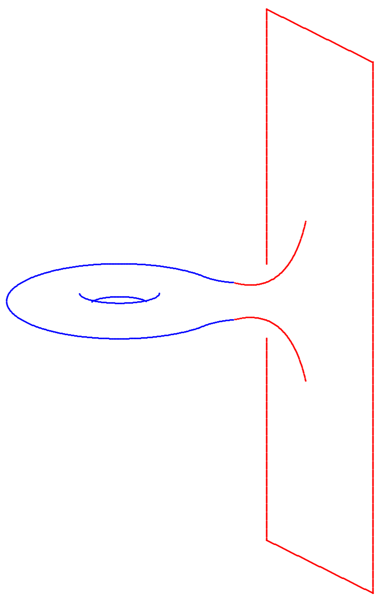
Mass, Kähler Manifolds, &

Symplectic Geometry

Claude LeBrun
Stony Brook University

Simons Center for Geometry and Physics
Weekly Talk in Mathematics
March 25, 2021

Definition. *A complete, non-compact Riemannian* n*-manifold* (M^n, g)



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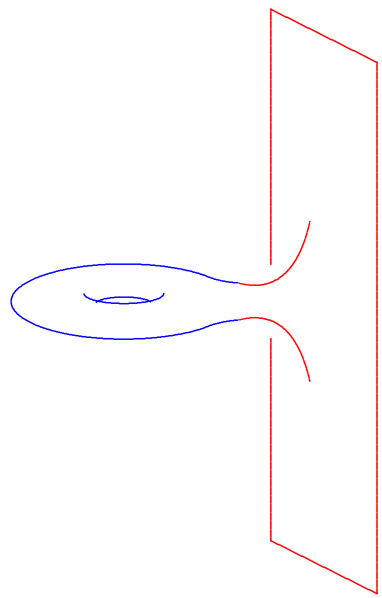
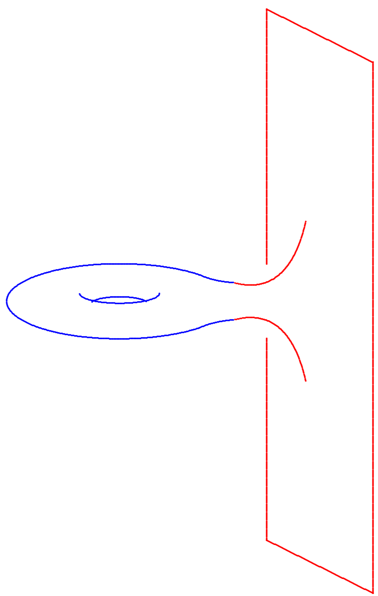


$$\boxed{n \geq 3}$$

Definition. *A complete, non-compact Riemannian* n*-manifold* (M^n, g) *is called* asymptotically Euclidean



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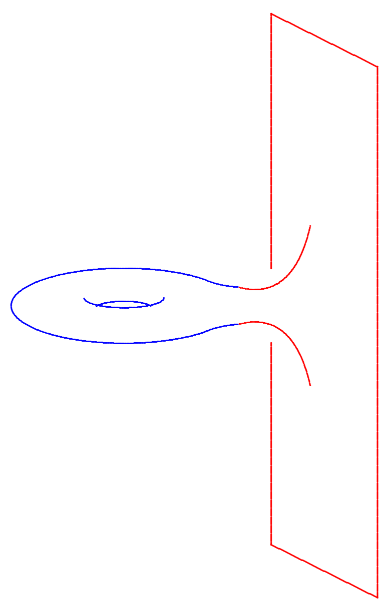
Definition. *A complete, non-compact Riemannian n-manifold (M^n, g) is called* asymptotically Euclidean *(AE)*



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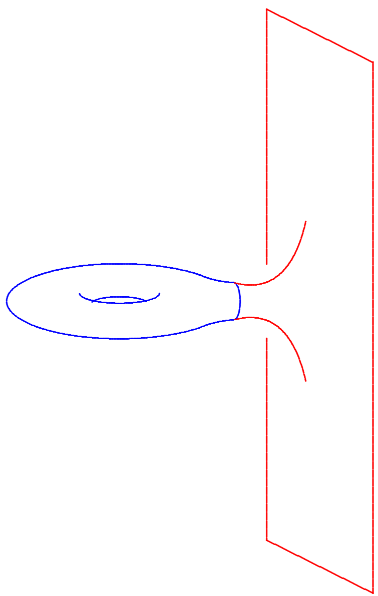

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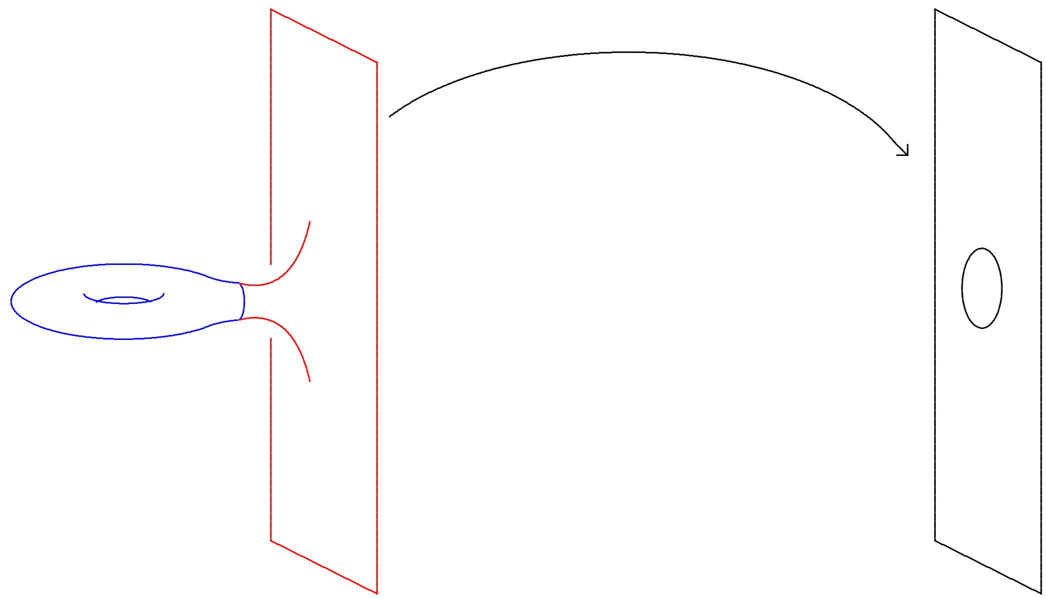

