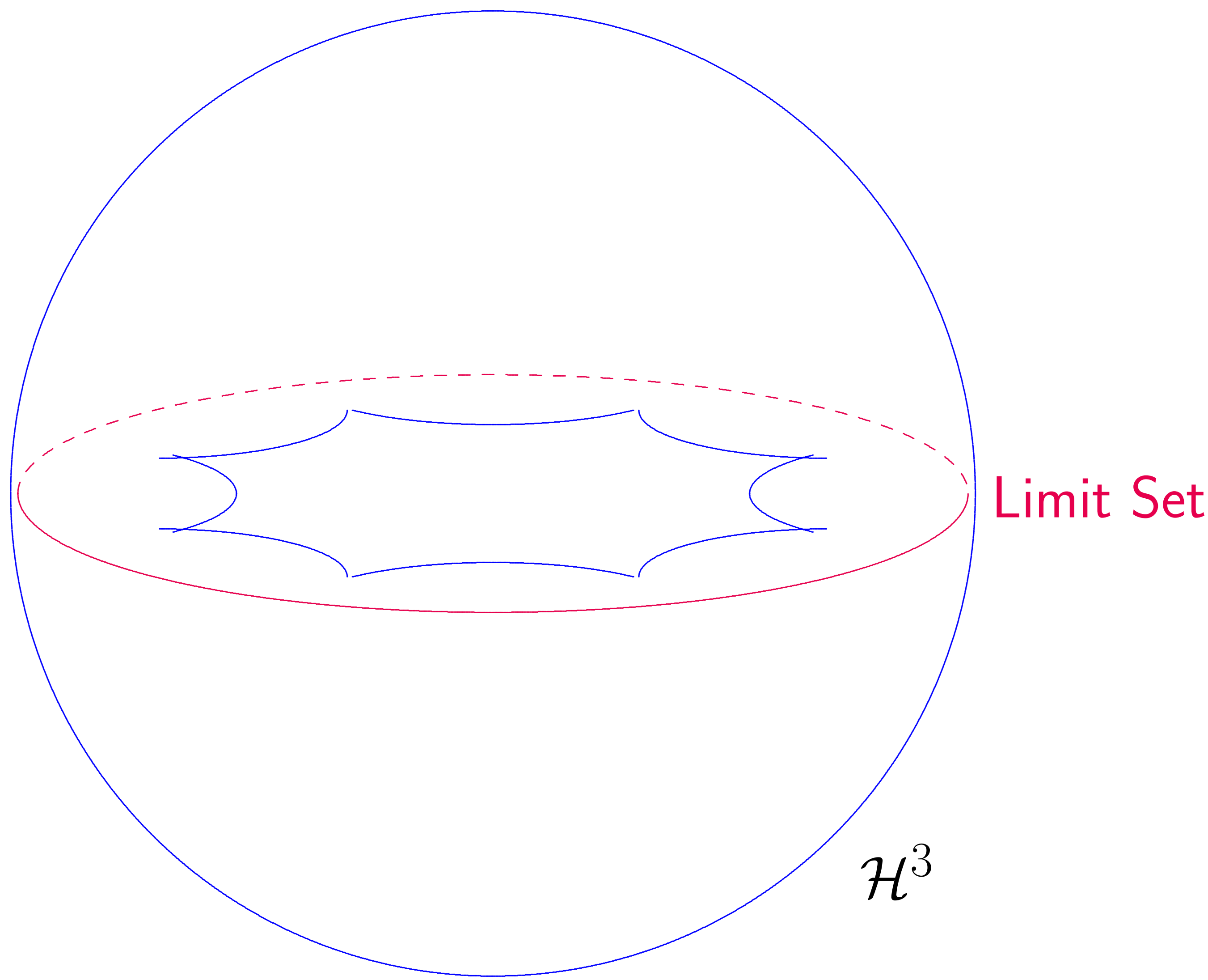
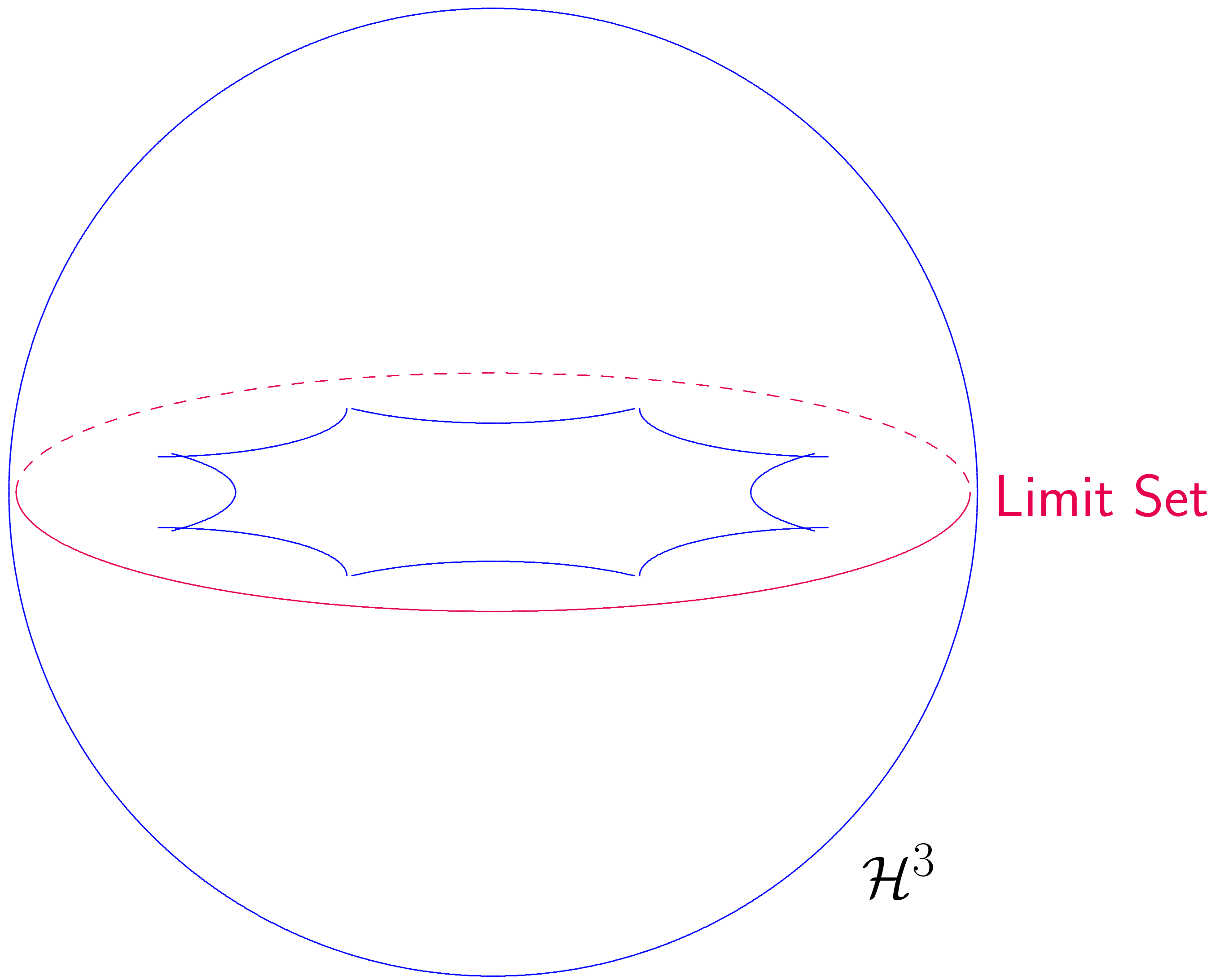

$\mathcal{H}^3$


Limit Set
$\mathcal{H}^3$



$\pi_1(\Sigma) \xrightarrow{\cong} \Gamma \subset \mathbf{PSL}(2, \mathbb{R})$ Fuchsian group



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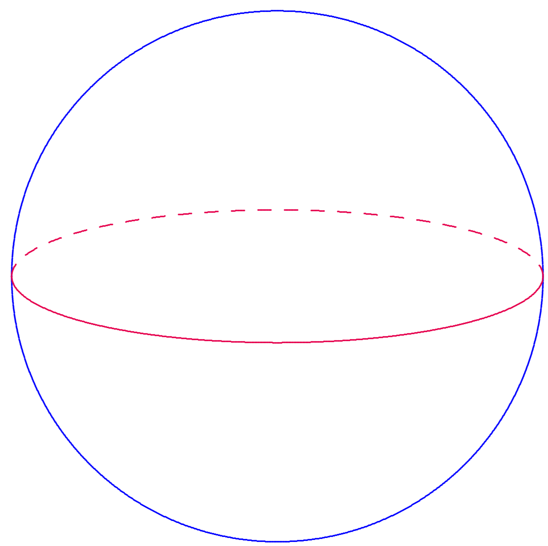
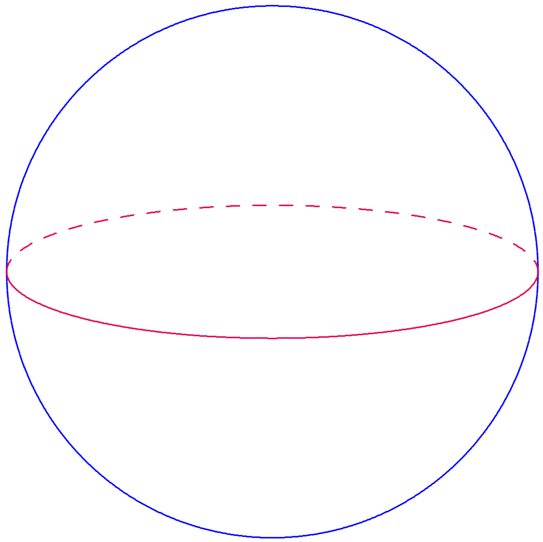
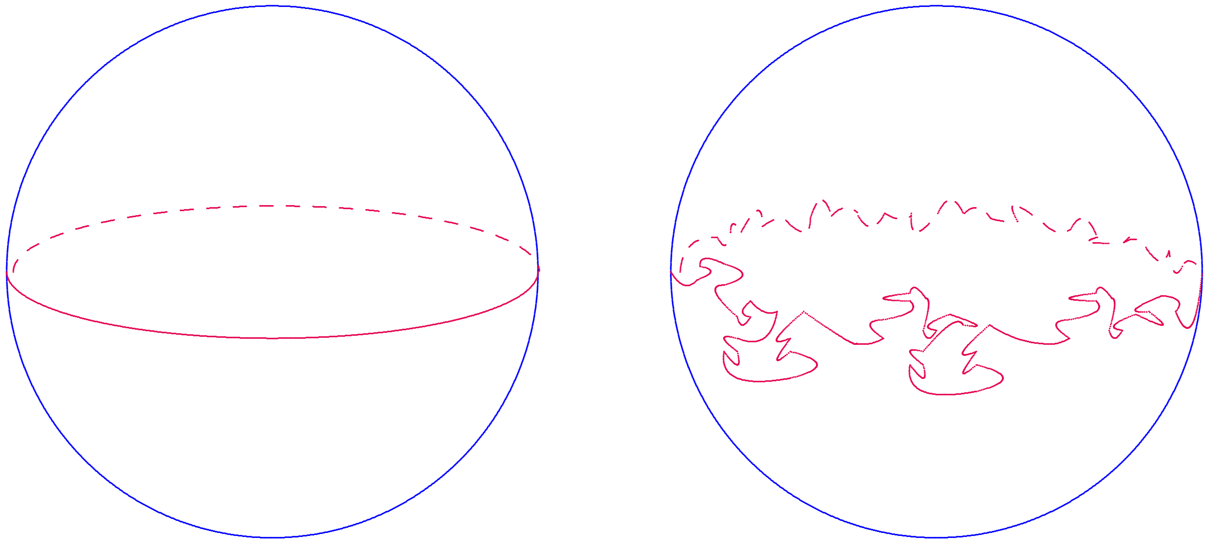
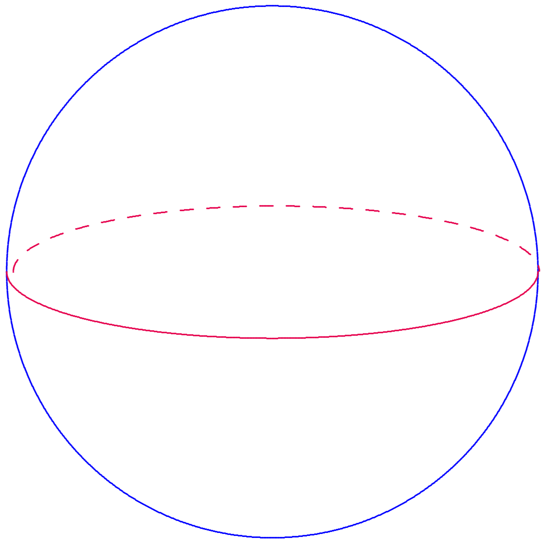




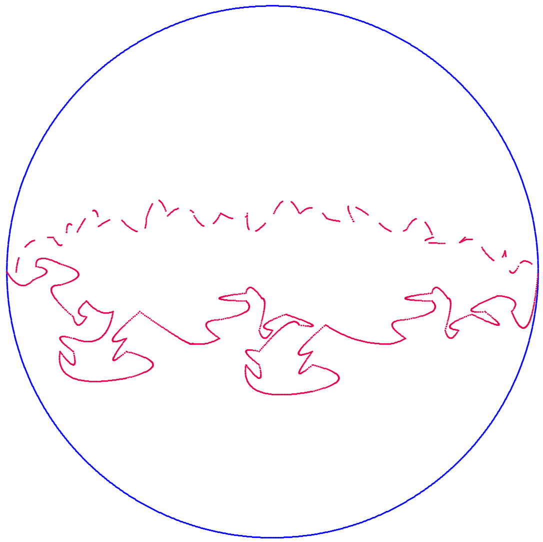
Fuchsian



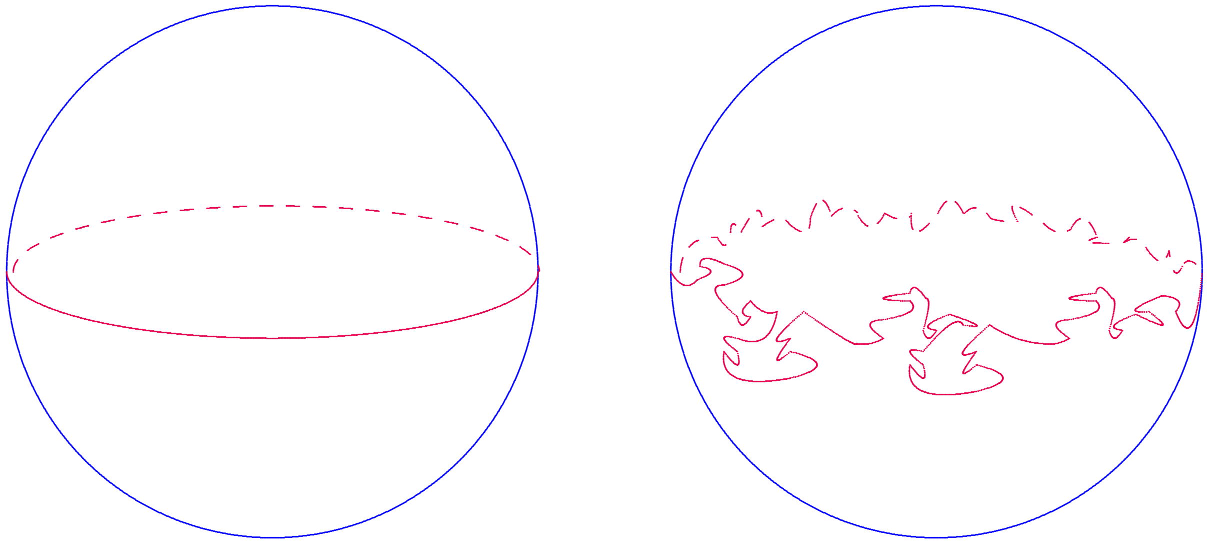
Fuchsian



Fuchsian

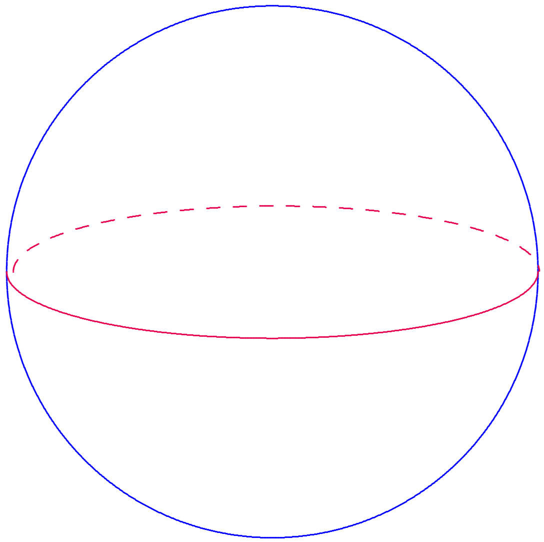


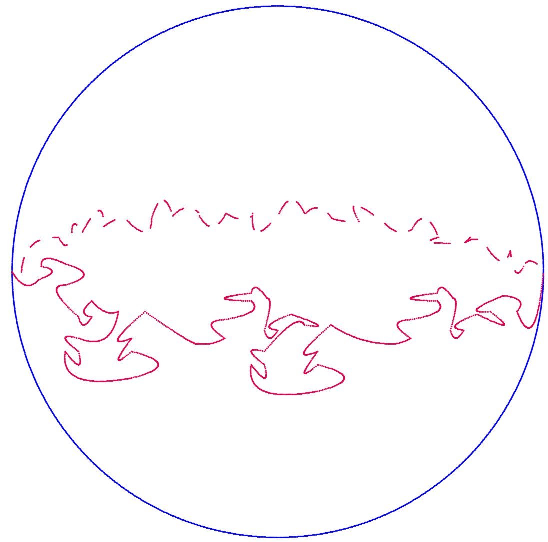
quasi-Fuchsian



Fuchsian quasi-Fuchsian

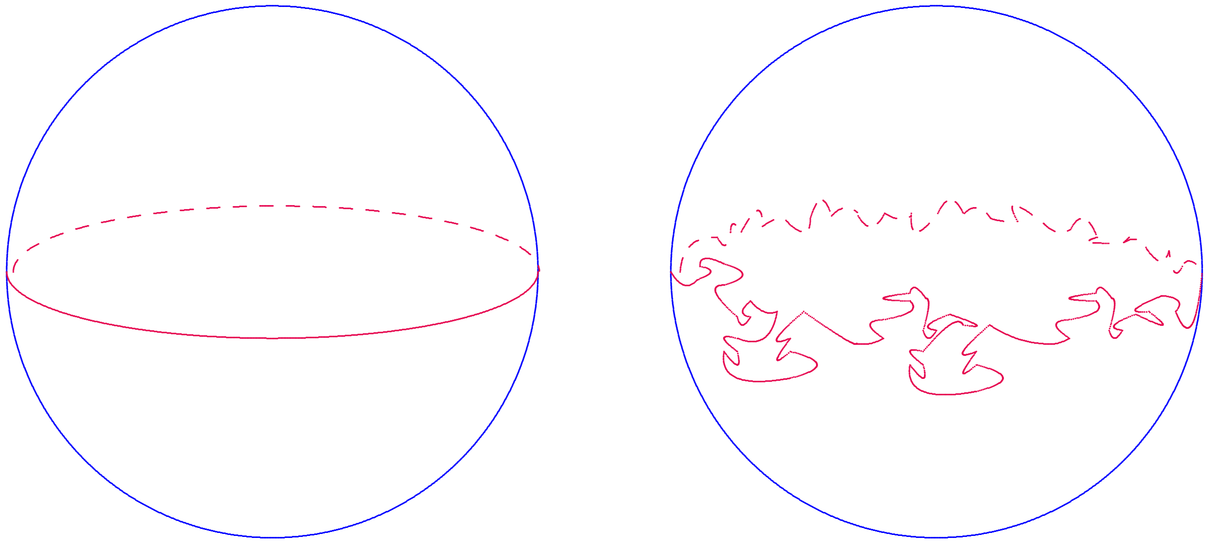
$$\pi_1(\Sigma) \xrightarrow{\cong} \Gamma \subset \mathbf{PSL}(2, \mathbb{C})$$ quasi-Fuchsian group