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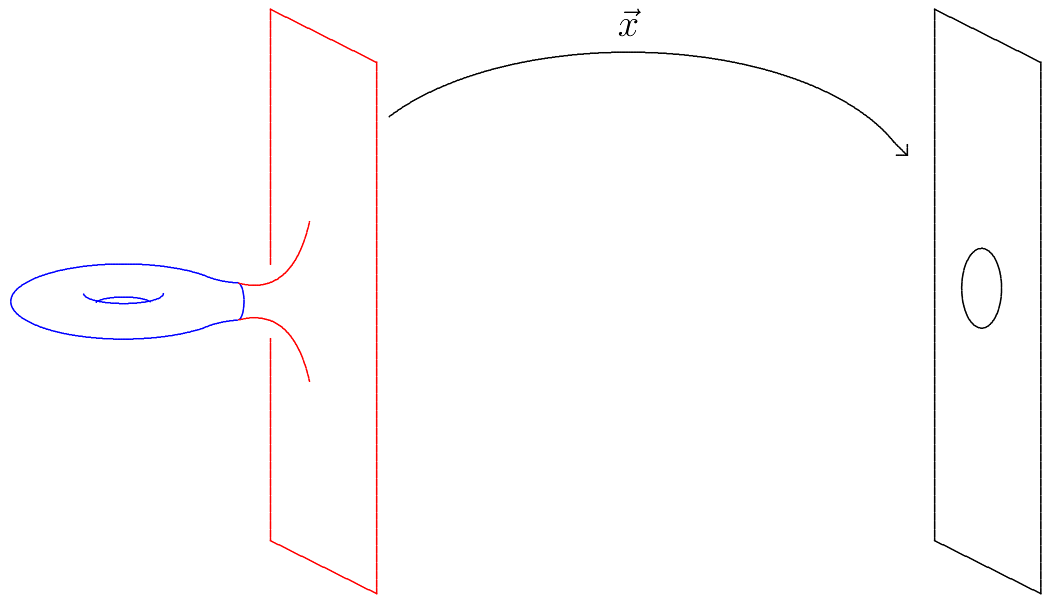

$$g_{jk} = \delta_{jk} + \text{terms that fall-off at infinity}$$

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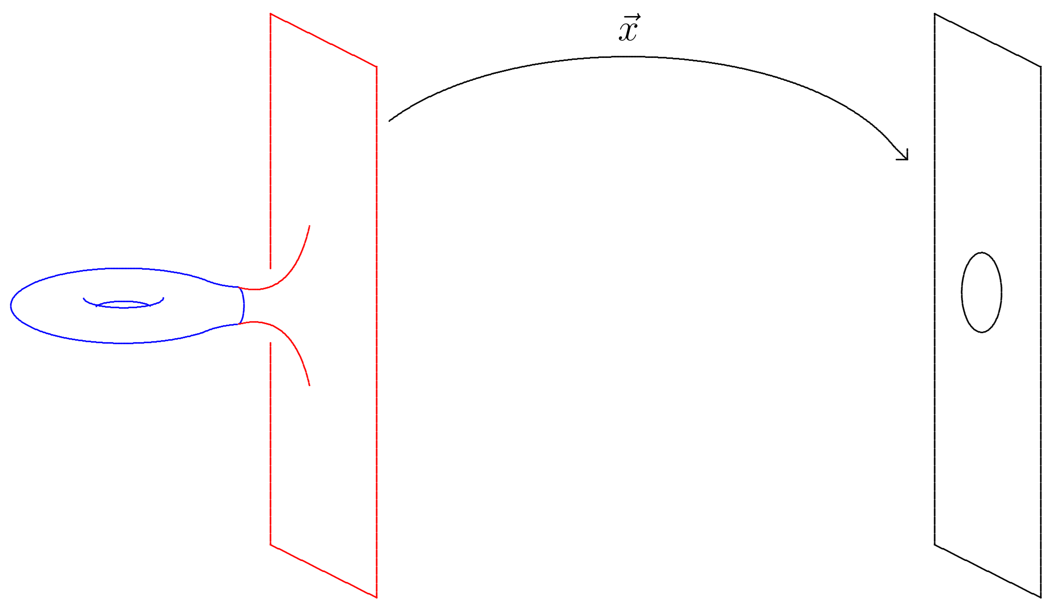

Weakest reasonable assumption:

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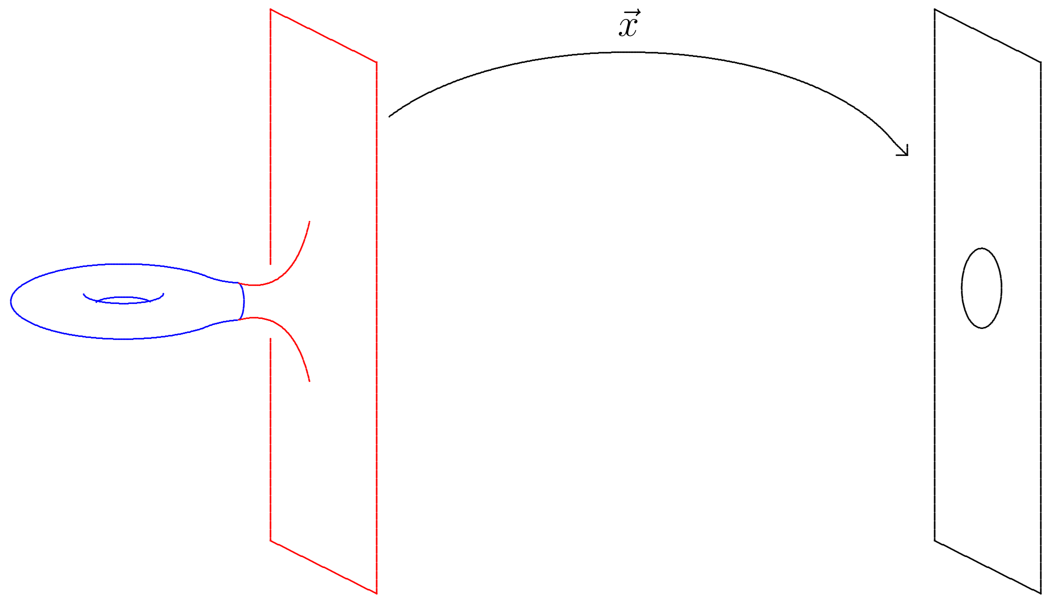

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Chruściel-type fall-off:

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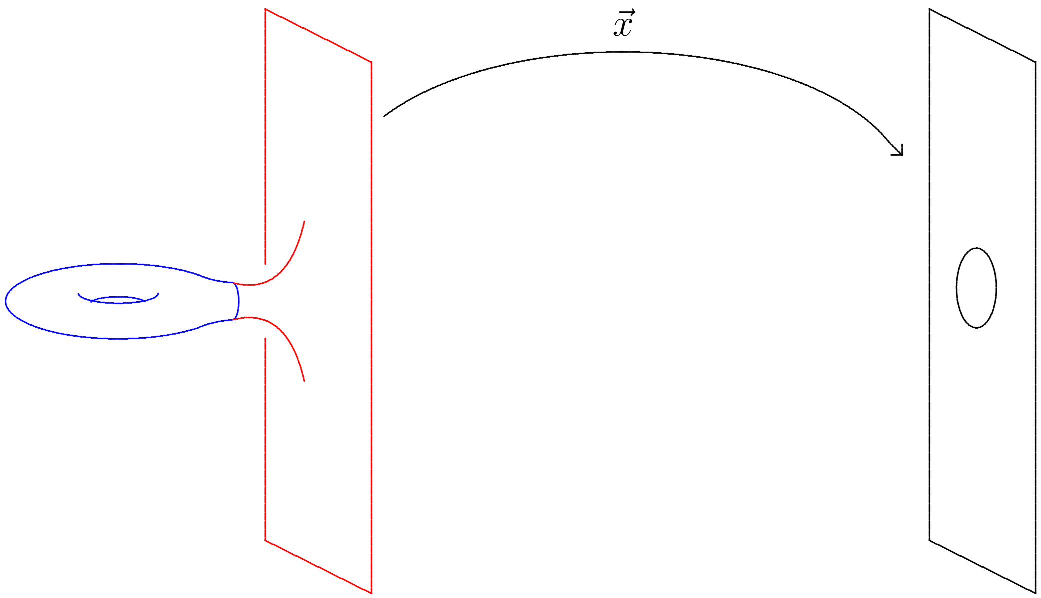

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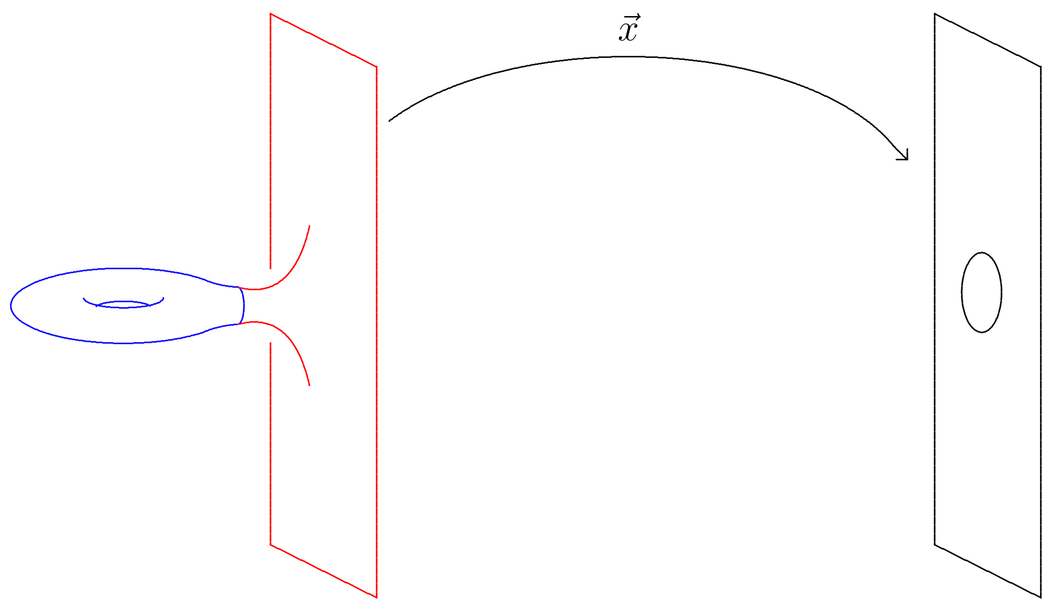

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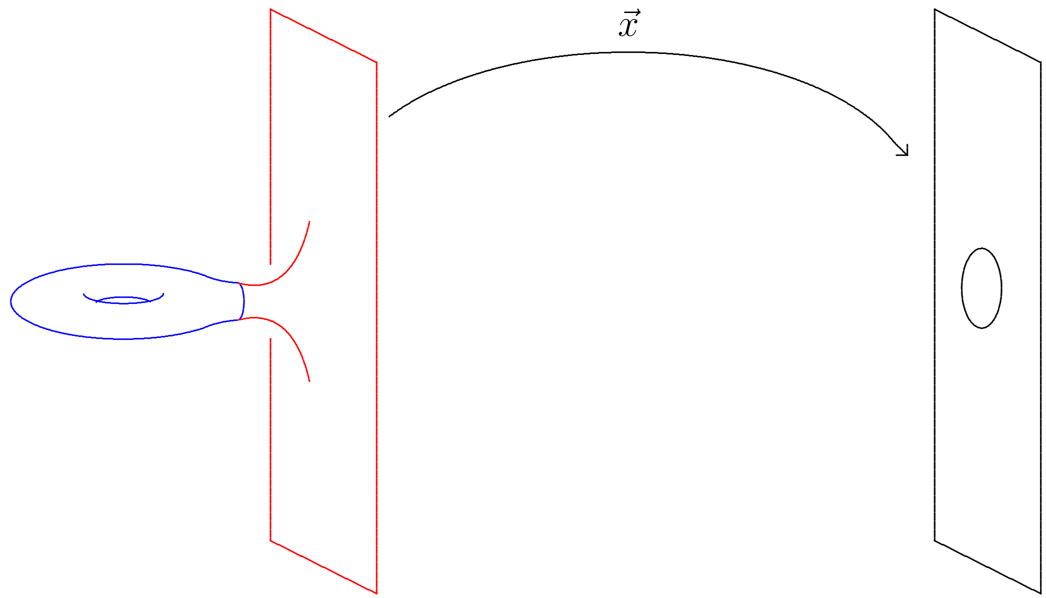

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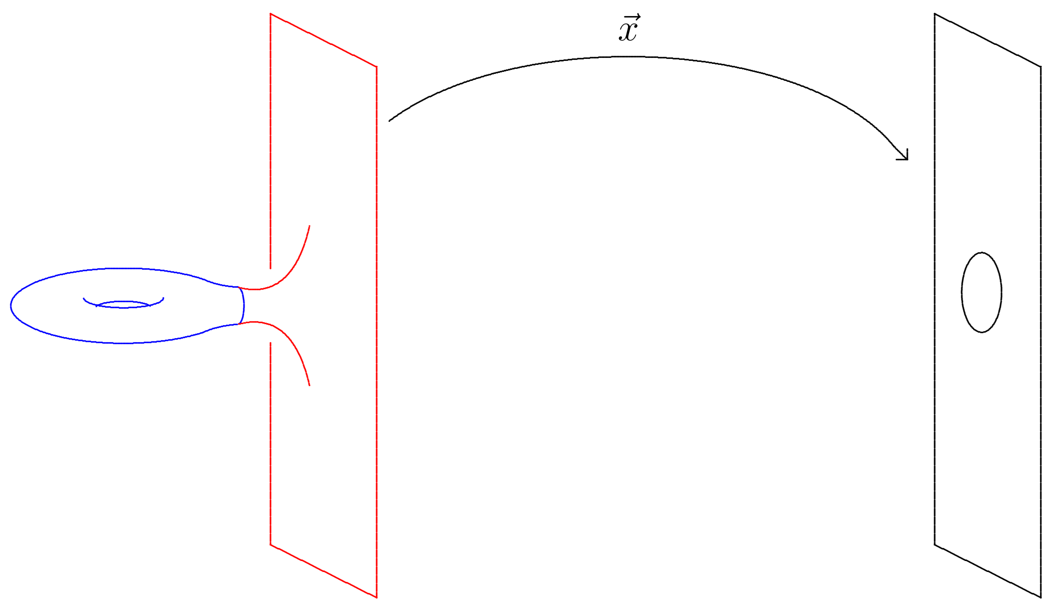
Definition. *A complete, non-compact Riemannian n-manifold (M^n, g) is called* asymptotically Euclidean *(AE) if there is a compact set $K \subset M$ such that*  *$M - K$ is diffeomorphic to $\mathbb{R}^n - D^n$ in such a manner that*