Fuchsian quasi-Fuchsian

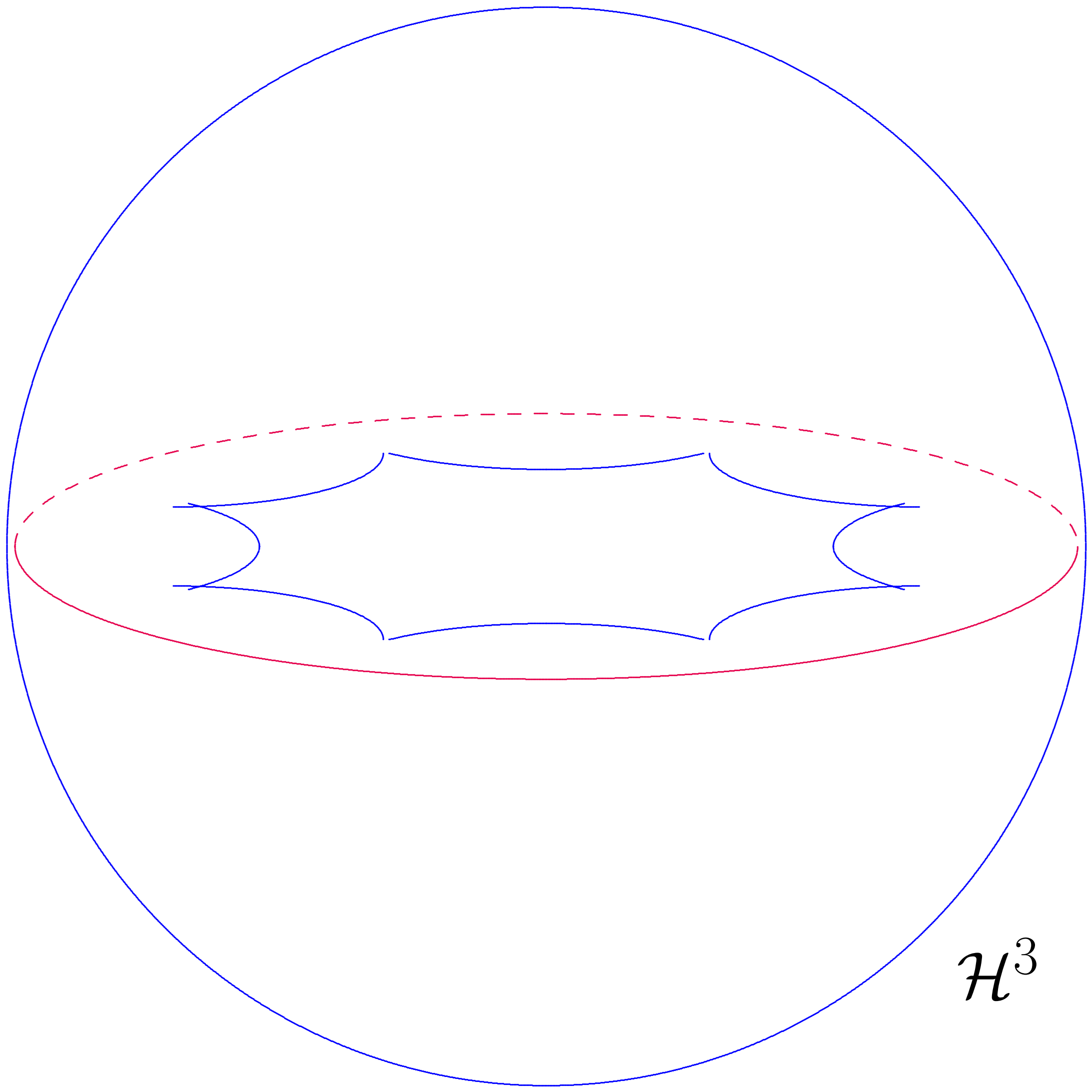
$$\pi_1(\Sigma) \xrightarrow{\cong} \Gamma \subset \mathbf{PSL}(2,\mathbb{C})$$ quasi-Fuchsian group of Bers type

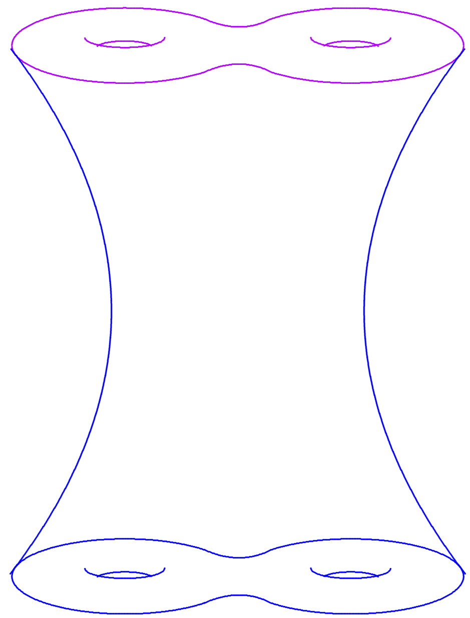


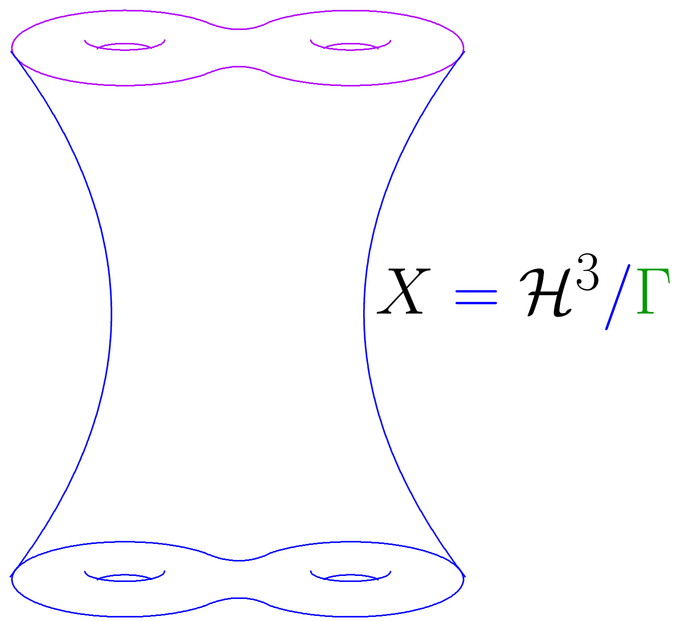
Fuchsian quasi-Fuchsian

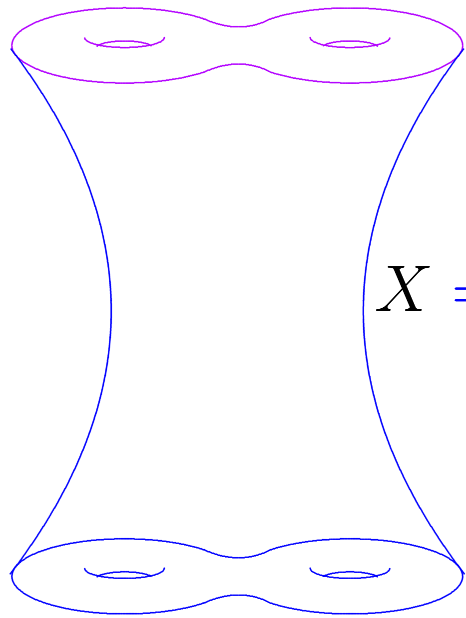
$\pi_1(\Sigma) \xrightarrow{\cong} \Gamma \subset \mathbf{PSL}(2, \mathbb{C})$ quasi-Fuchsian group of Bers type

Quasi-conformally conjugate to Fuchsian.


$\mathcal{H}^3$

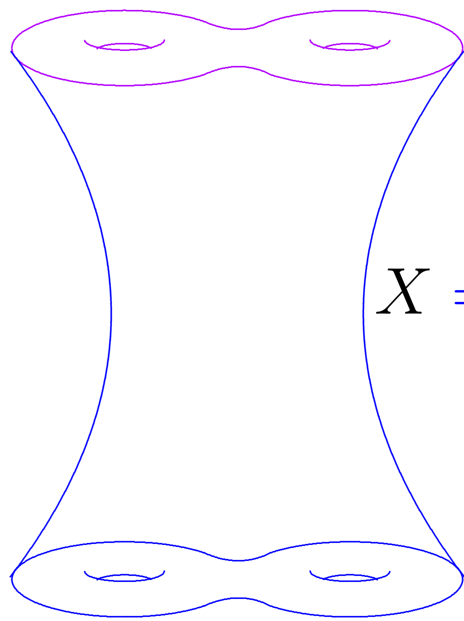



$X = \mathcal{H}^3/\Gamma$



$$X = \mathcal{H}^3/\Gamma$$

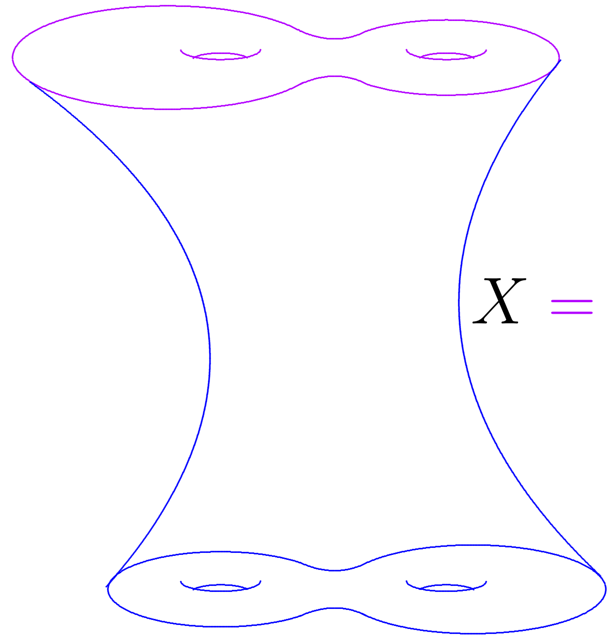
Γ Fuchsian



$X = \mathcal{H}^3/\Gamma$

Γ Fuchsian

$X \approx \Sigma \times \mathbb{R}$



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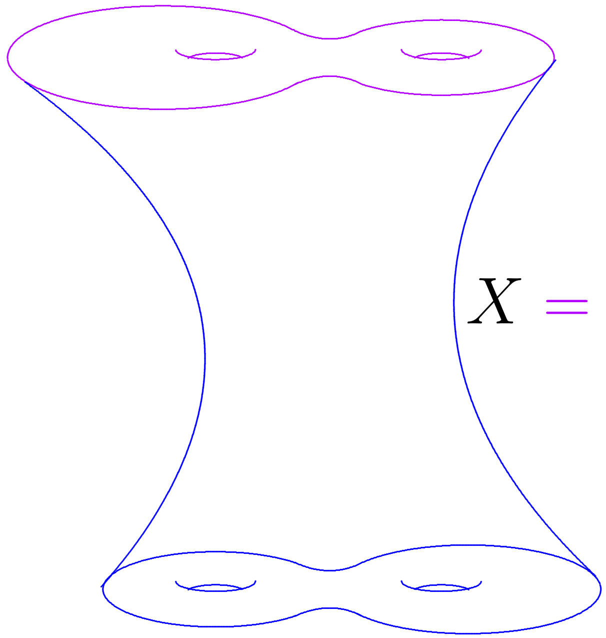
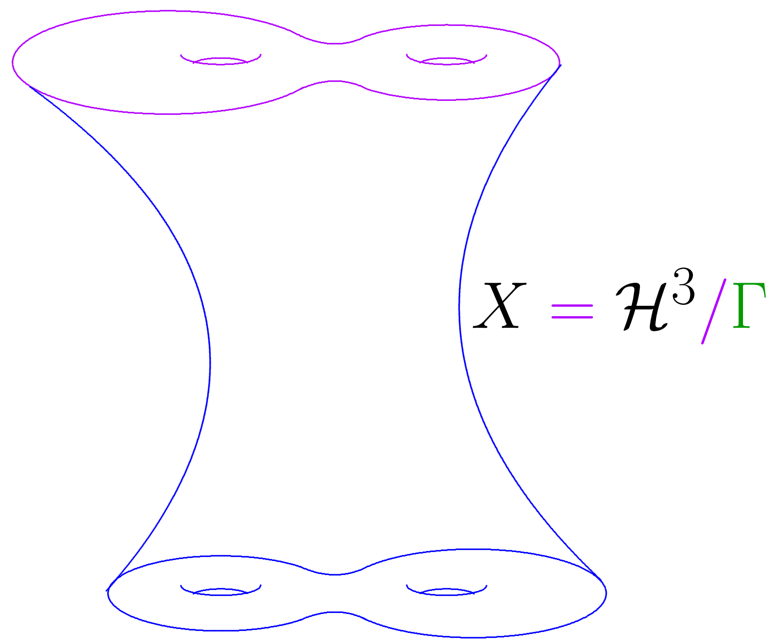


$X = \mathcal{H}^3/\Gamma$

Γ quasi-Fuchsian

$X \approx \Sigma \times \mathbb{R}$

Freedom: two points in Teichmüller space.



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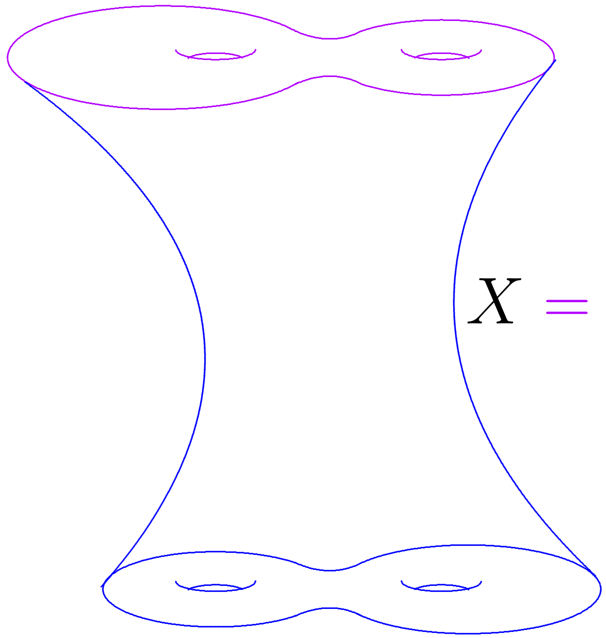

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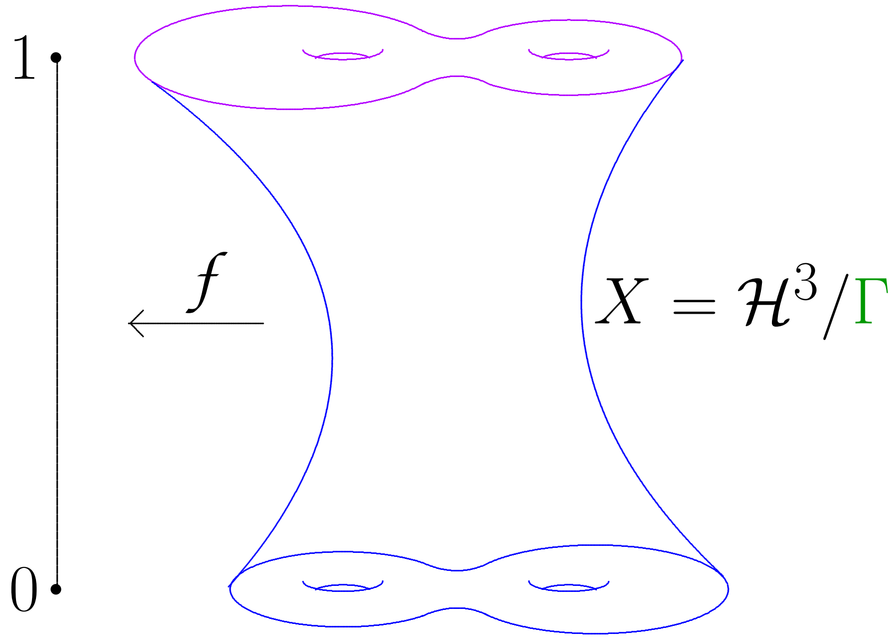
$$\overline{X} \approx \Sigma \times [0,1]$$



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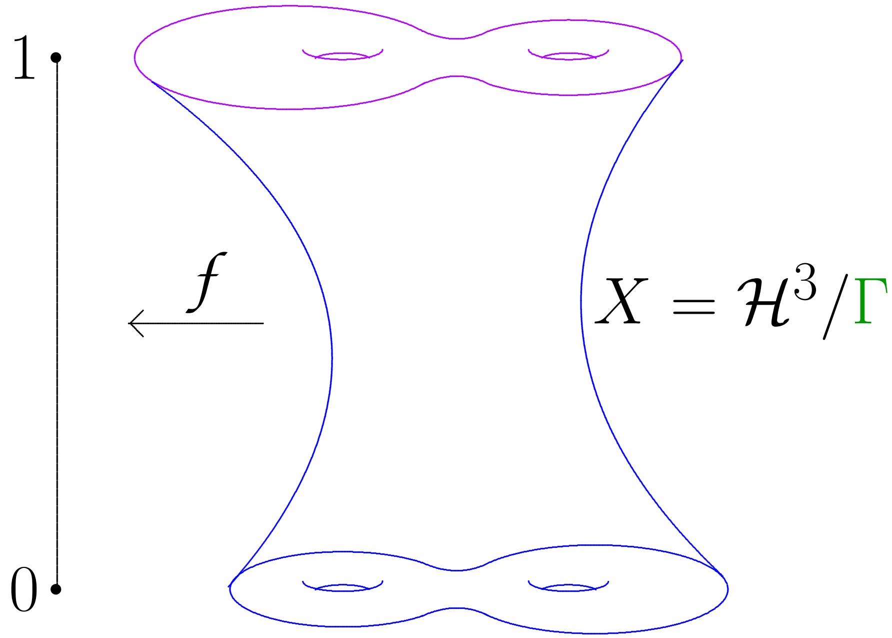
Tunnel-Vision function:

$$f : \overline{X} \to [0,1]$$



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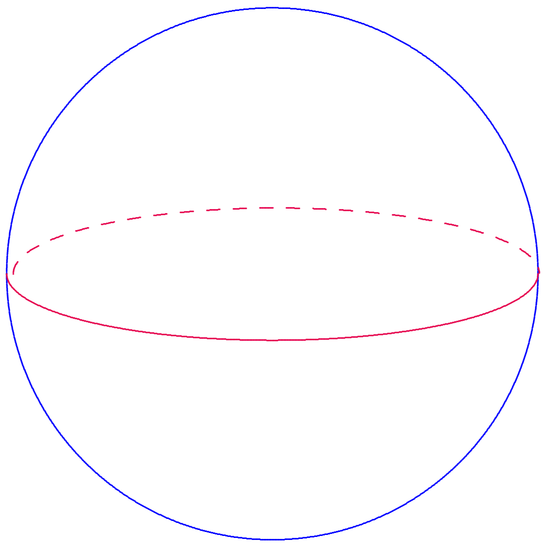
$$\Delta f = 0$$