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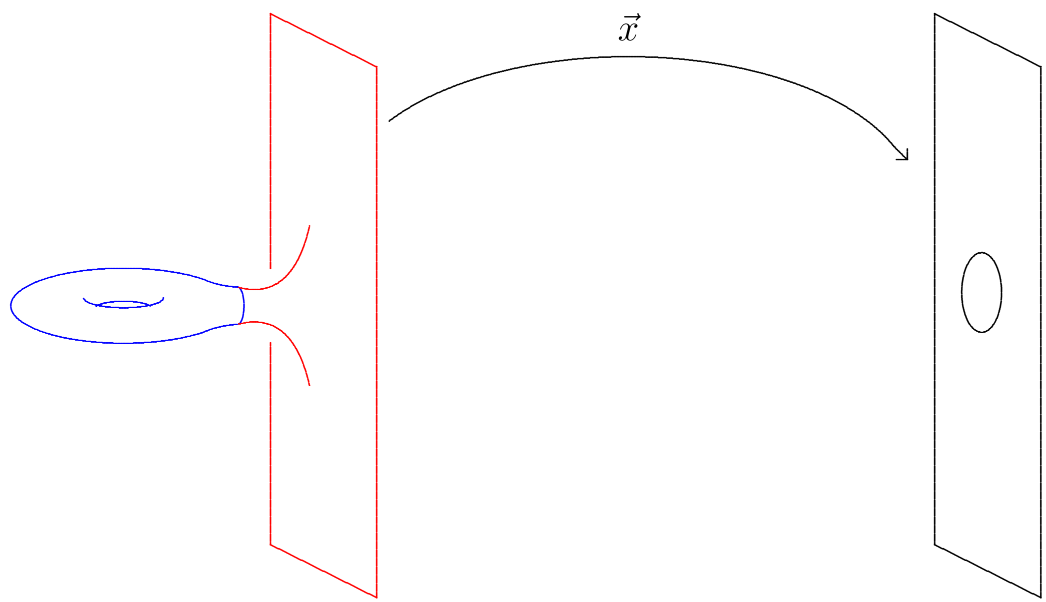

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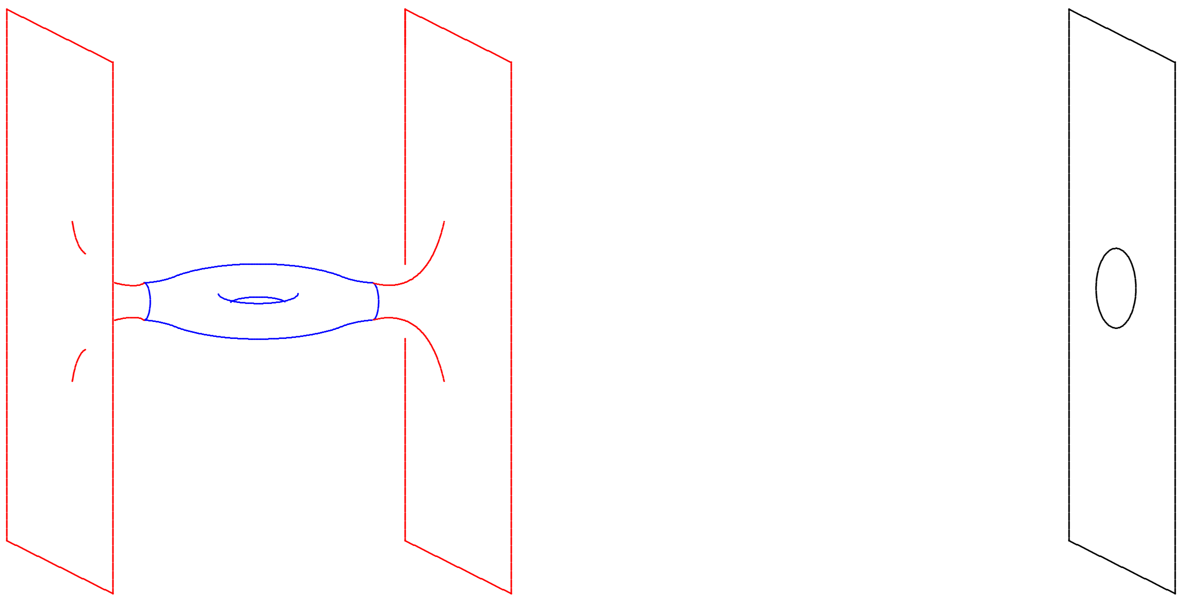

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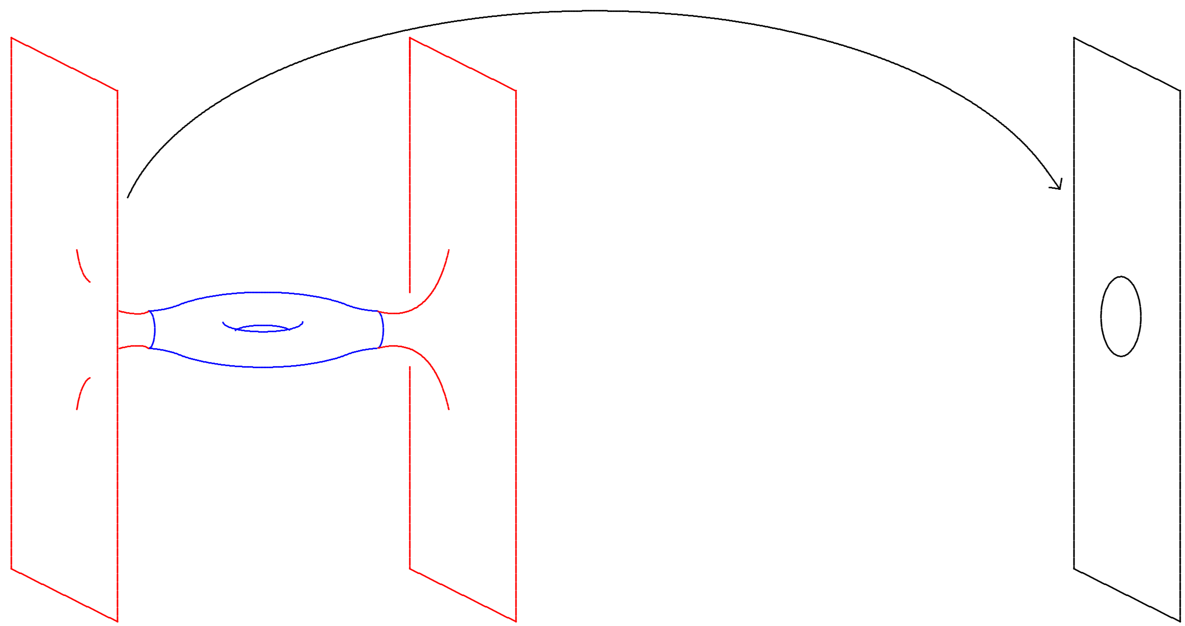

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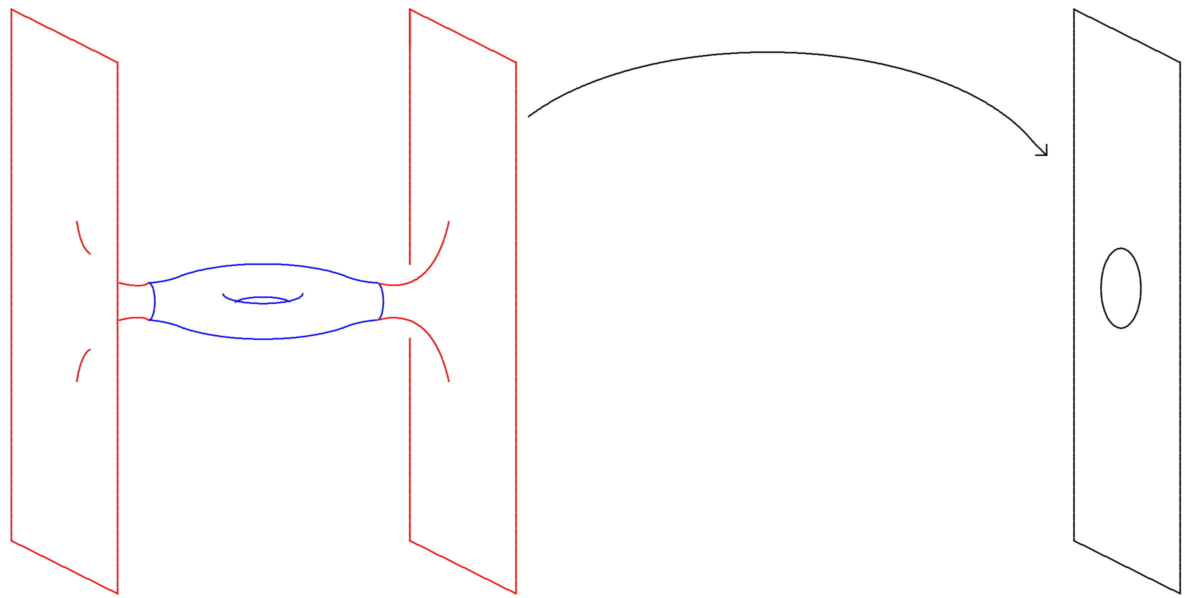

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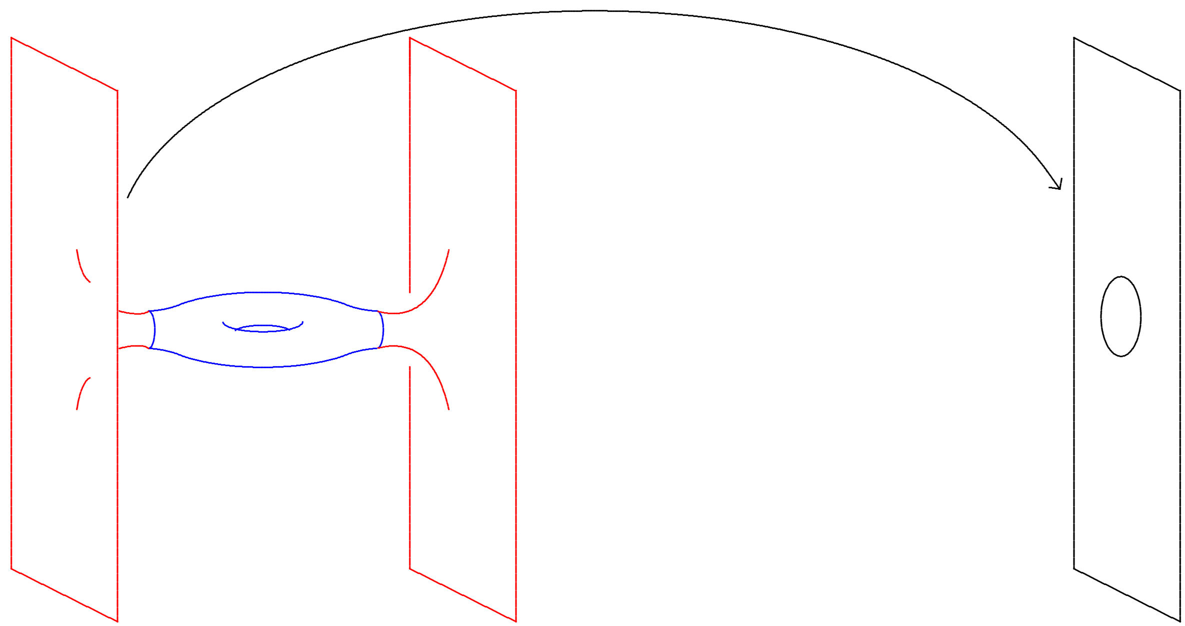

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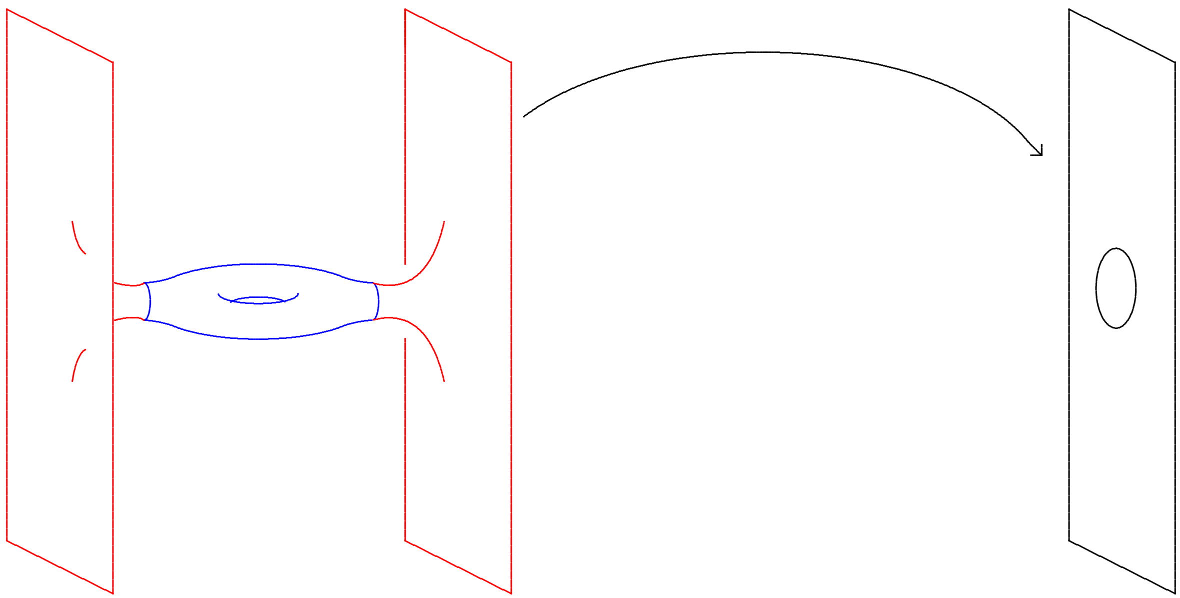