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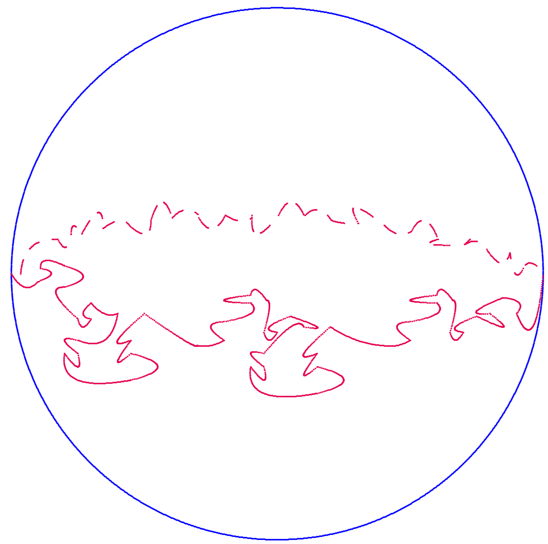
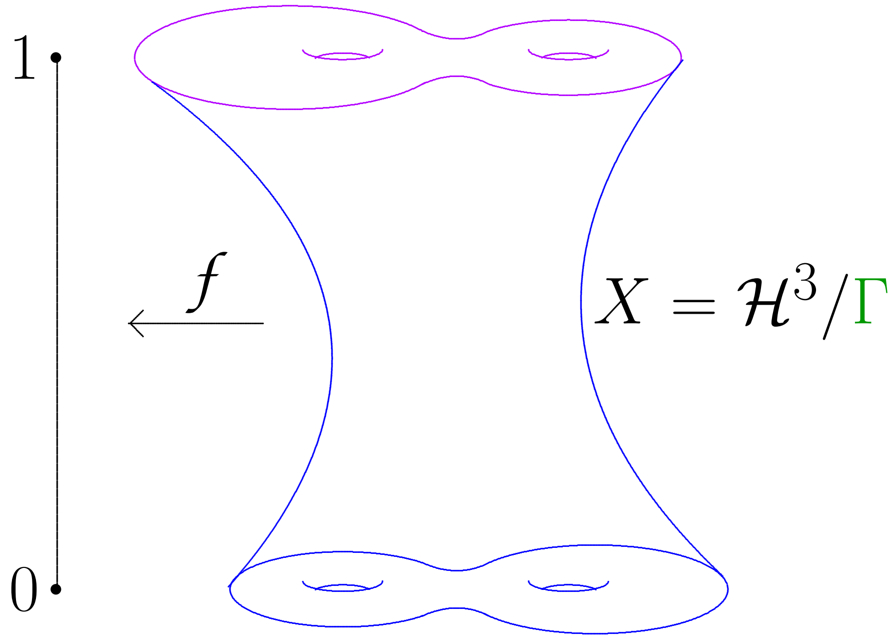


quasi-Fuchsian



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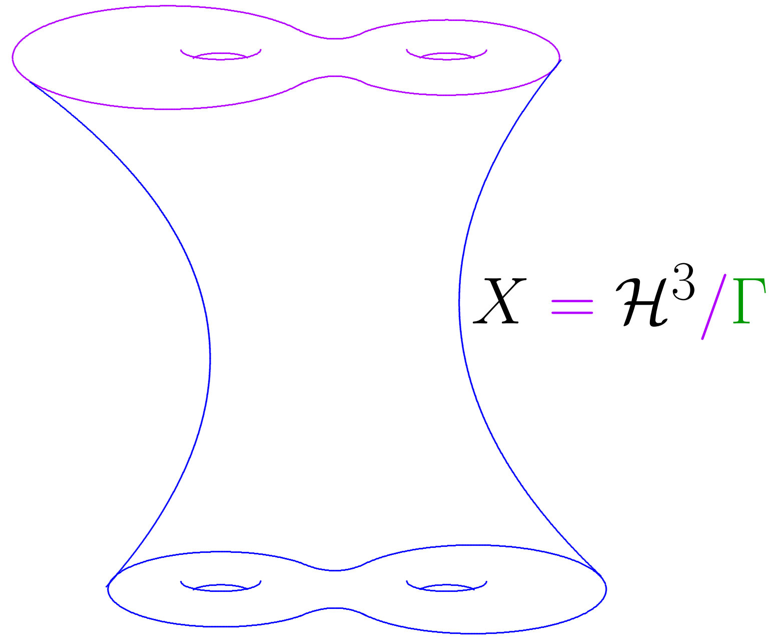
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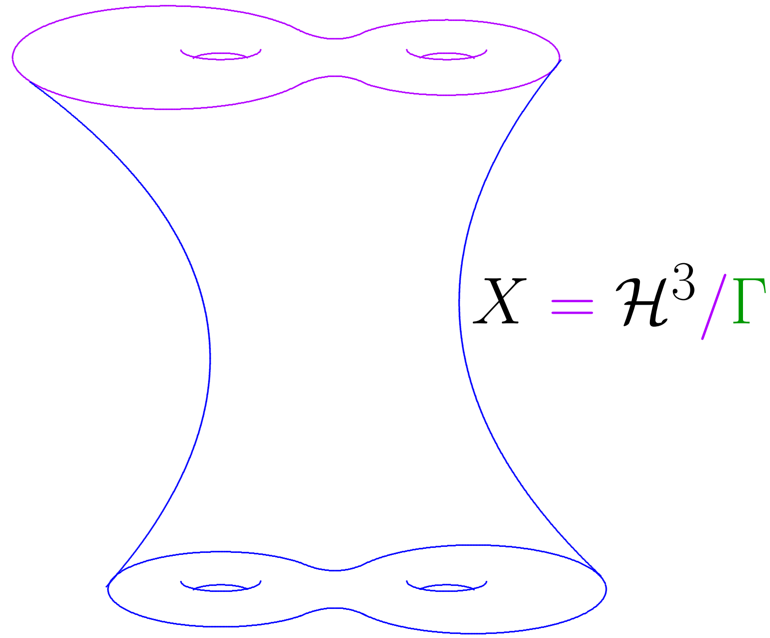
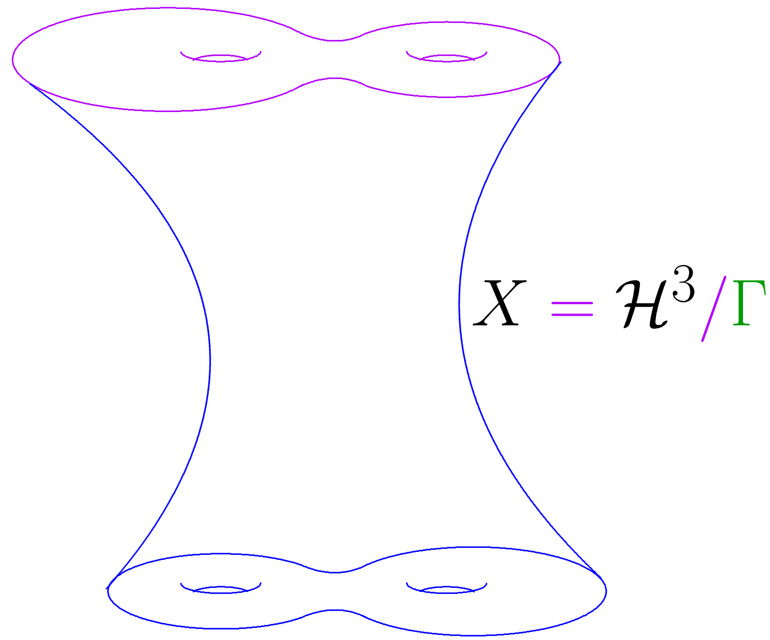
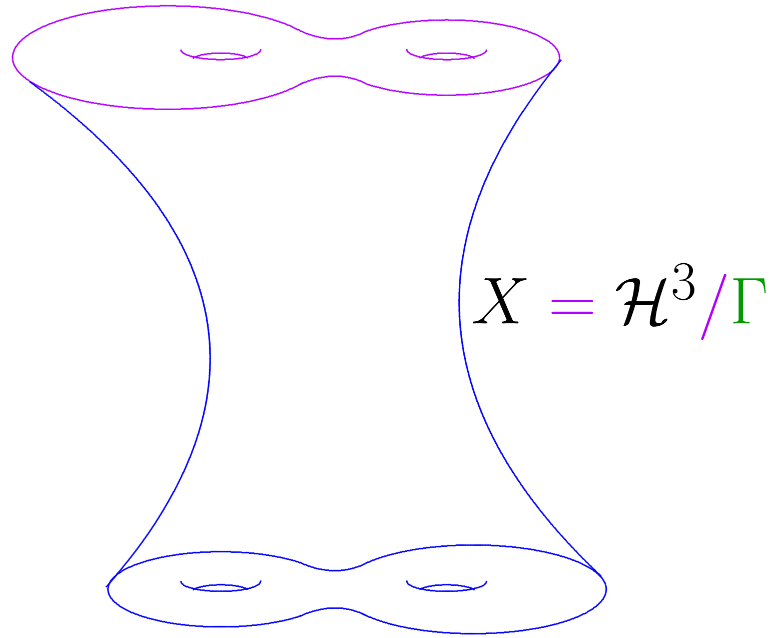


Construction of conformally flat 4-manifolds:



Construction of conformally flat 4-manifolds:

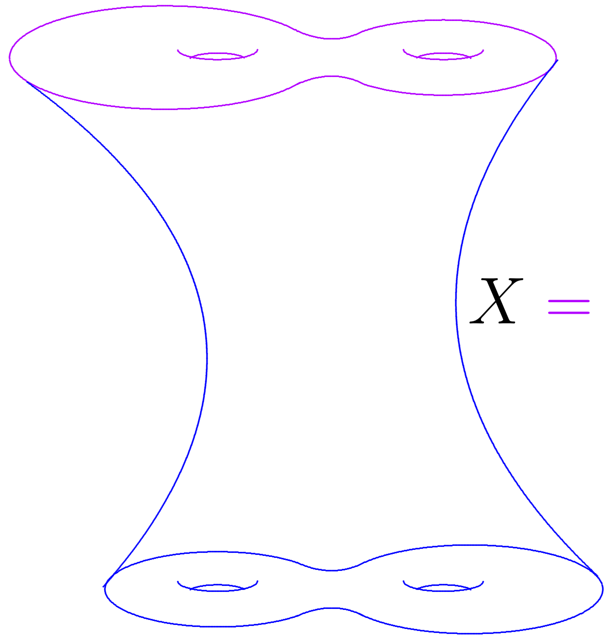
$$M = [\overline{X} \times S^1]/\sim$$



Construction of conformally flat 4-manifolds:

$$M = [\overline{X} \times S^1]/\sim$$

$\sim$: crush $\partial\overline{X} \times S^1$ to $\partial\overline{X}$.



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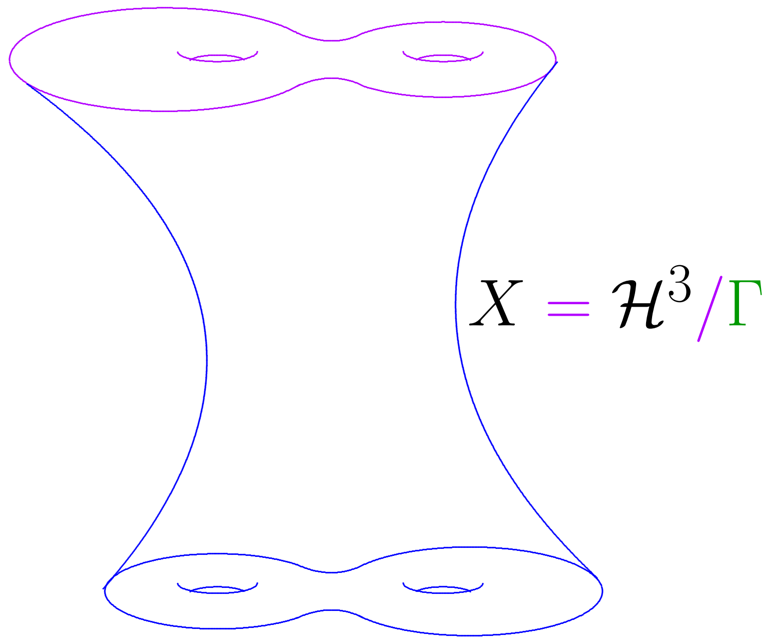
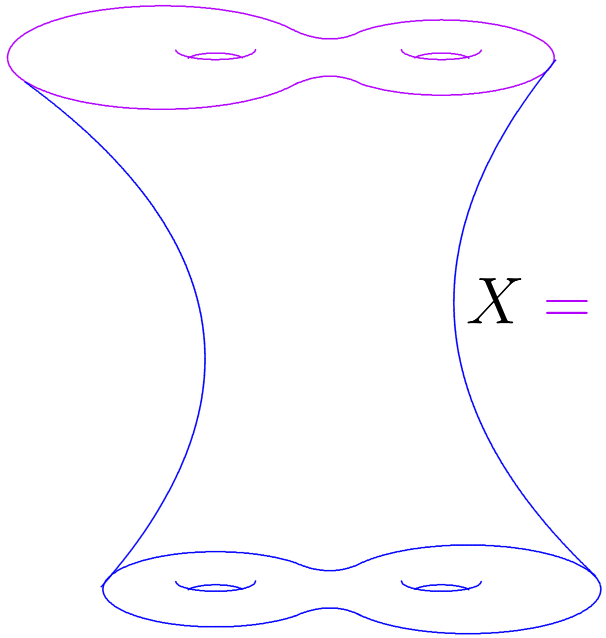


Construction of conformally flat 4-manifolds:

$$M = [\overline{X} \times S^1]/\sim$$

$$g = \qquad\qquad h + dt^2$$



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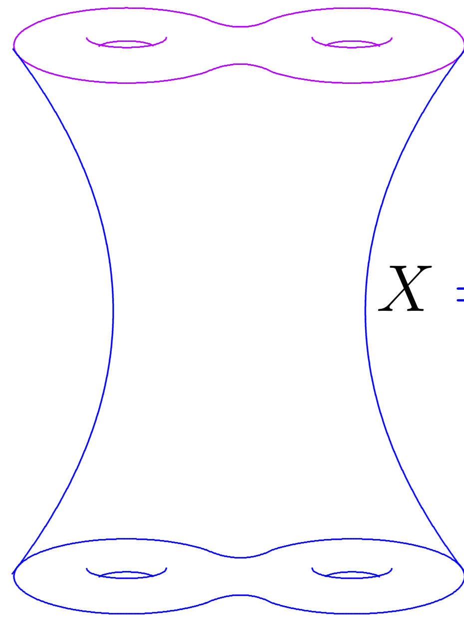

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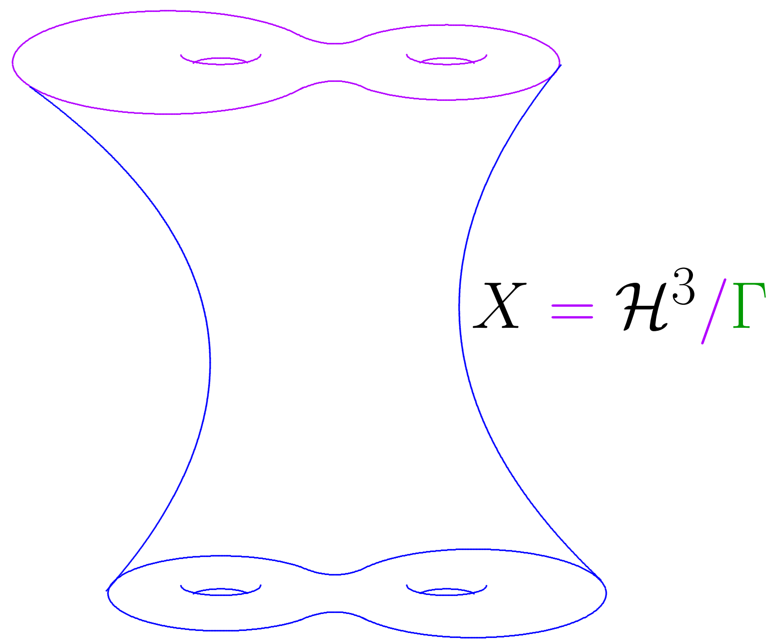
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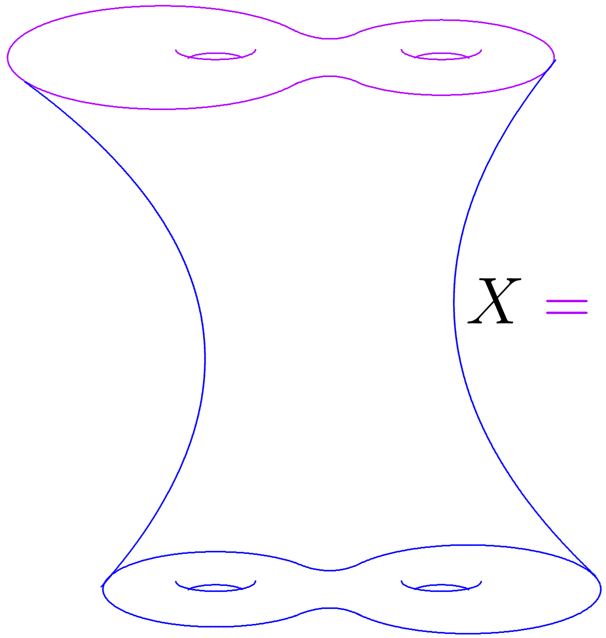

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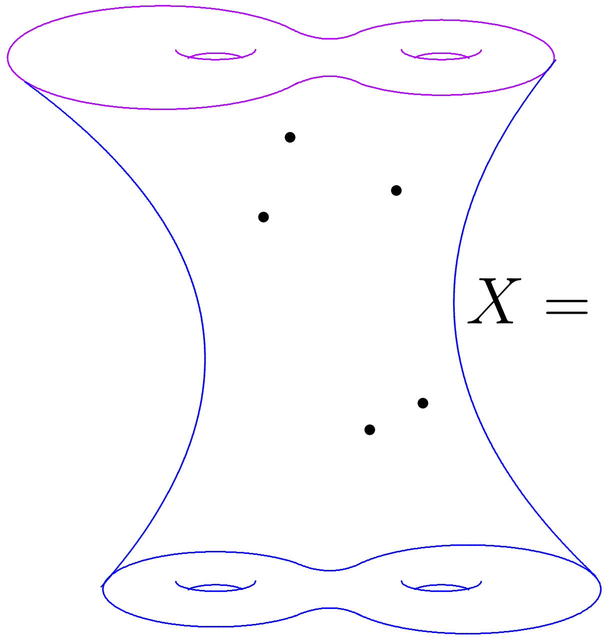
Construction of ASD 4-manifolds:



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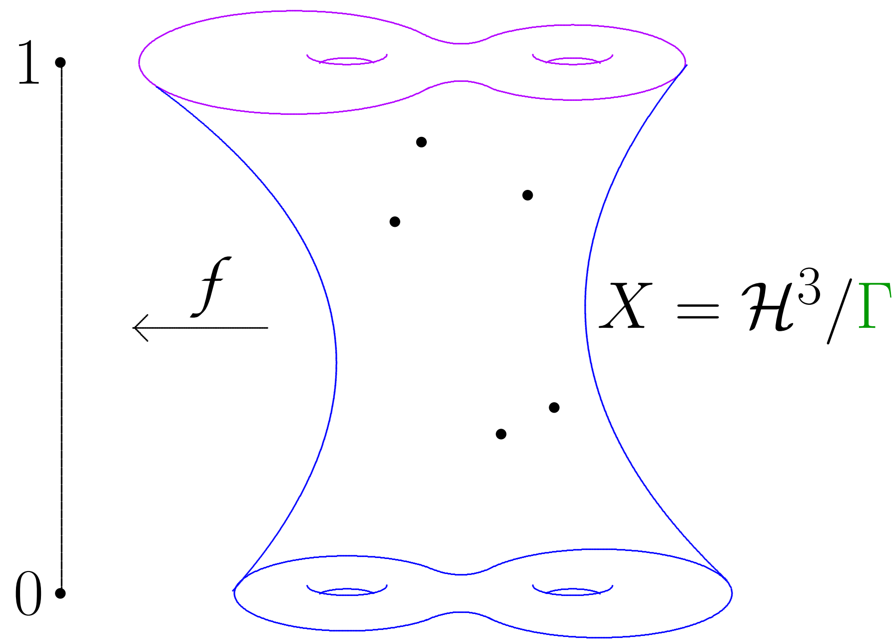
Construction of ASD 4-manifolds:

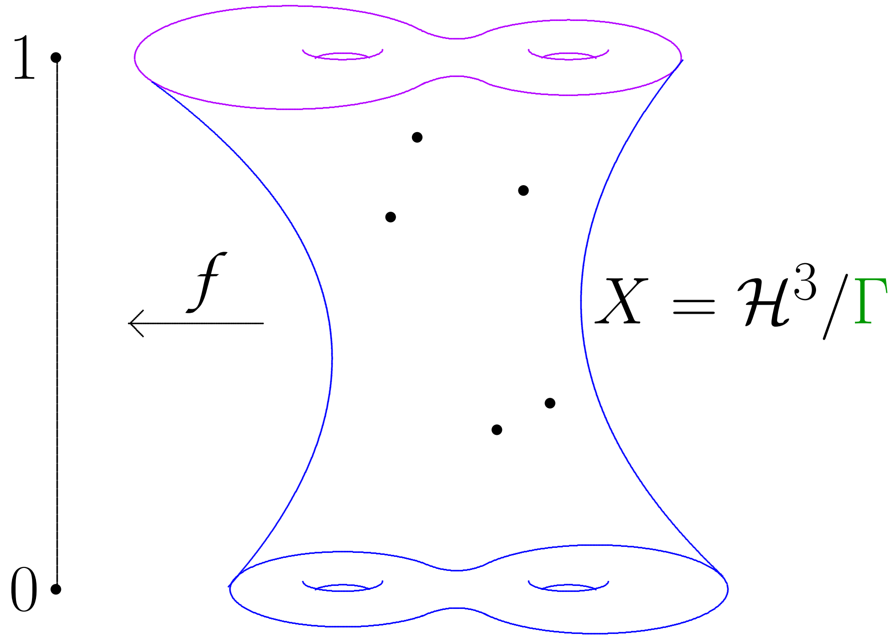
Choose k points $p_1, \ldots, p_k \in X$



Construction of ASD 4-manifolds:

Choose k points $p_1, \ldots, p_k \in X$

satisfying $\sum_{j=1}^{k} f(p_j) \in \mathbb{Z}$.