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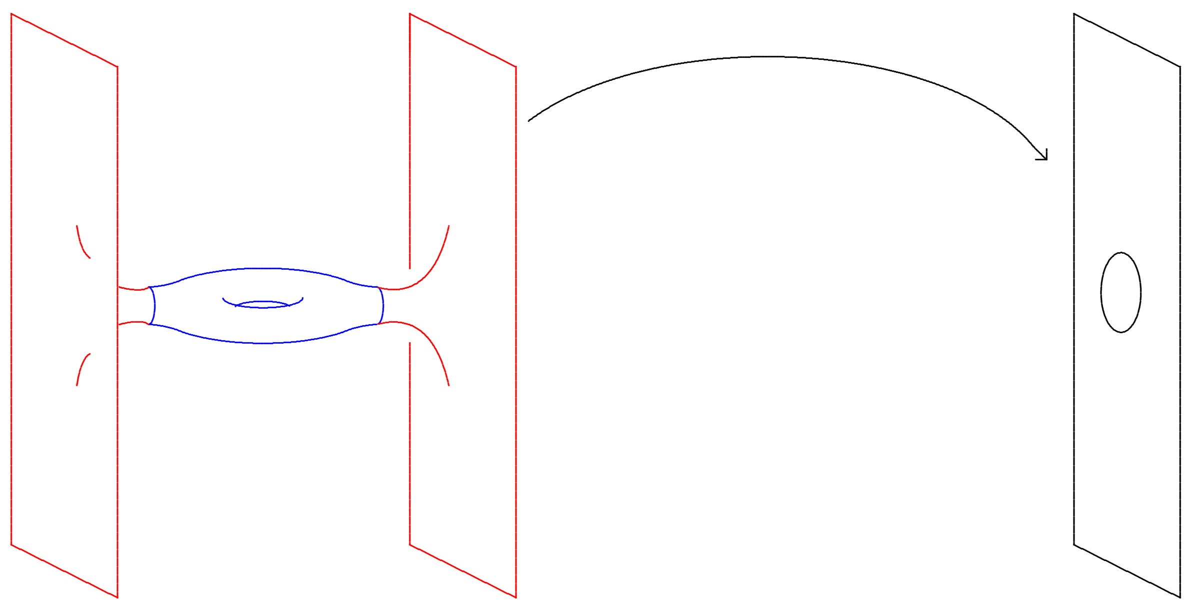

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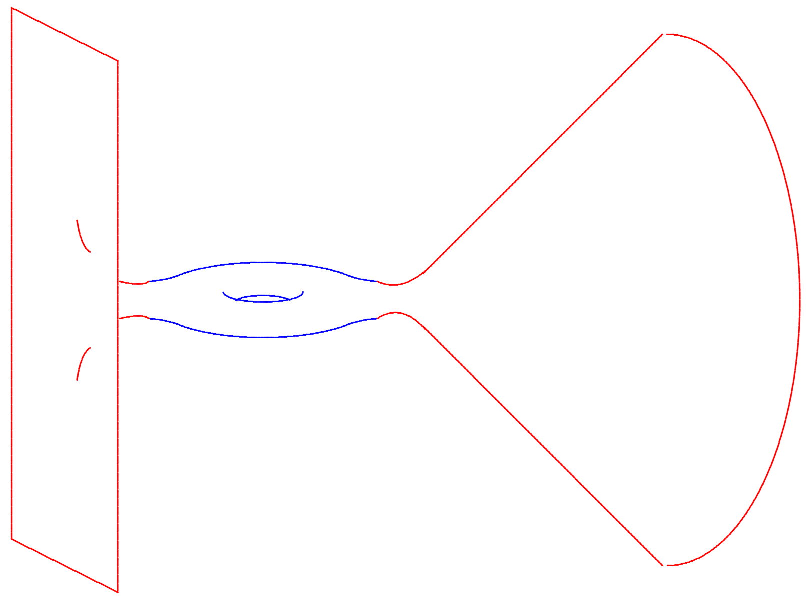

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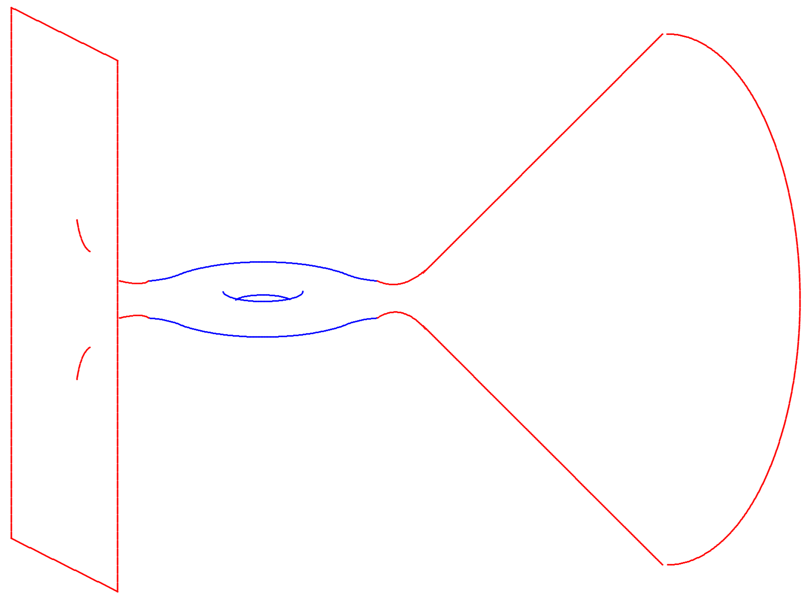

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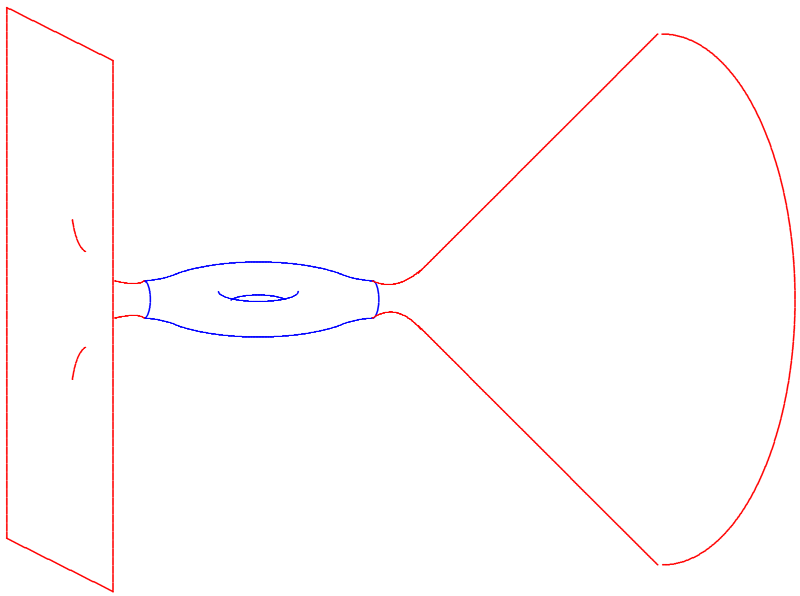