Construction of ASD 4-manifolds:

Choose k points $p_1, \ldots, p_k \in X$

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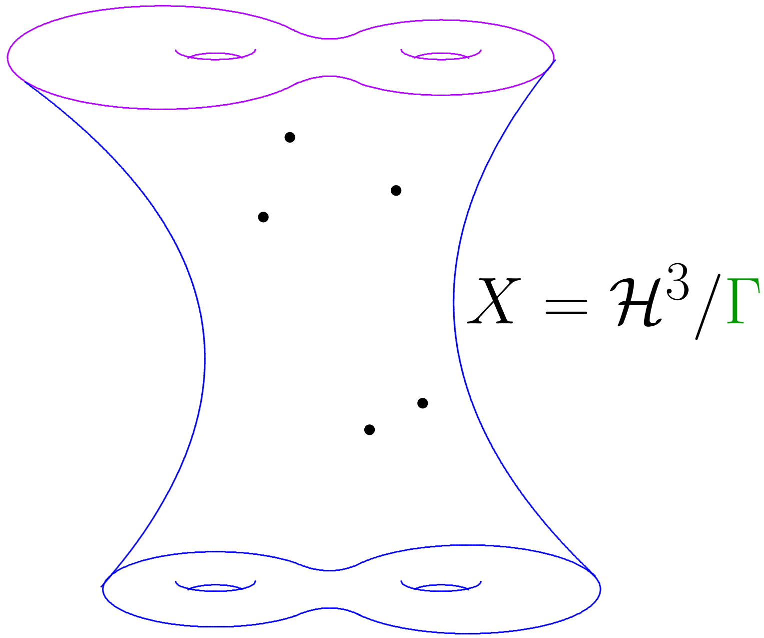
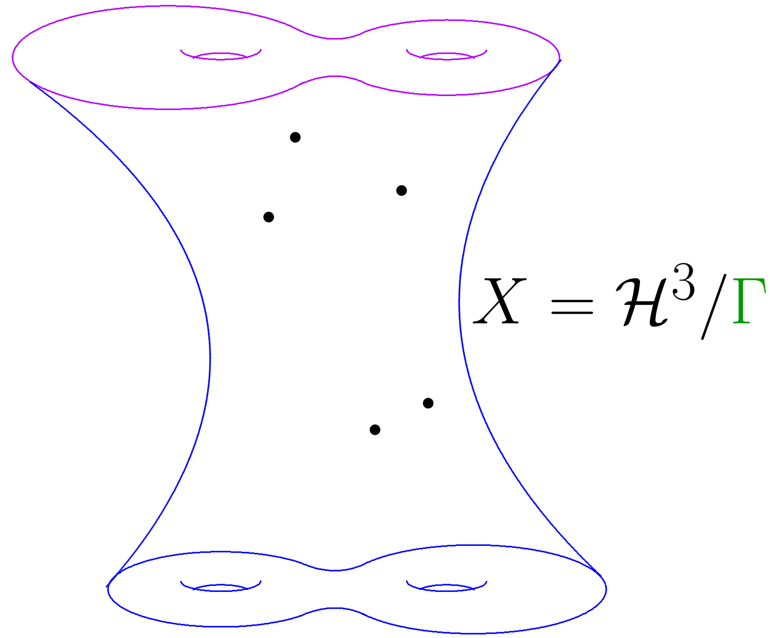
Can do if $k \neq 1$.



Construction of ASD 4-manifolds:

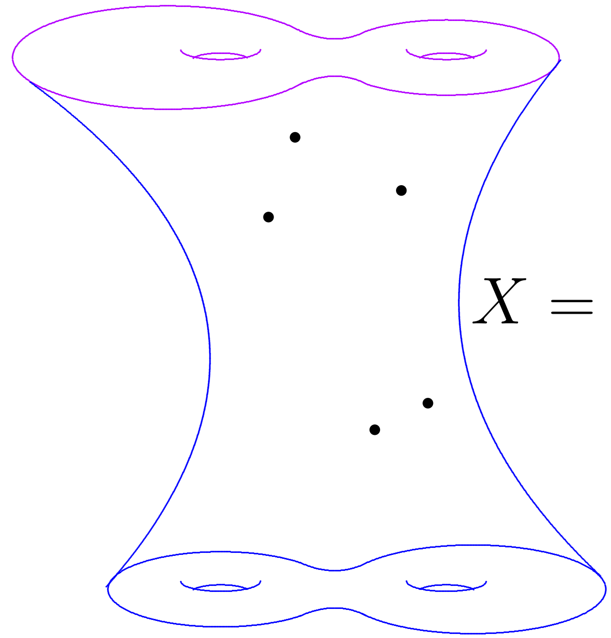
Let G_j be the Green's function of p_j:



Construction of ASD 4-manifolds:

Let G_j be the Green's function of p_j:

$$\Delta G_j = 2\pi\delta_{p_j}, \qquad G_j \to 0 \text{ at } \partial\overline{X}$$

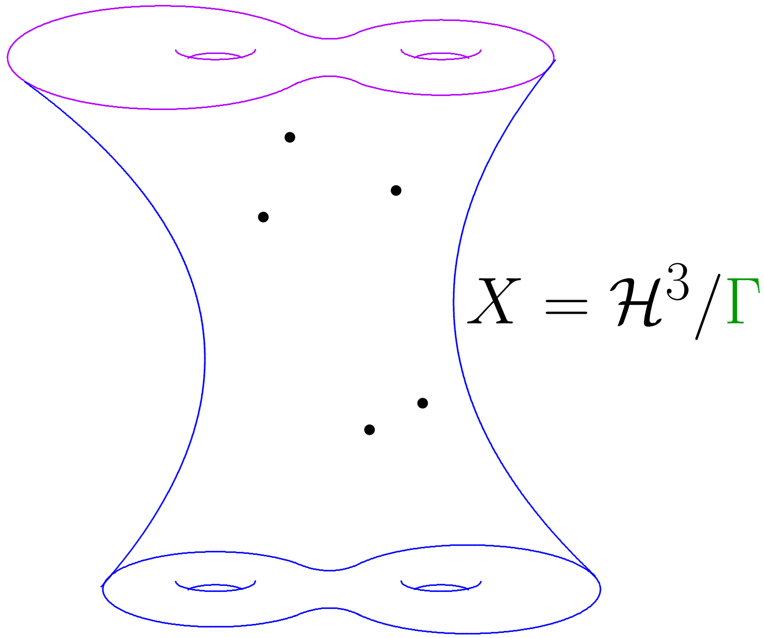


$X = \mathcal{H}^3/\Gamma$

Construction of ASD 4-manifolds:

Let G_j be the Green's function of p_j, and set

$$V = 1 + \sum_{j=1}^{k} G_j .$$



Construction of ASD 4-manifolds:

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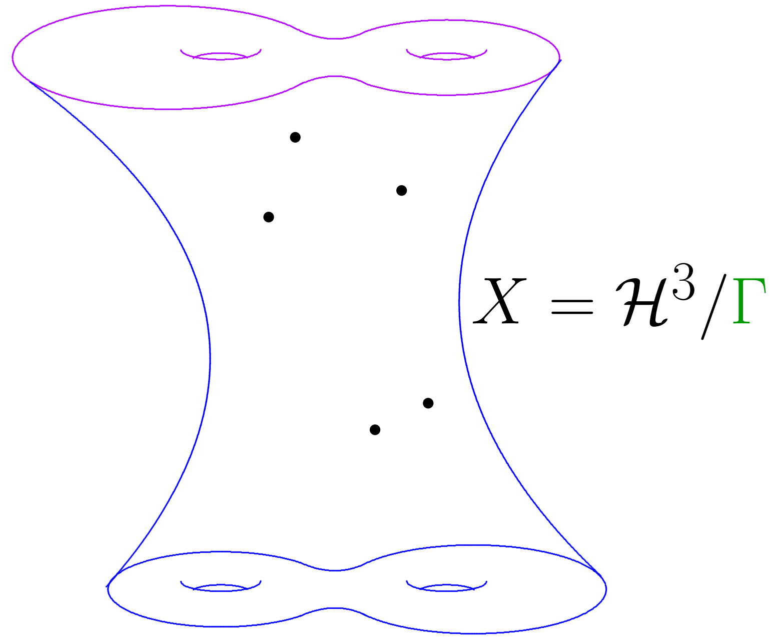
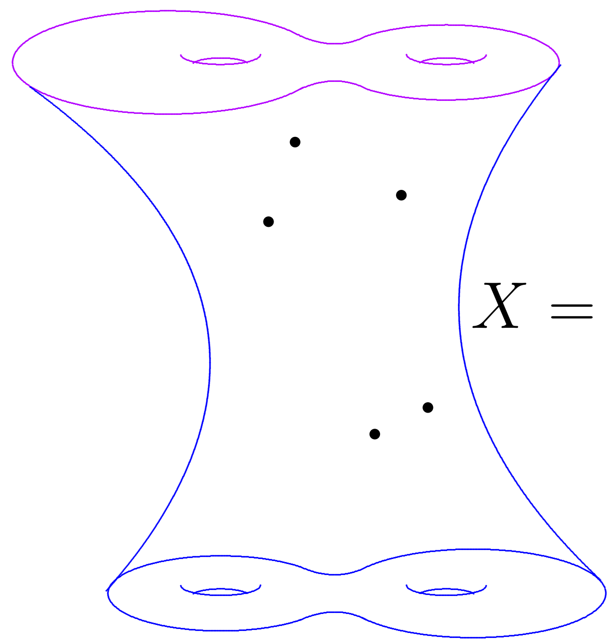


Construction of ASD 4-manifolds:

$$V = 1 + \sum_{j=1}^{k} G_j .$$

Choose $P \to (X - \{p_1, \dots, p_k\})$ circle bundle with connection form θ such that

$$d\theta = \star dV .$$



$$X = \mathcal{H}^3/\Gamma$$

Construction of ASD 4-manifolds:

$$g = \qquad V h + V^{-1}\theta^2$$

$$V = 1 + \sum_{j=1}^{k} G_j$$

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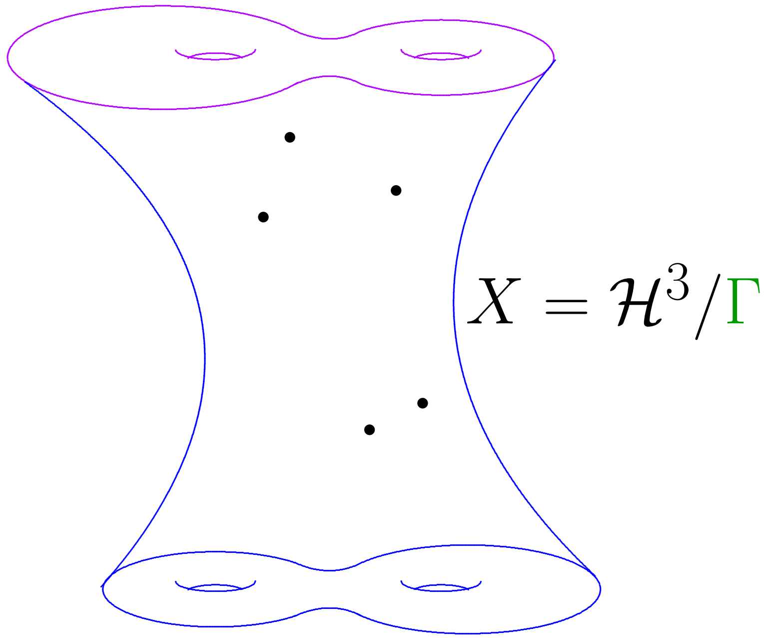

Construction of ASD 4-manifolds:

$$g = f(1-f)[Vh + V^{-1}\theta^2]$$

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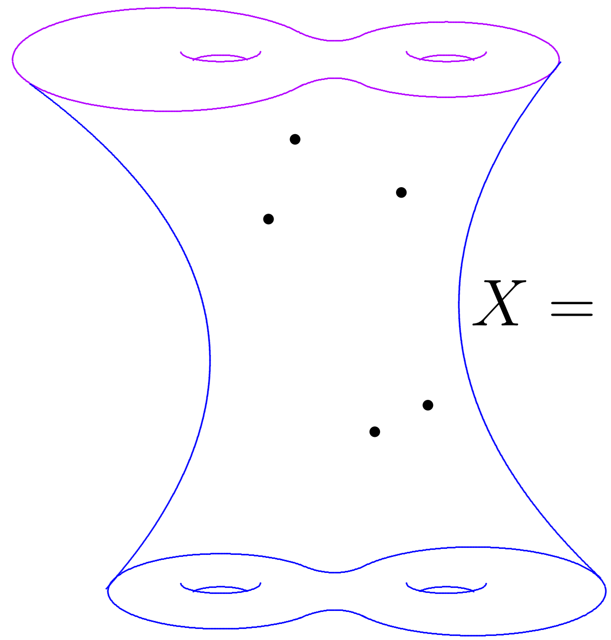

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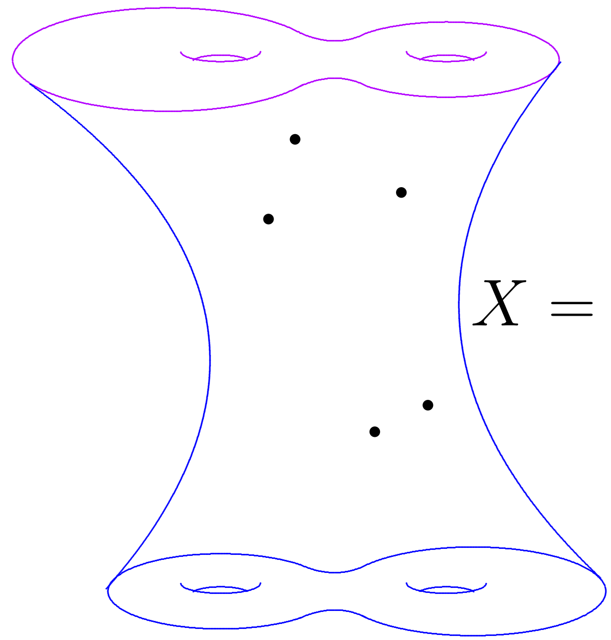
Construction of ASD 4-manifolds:

$$g = f(1-f)[Vh + V^{-1}\theta^2]$$

$$M = P \cup \{\hat{p}_1, \ldots, \hat{p}_k\} \cup \partial\overline{X}$$



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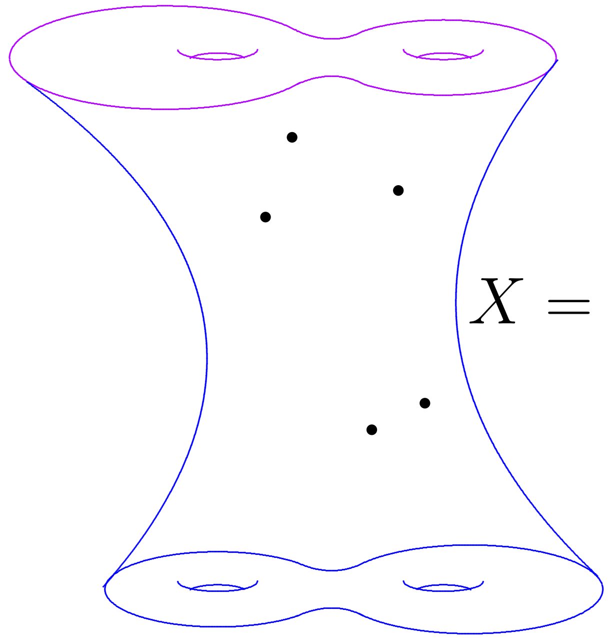
$$\begin{array}{ccccccc}
M & = & P & \cup & \{\hat{p}_1, \ldots, \hat{p}_k\} & \cup & \partial\overline{X} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\overline{X} & = & X - \{p_1, \ldots, p_k\} & \cup & \{p_1, \ldots, p_k\} & \cup & \partial\overline{X}
\end{array}$$



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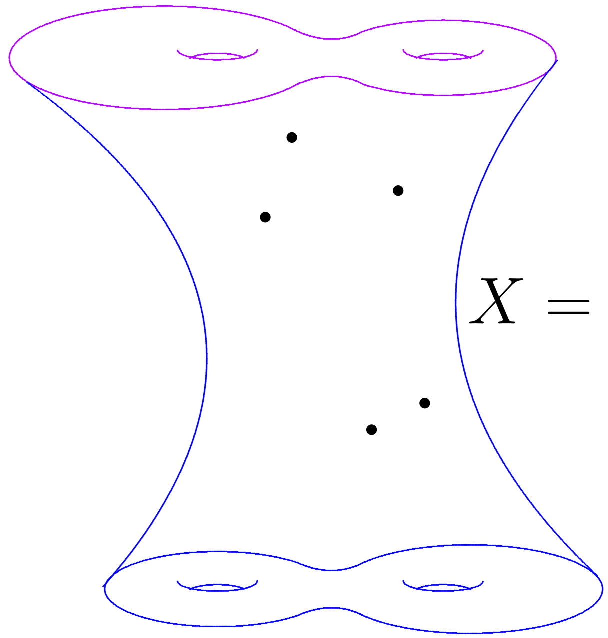