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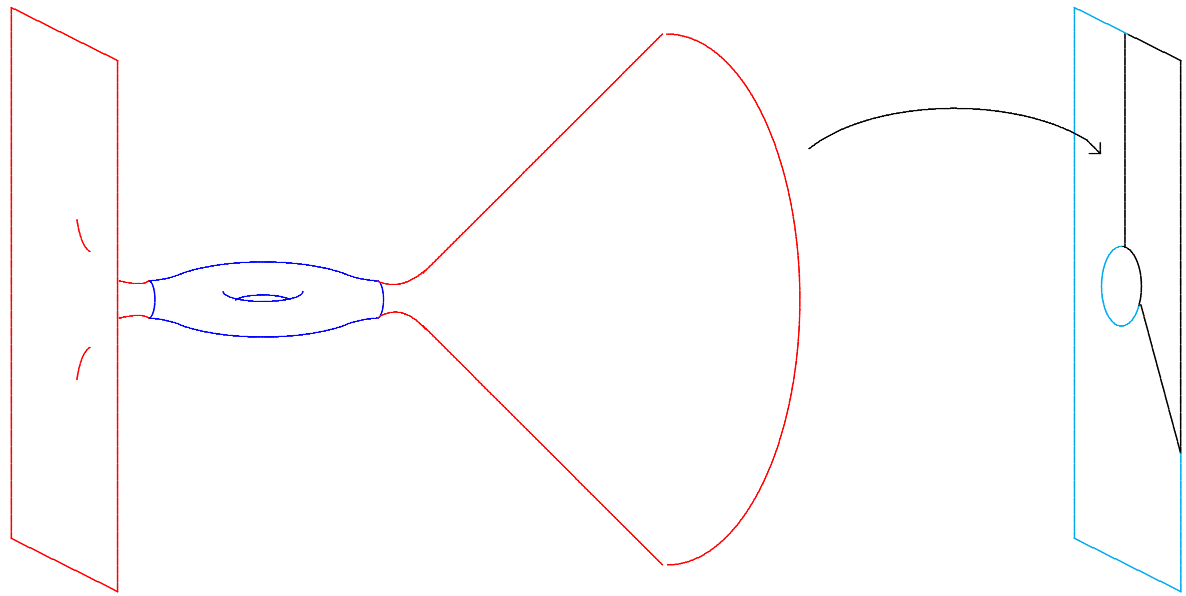

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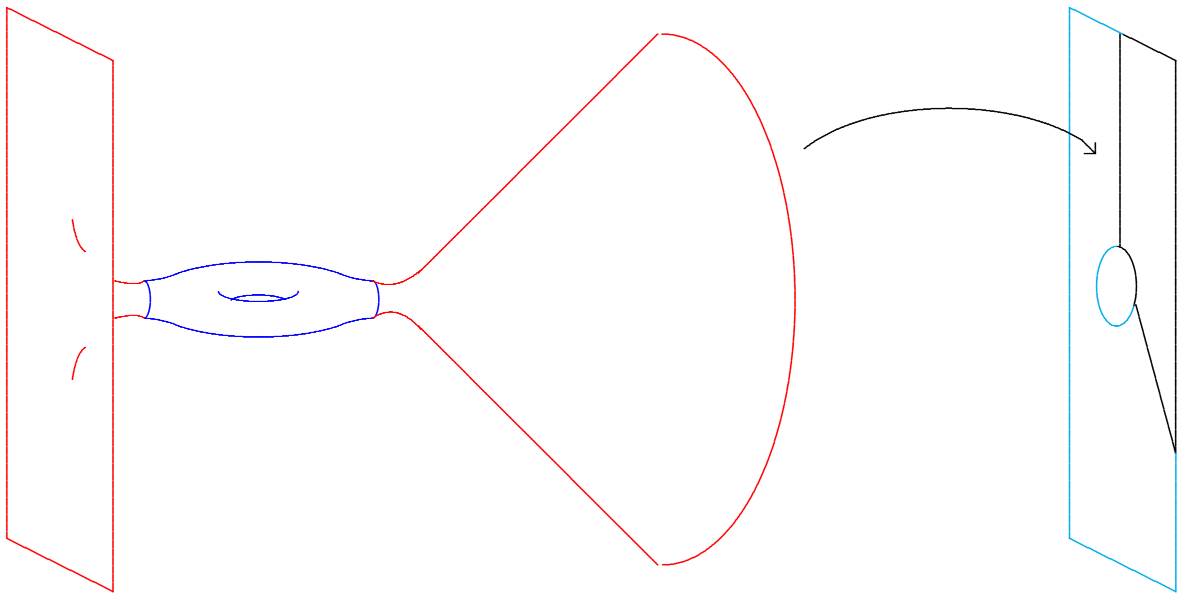

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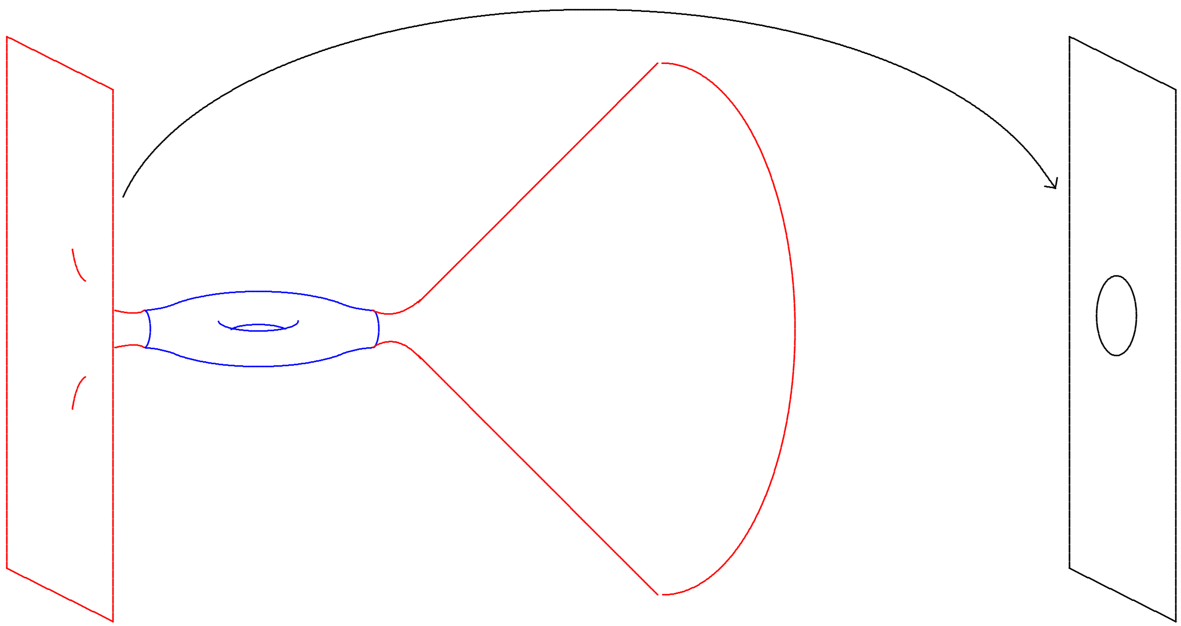