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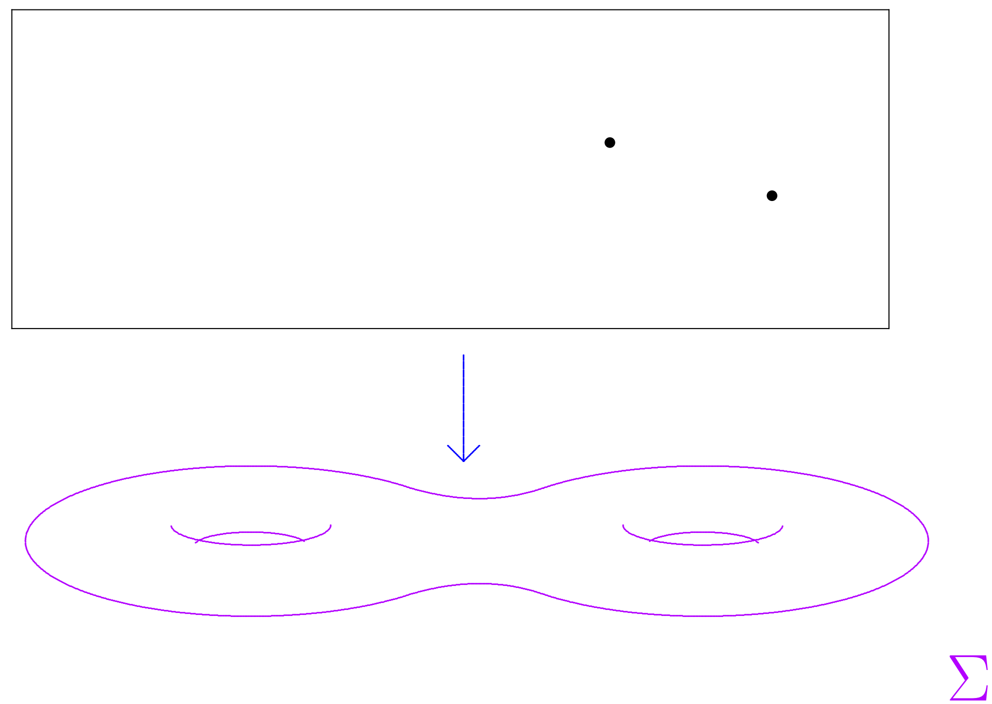
$$\approx (\Sigma \times S^2)\#k\overline{\mathbb{CP}}_2$$

$$\overline{X} = \Sigma \times [0,1]$$



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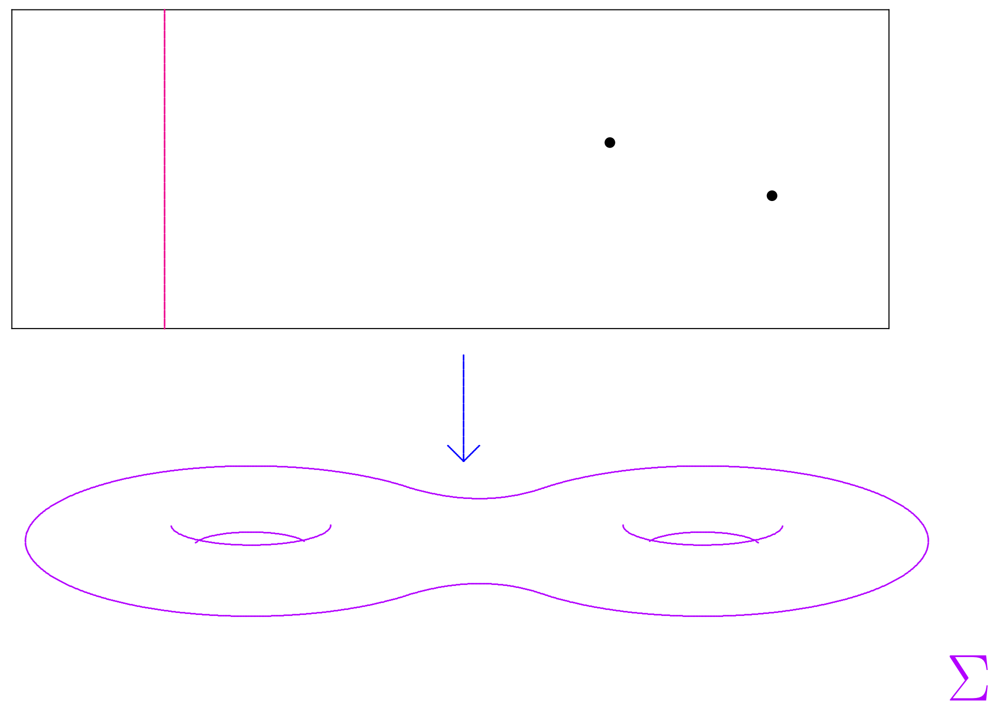

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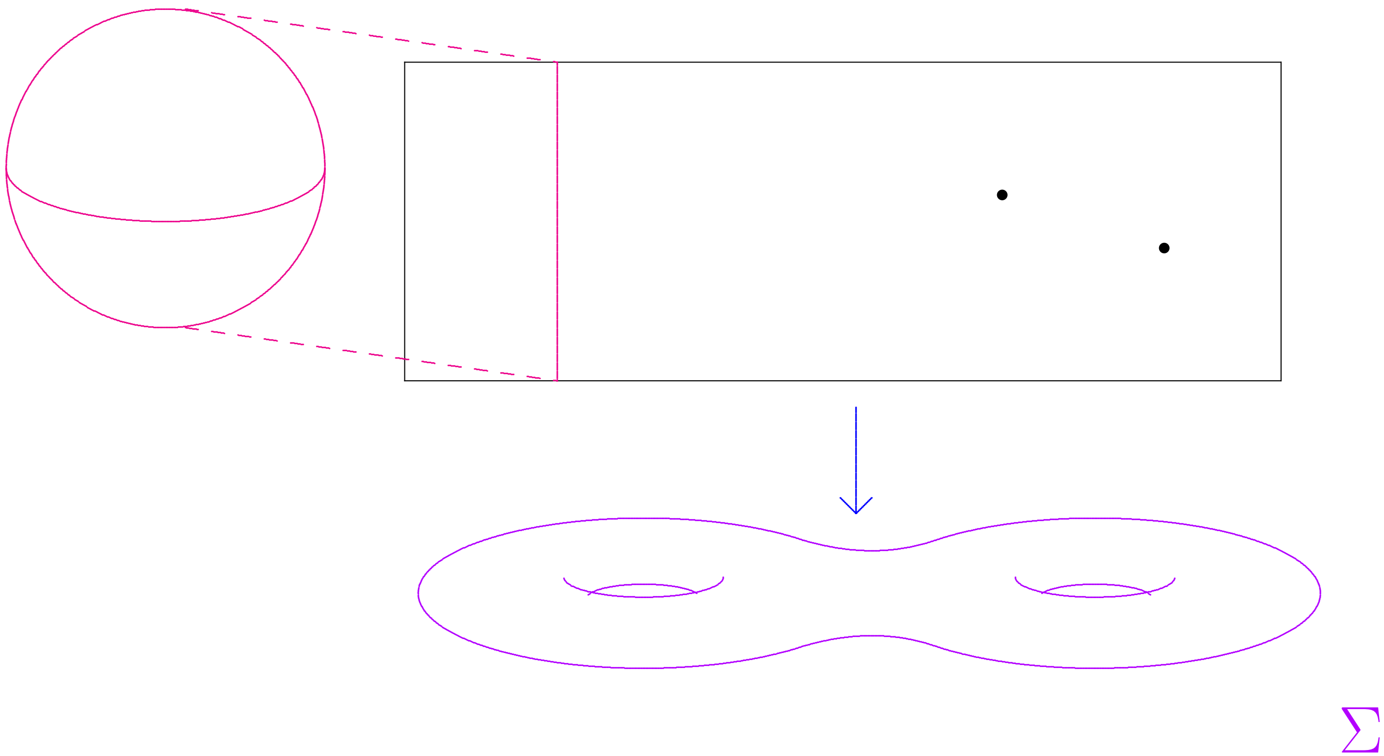

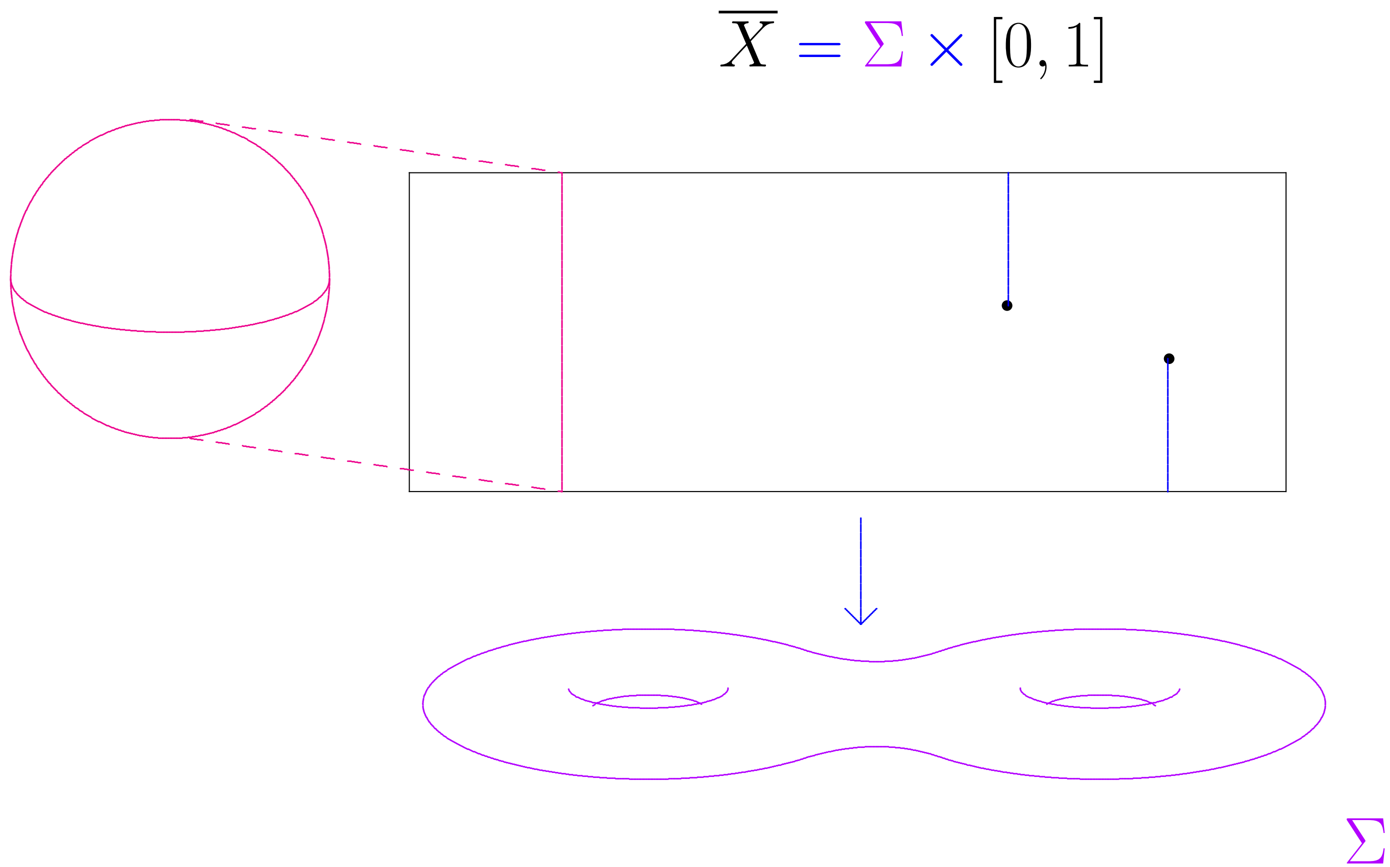
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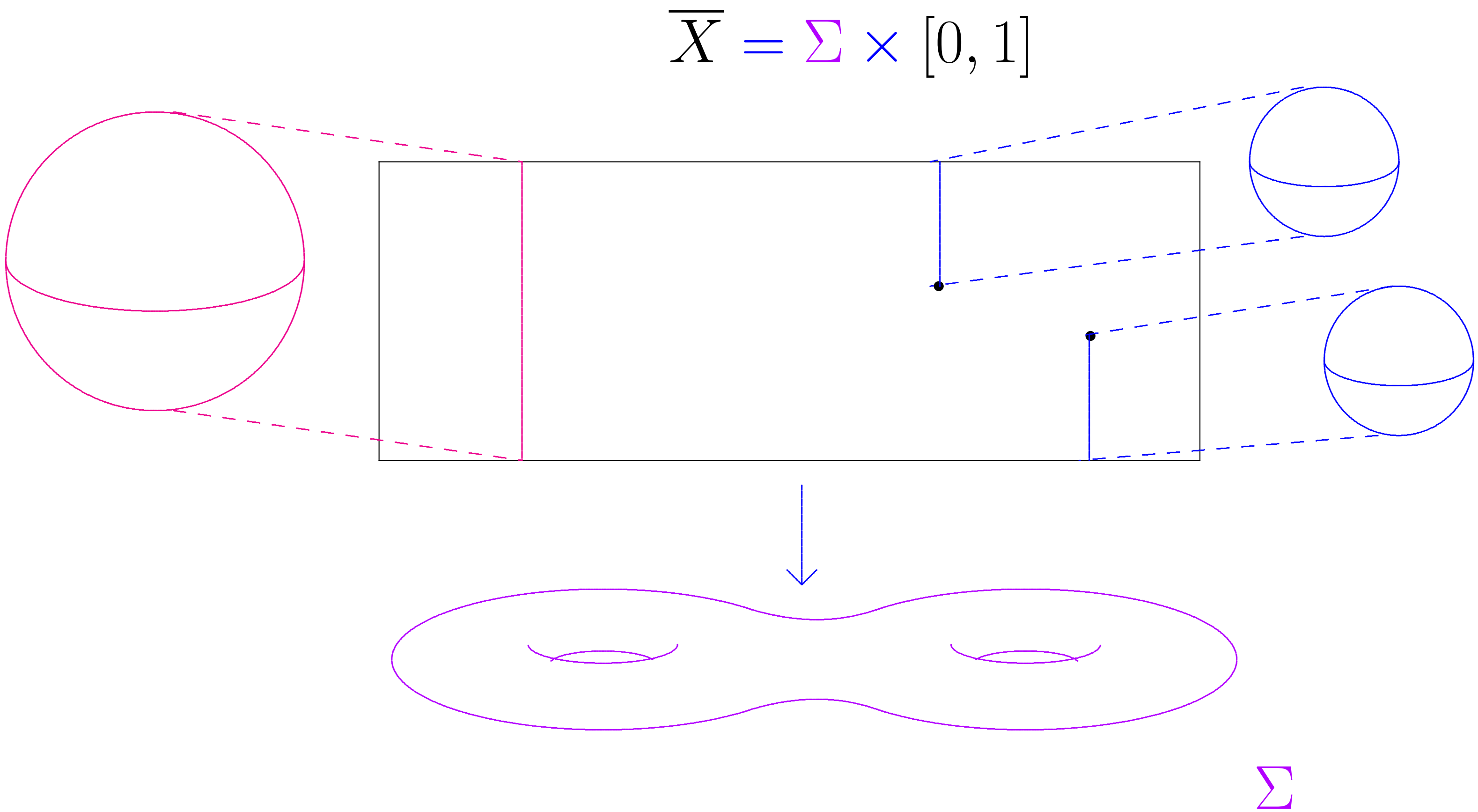
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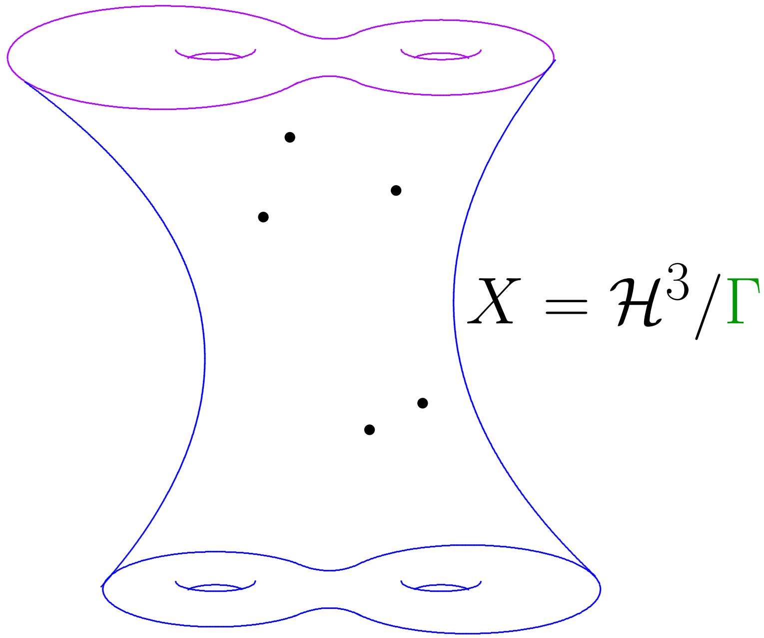

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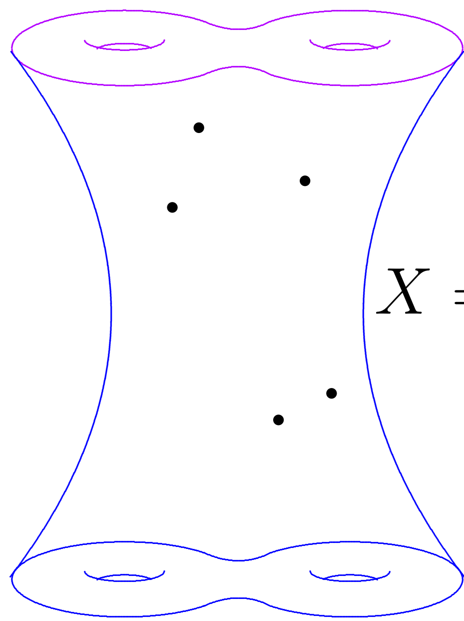
$$\approx (\Sigma \times S^2) \# k\overline{\mathbb{CP}}_2$$



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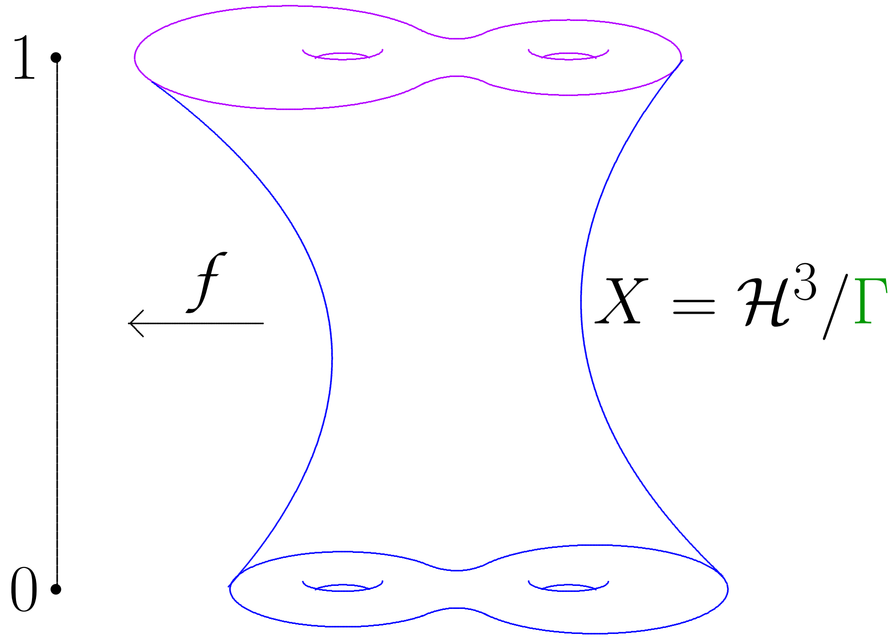
Fuchsian case: $(\Sigma \times S^2)\# k\overline{\mathbb{CP}}_2$ scalar-flat Kähler



Γ quasi-Fuchsian

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Tunnel-Vision function:

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$$\Delta f = 0$$

Theorem.

Theorem. *Let* $(M, [g])$ *be ASD manifold*

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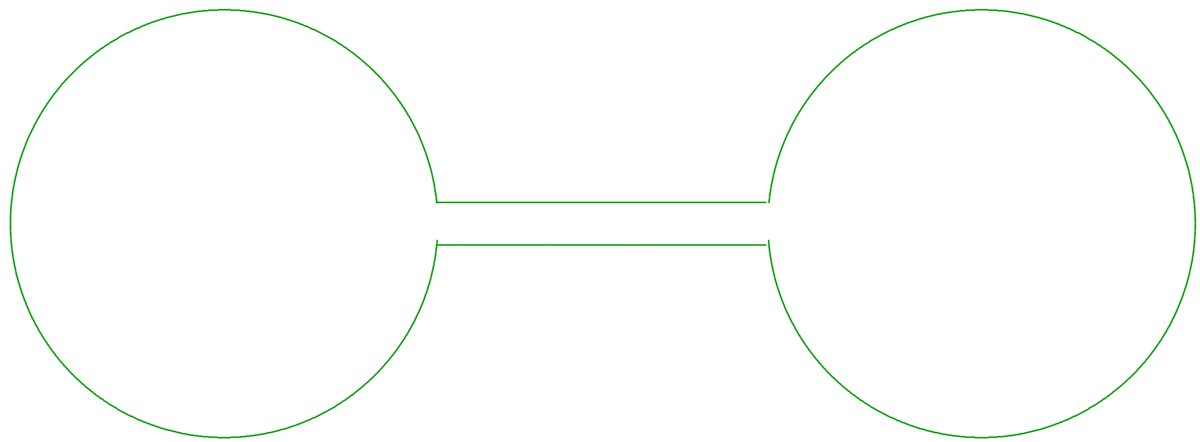
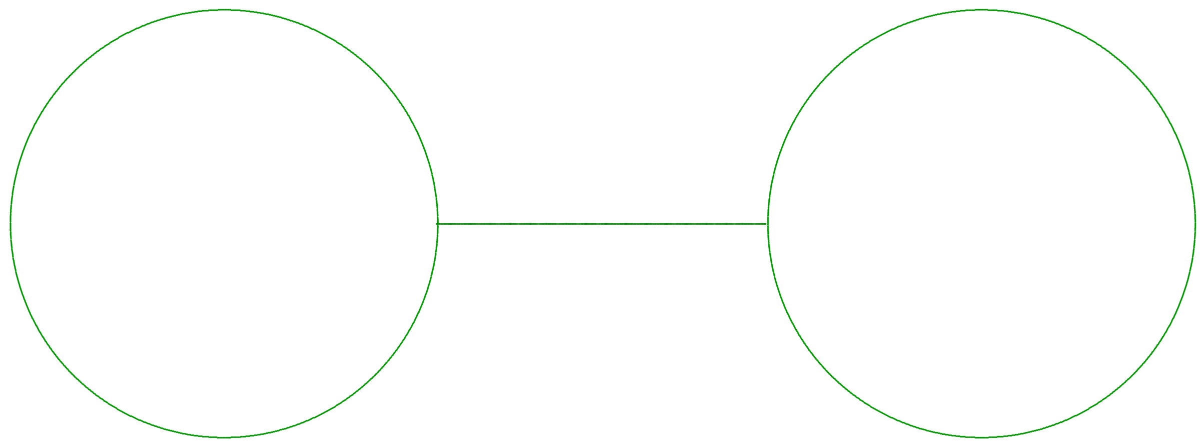
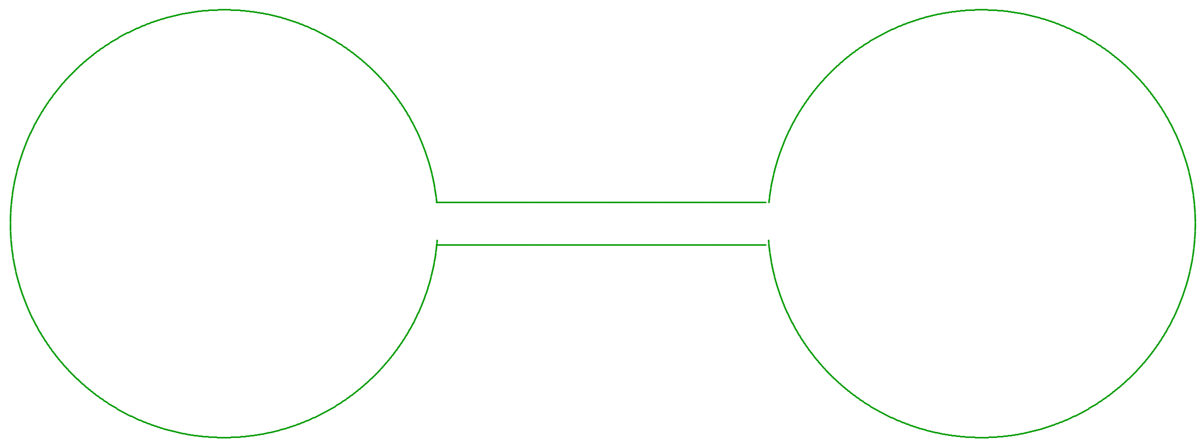
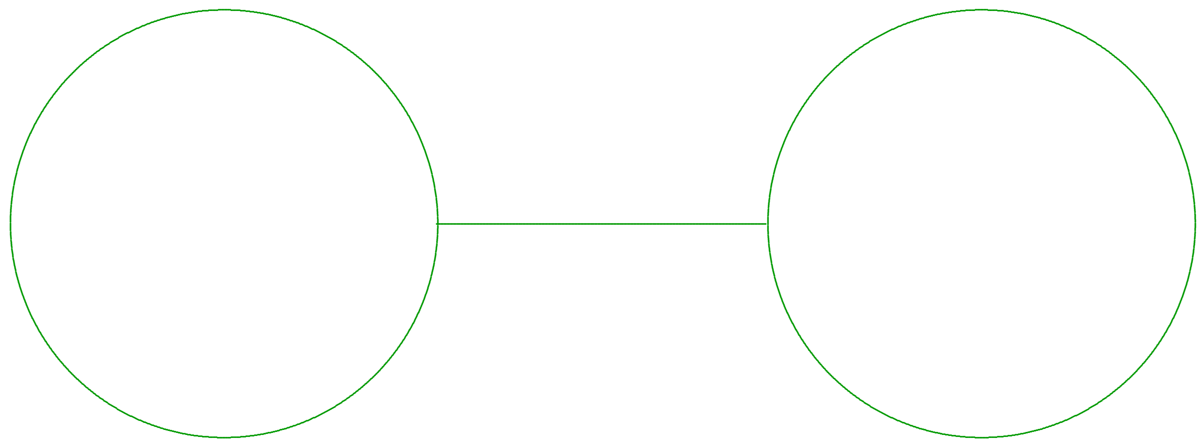
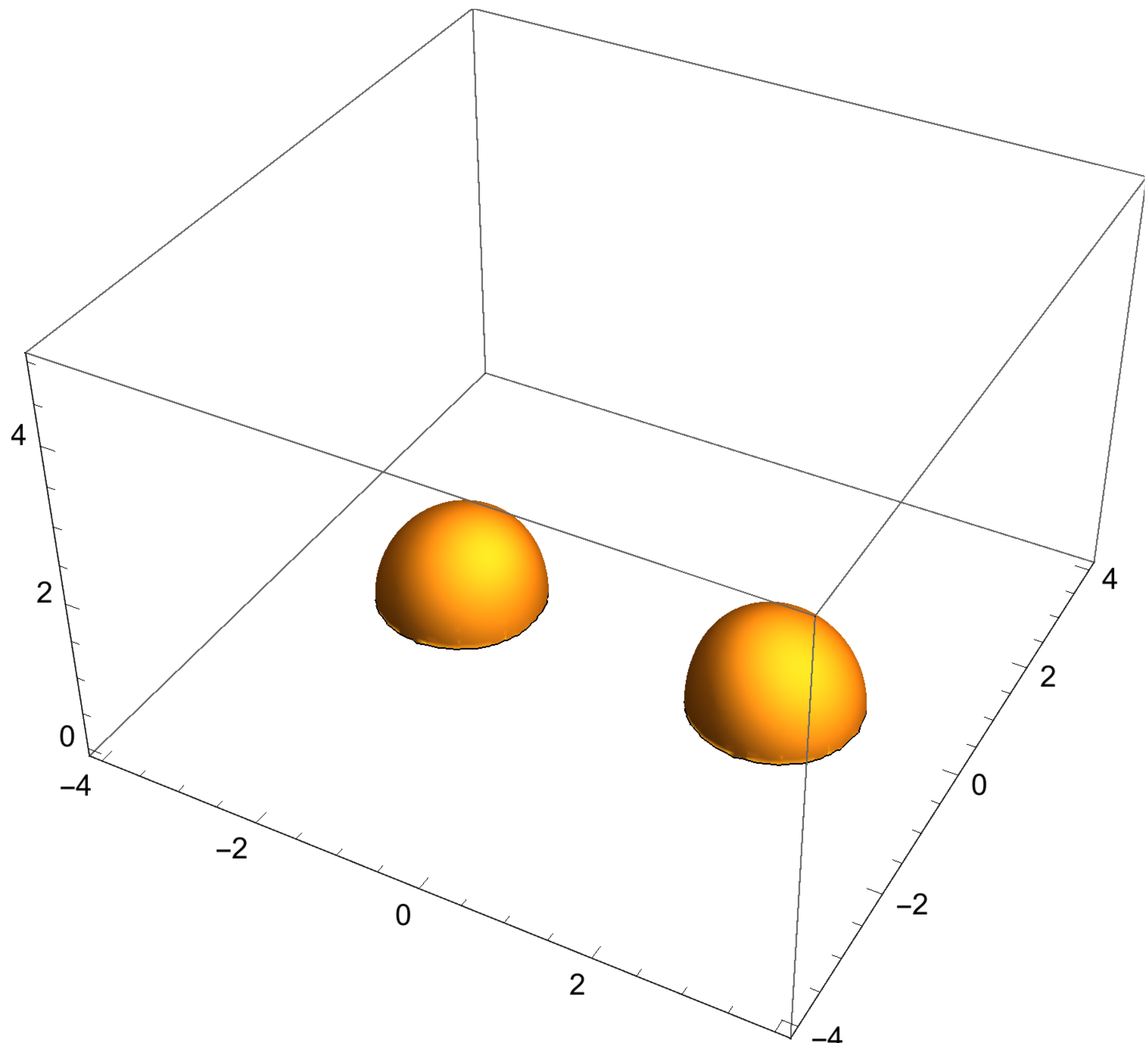
Ahlfors-Bers: Quasi-conformal mappings







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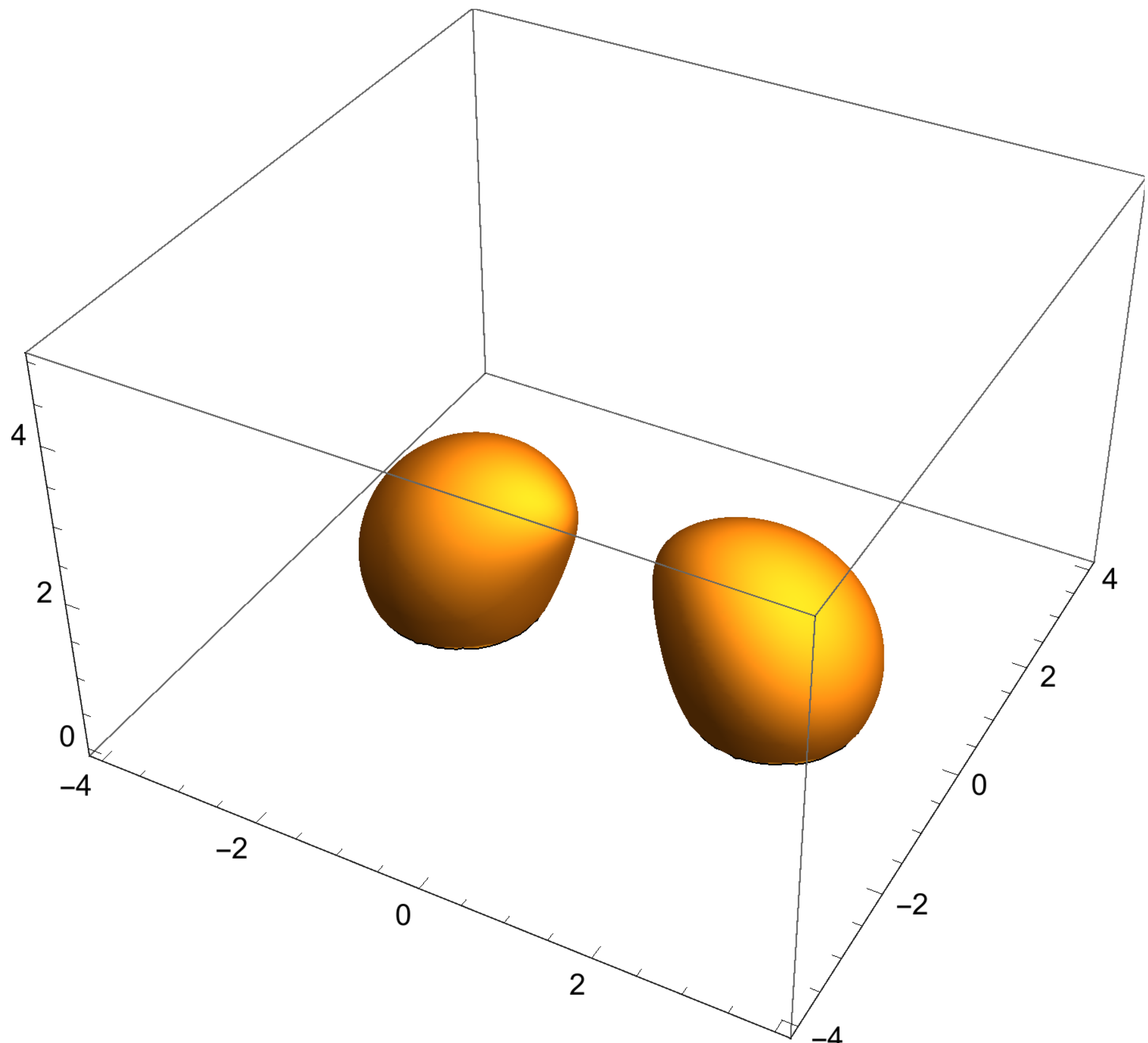

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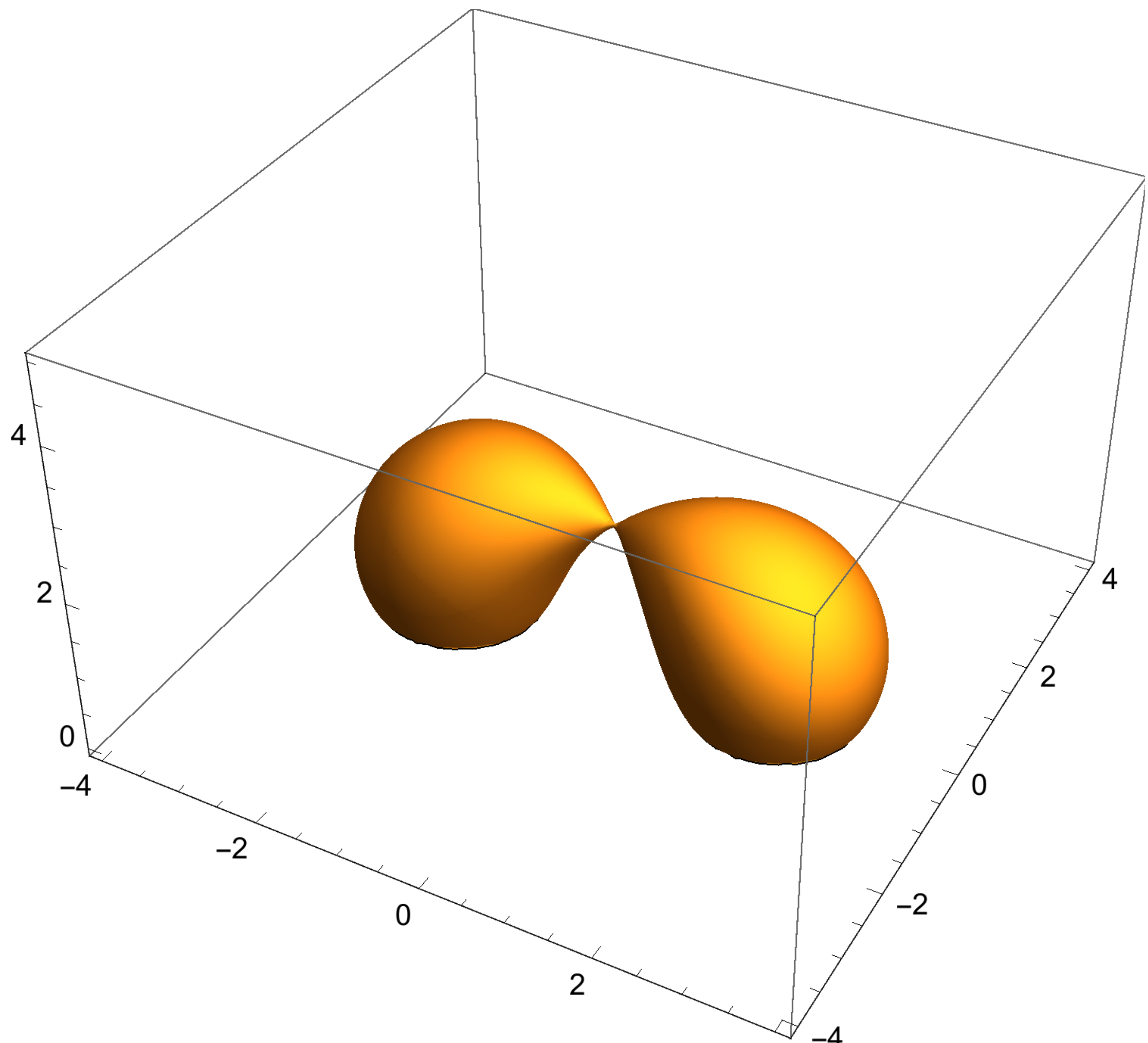

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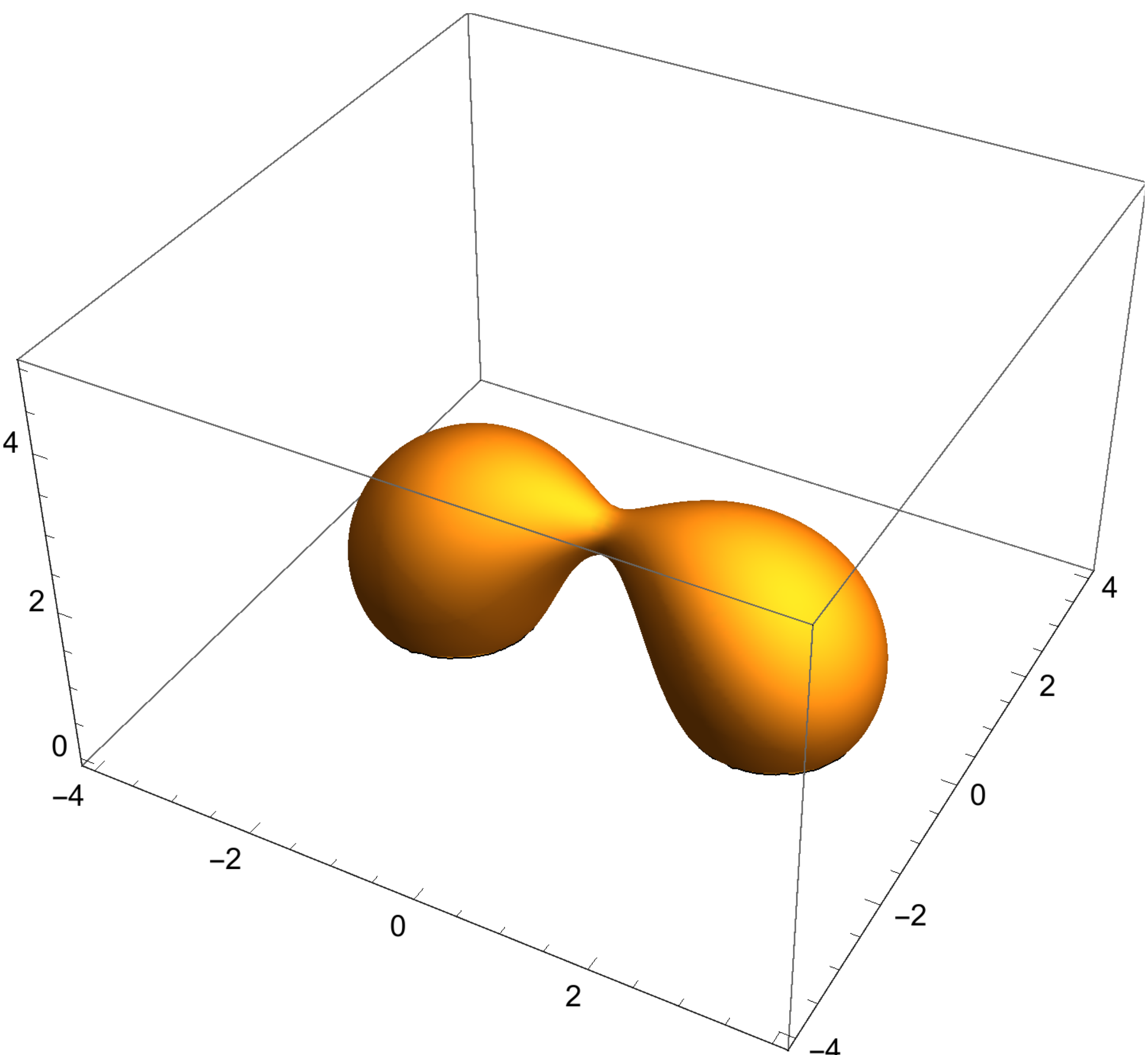
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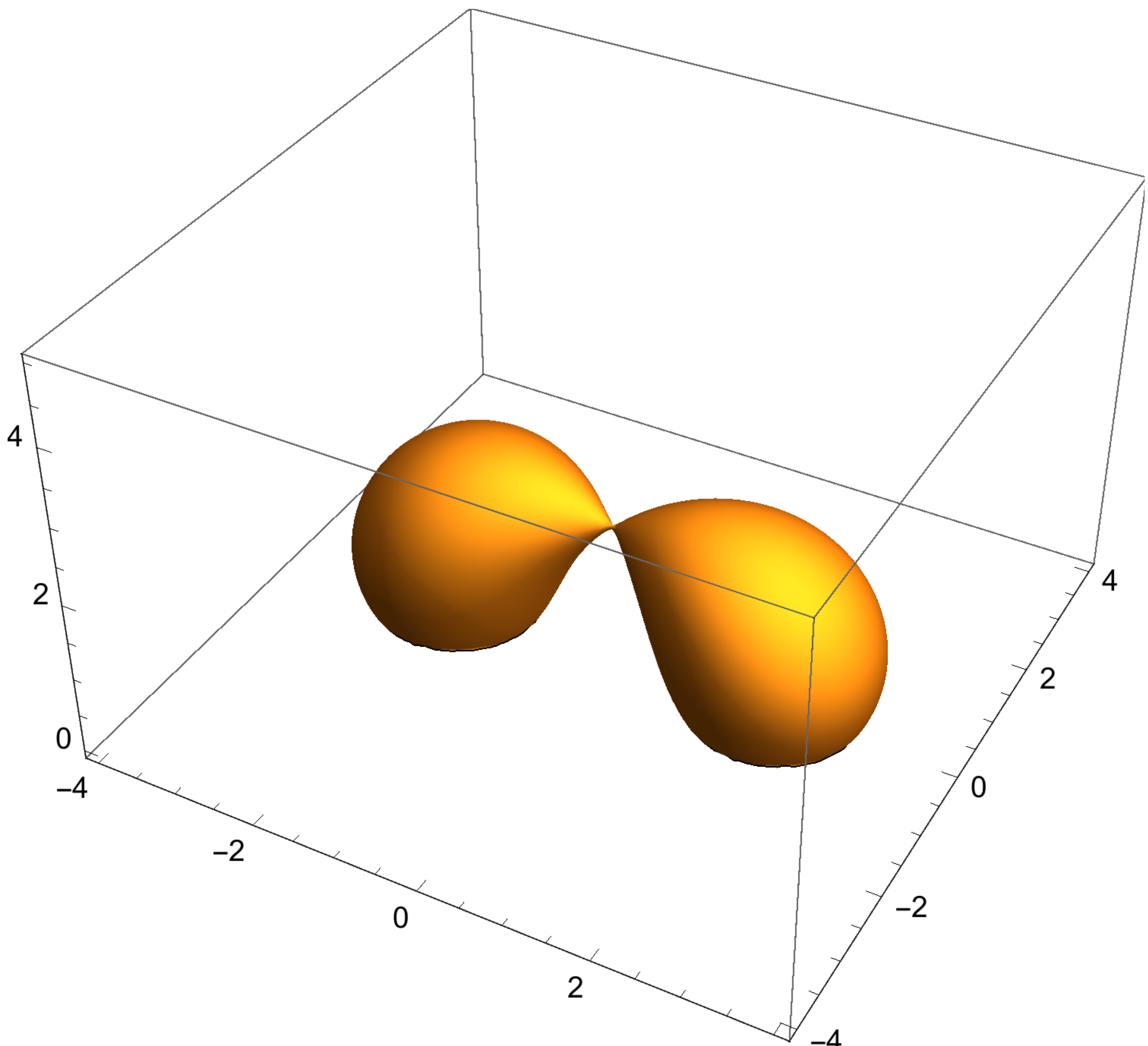
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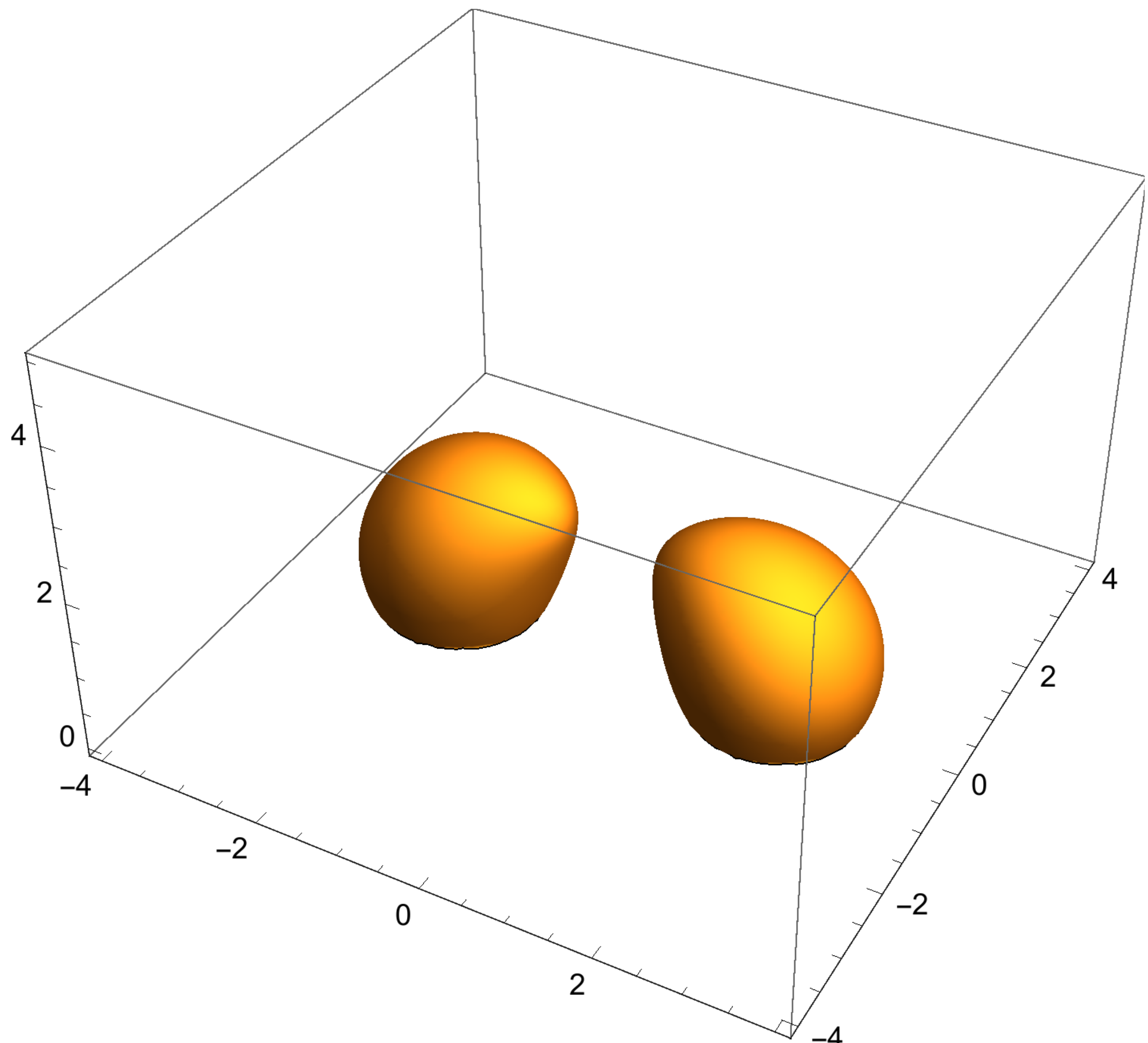
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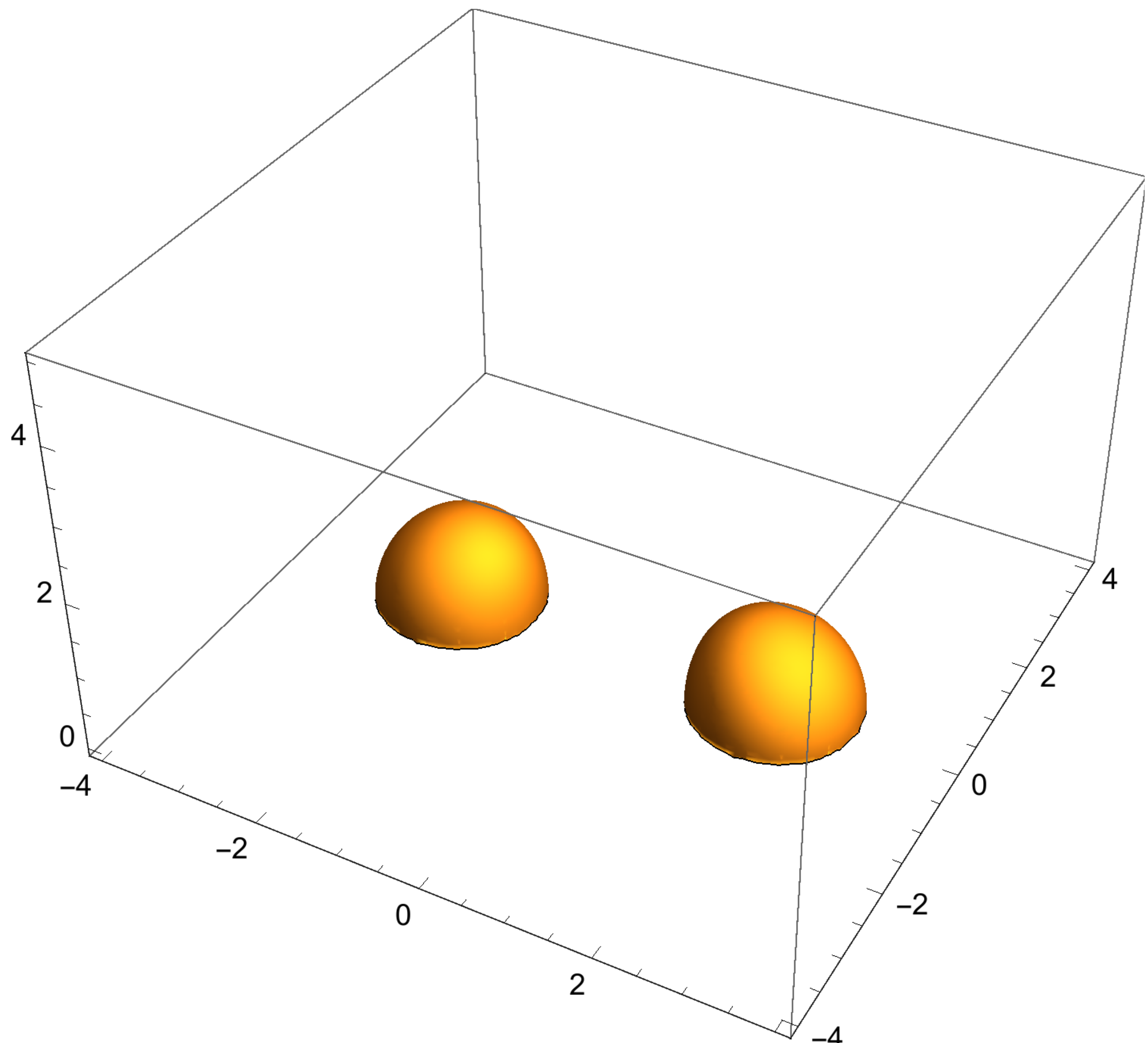

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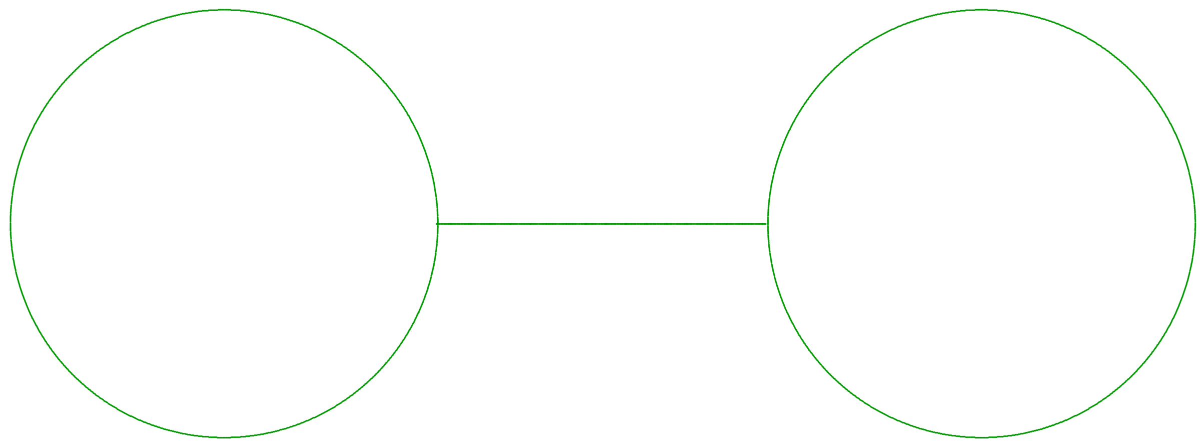

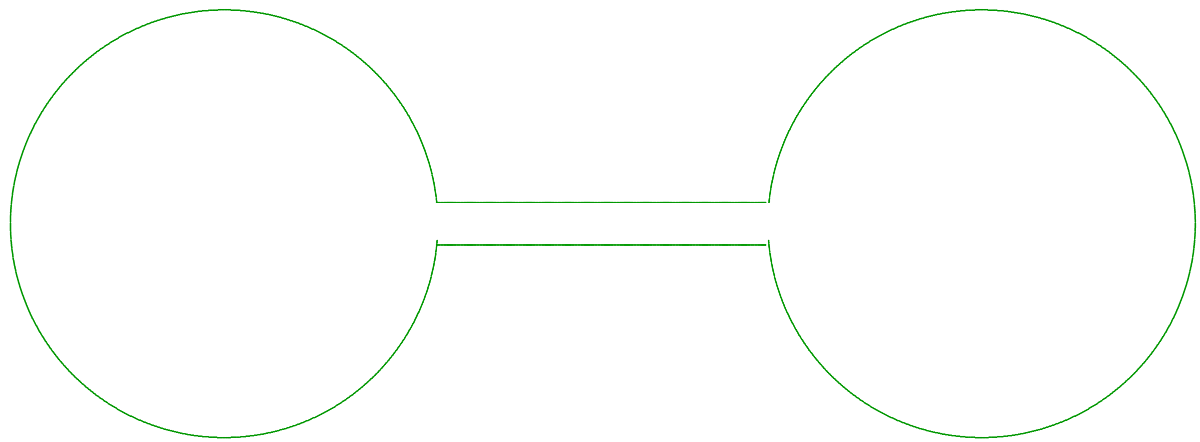

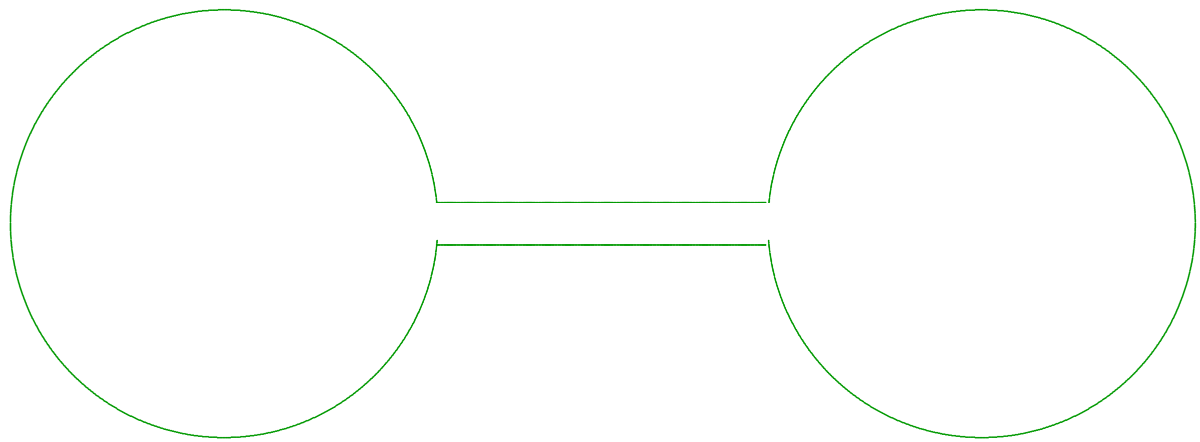

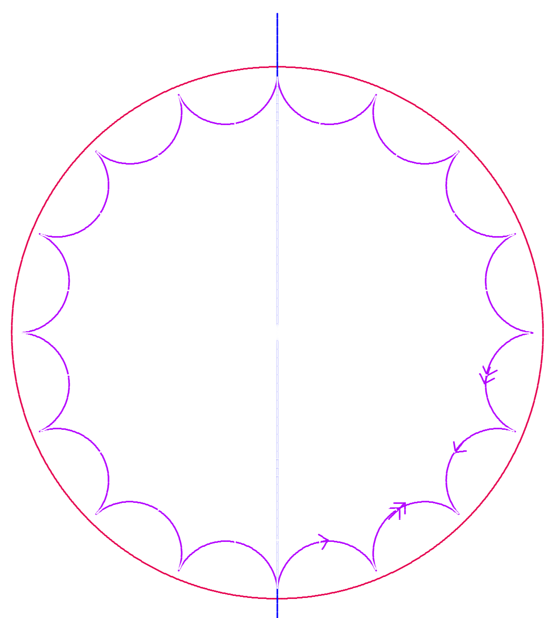

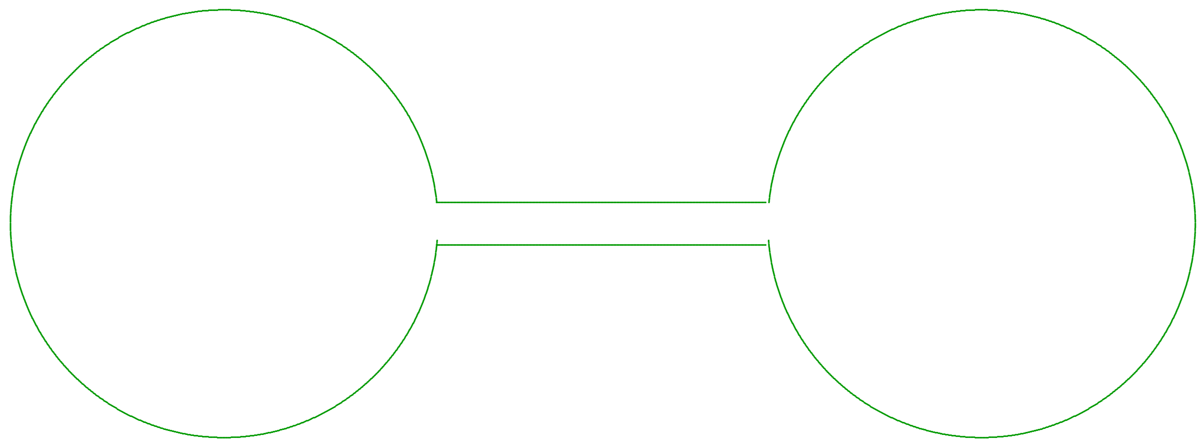

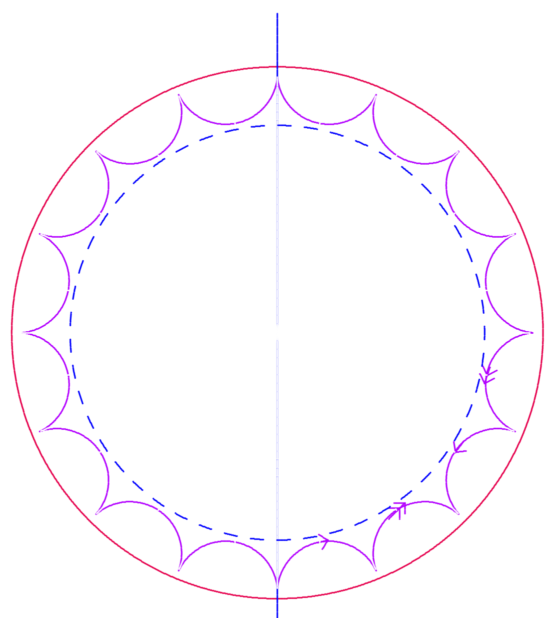

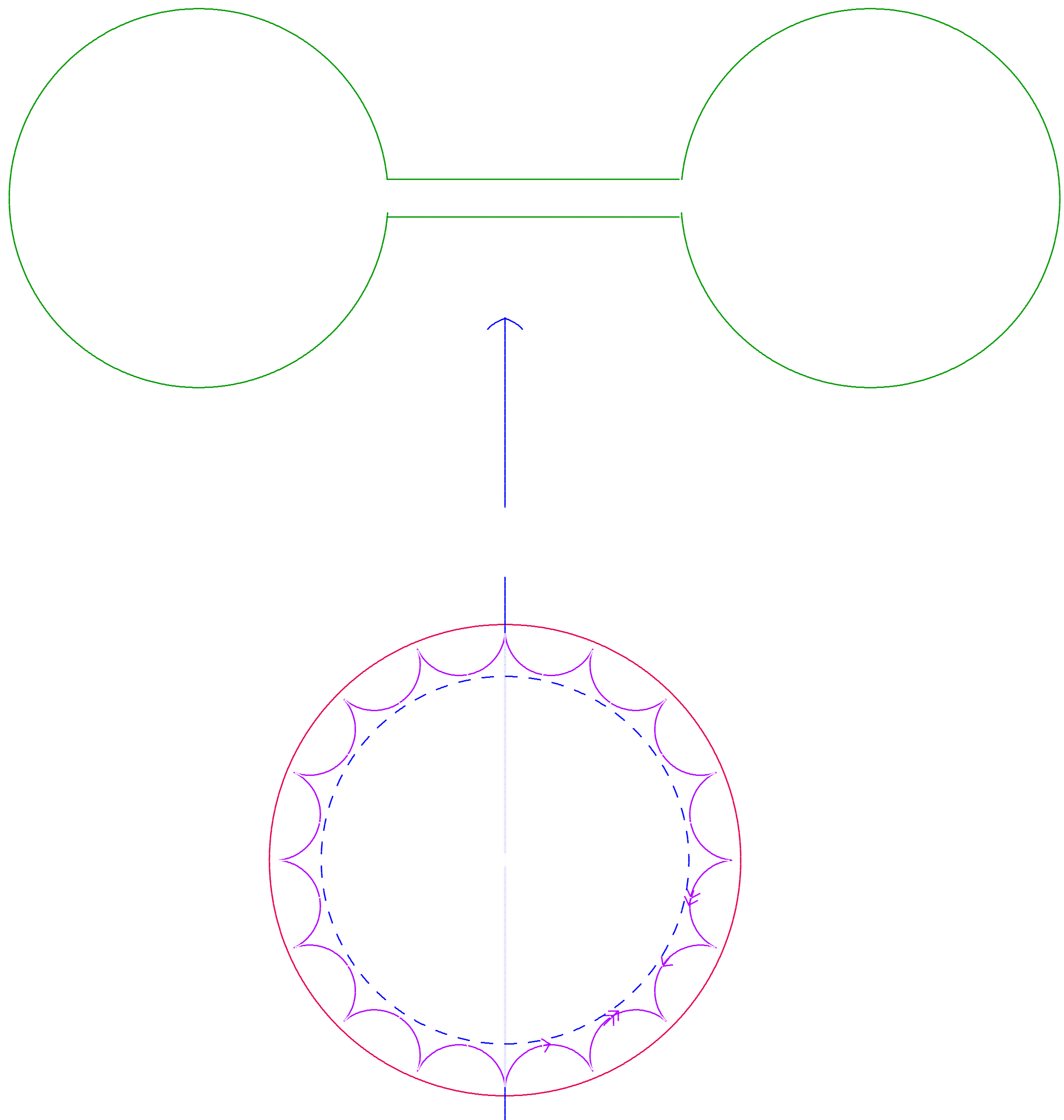

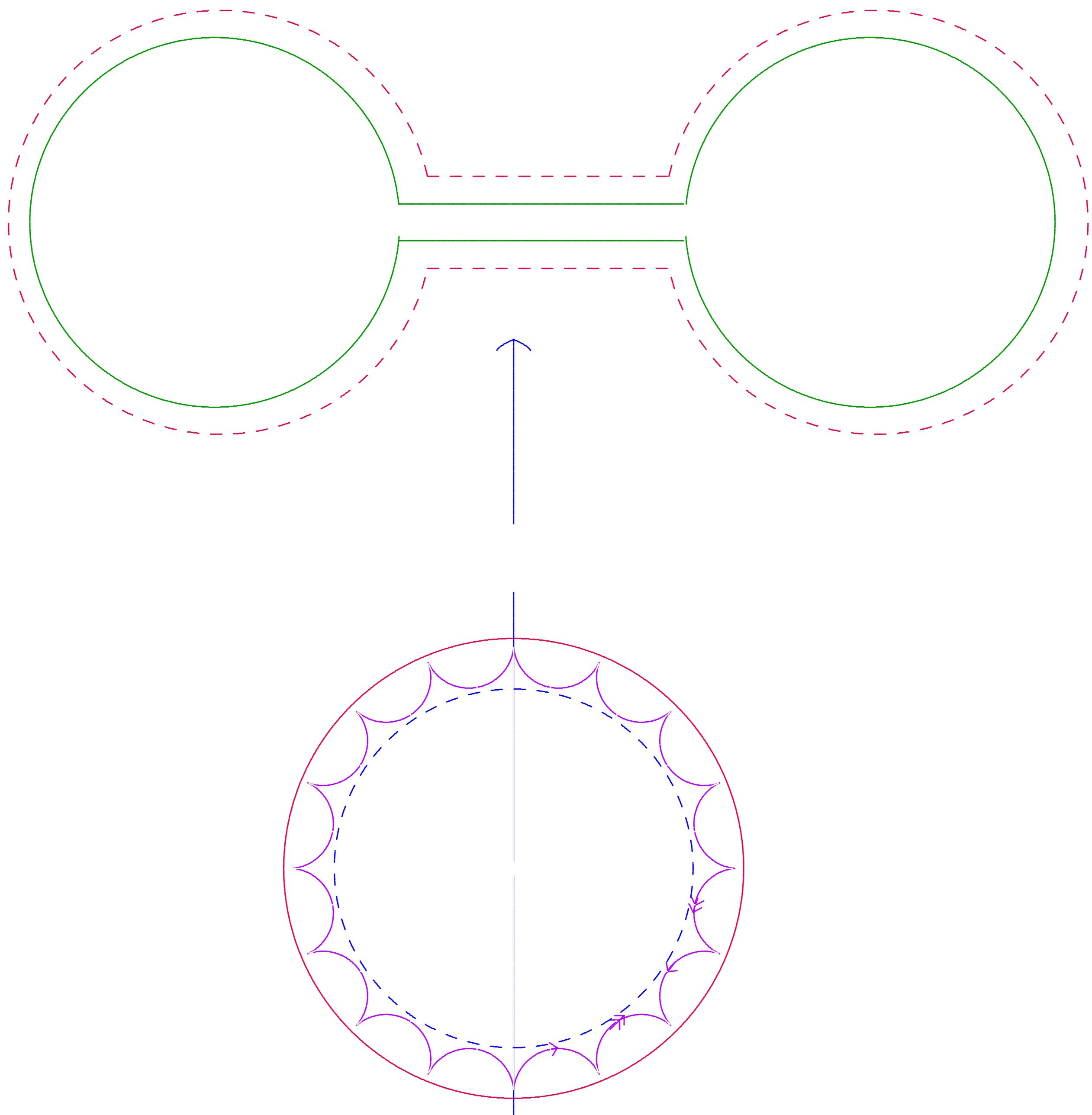

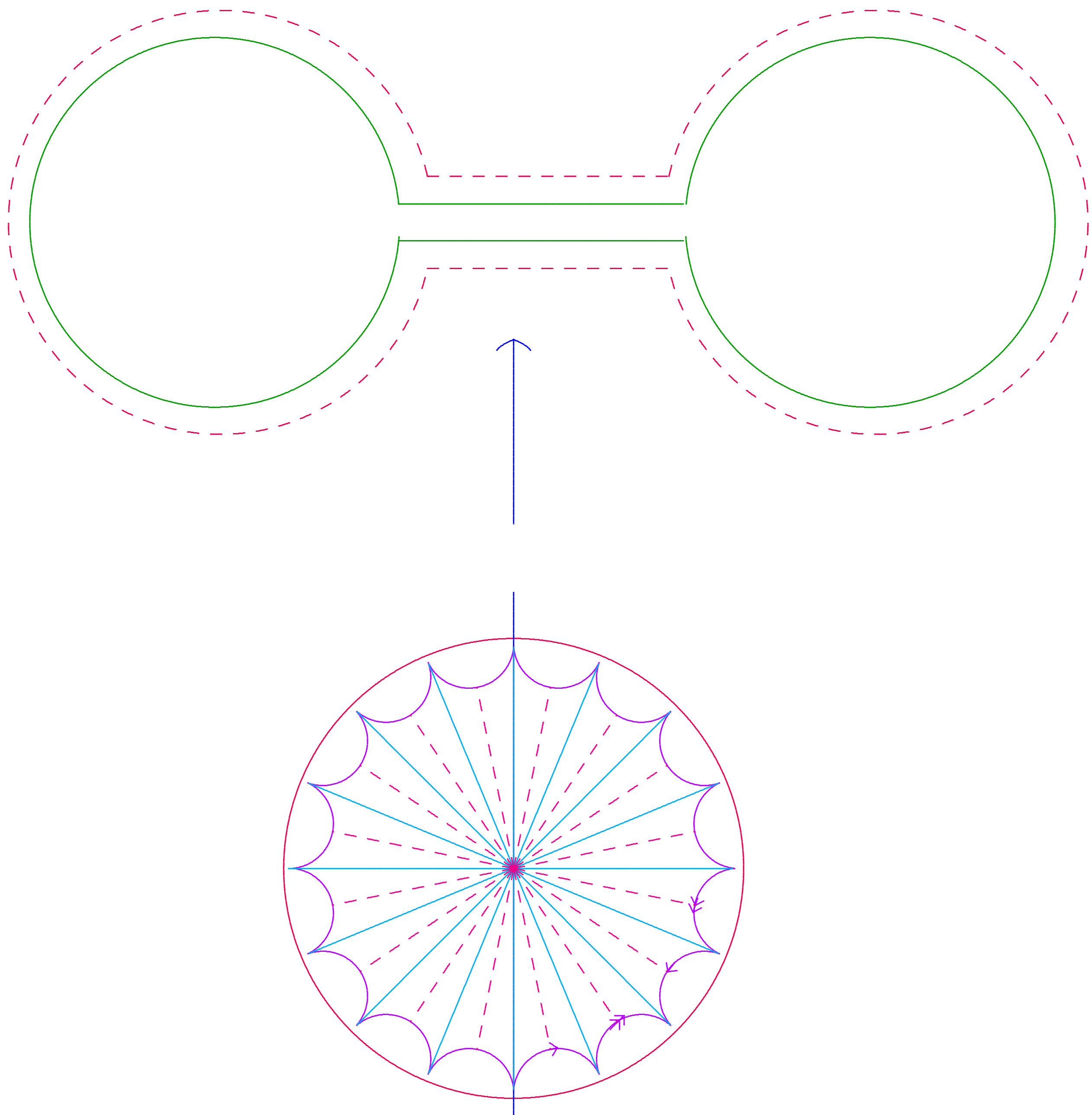

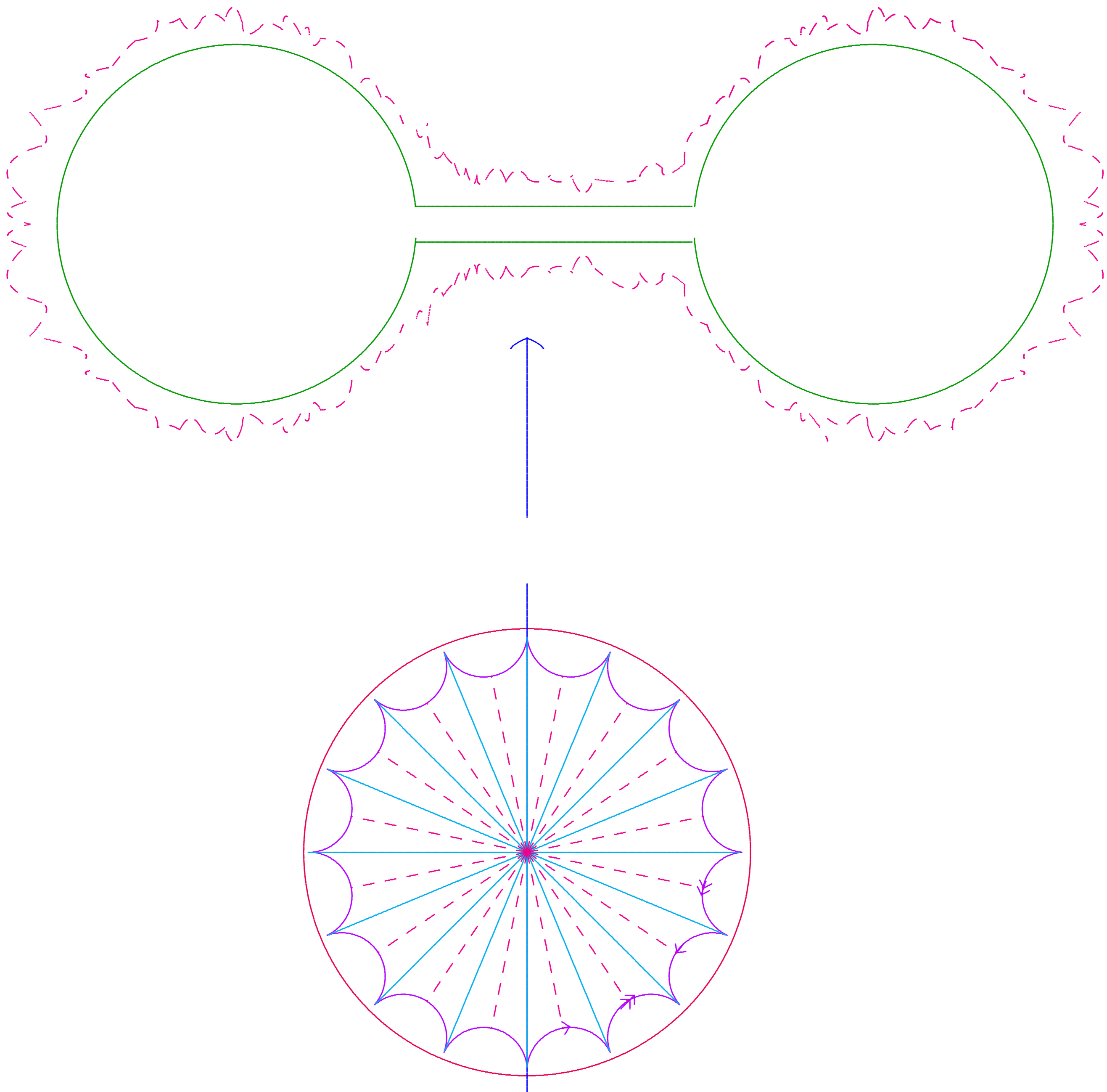

Theorem A. *Consider* 4*-manifolds* $M = \Sigma \times S^2$, *where* Σ *compact Riemann surface of genus* $\mathbf{g}$.

Then $\forall$ *even* $\mathbf{g} \gg 0$, $\exists$ *family* $[g_t]$, $t \in [0,1]$, *of locally-conformally-flat classes on* M, *such that*

- $\exists$ *scalar-flat Kähler metric* $g_0 \in [g_0]$; *but*
- $\nexists$ *almost-Kähler metric* $g \in [g_1]$.

Theorem B. *Fix an integer* $k \geq 2$*, and then consider the 4-manifolds* $M = (\Sigma \times S^2)\#k\overline{\mathbb{CP}}_2$*, where* Σ *compact Riemann surface of genus* $\mathfrak{g}$*.*

Then $\forall$ *even* $\mathfrak{g} \gg 0$, $\exists$ *family* $[g_t]$*,* $t \in [0,1]$*, of anti-self-dual conformal classes on* M*, such that*

- $\exists$ *scalar-flat Kähler metric* $g_0 \in [g_0]$*; but*
- $\nexists$ *almost-Kähler metric* $g \in [g_1]$*.*