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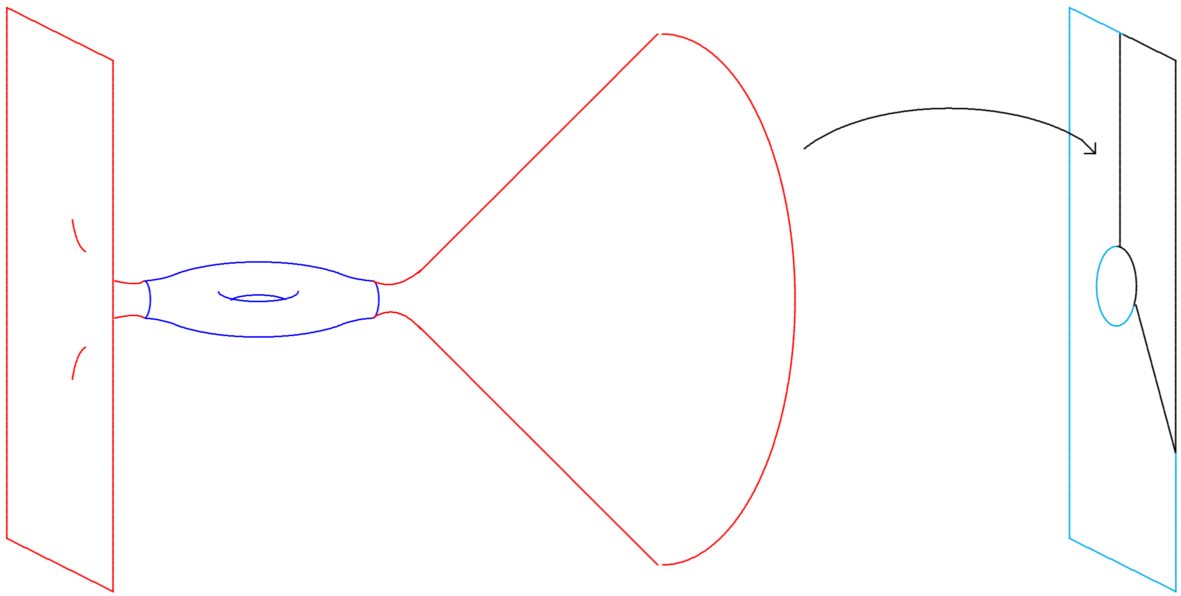

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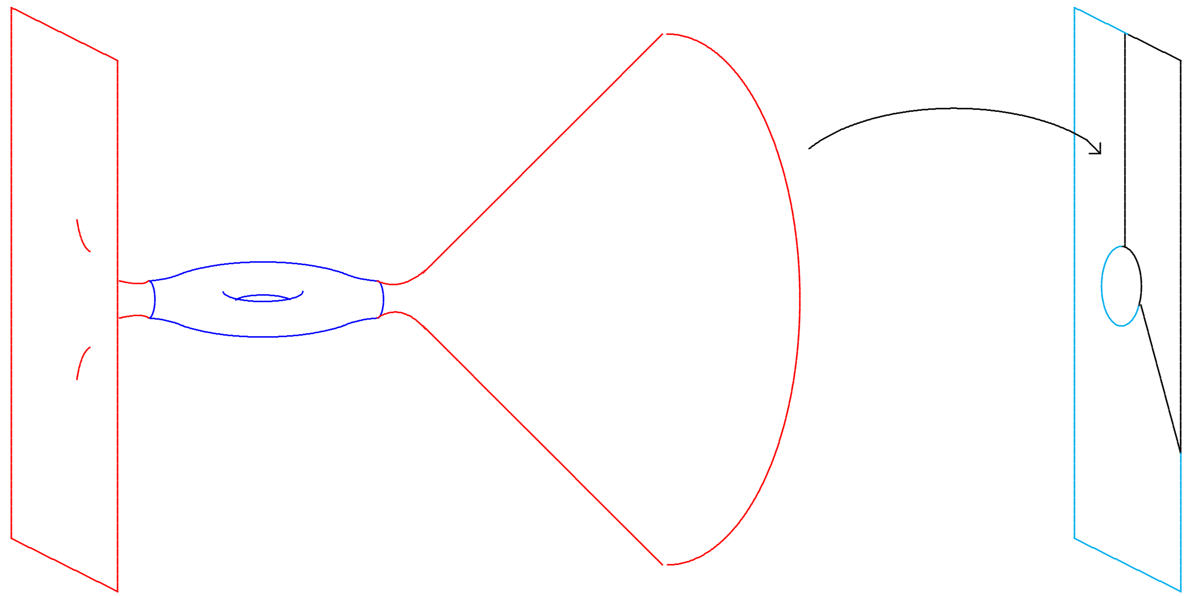

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Why consider ALE spaces?

Key examples:

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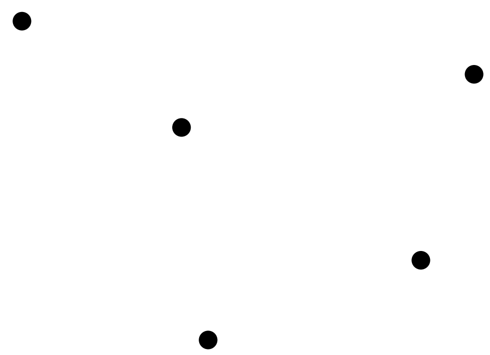
They wrote down various explicit Ricci-flat ALE 4-manifolds they called gravitational instantons.



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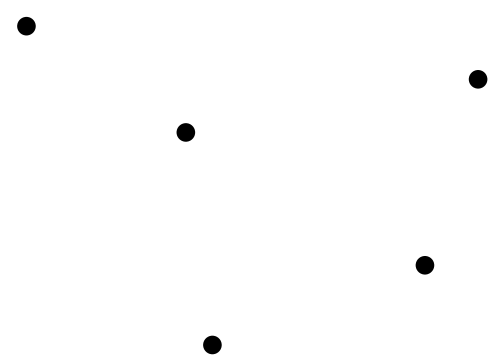
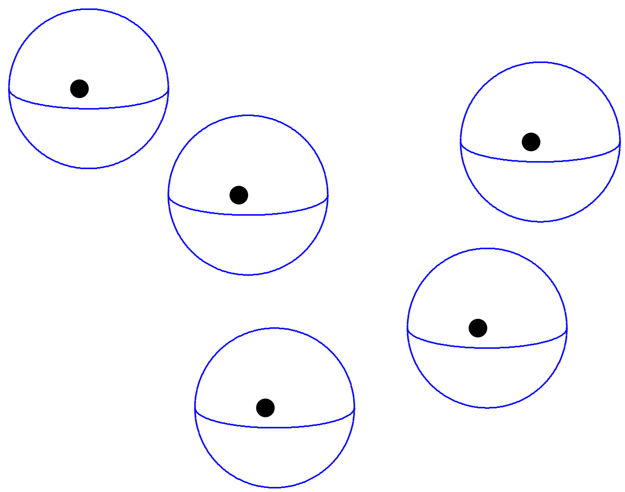


Data: ℓ points in $\mathbb{R}^3$. $\Longrightarrow$ V with $\Delta V = 0$

$$V = \sum_{j=1}^{\ell} \frac{1}{2\varrho_j}$$



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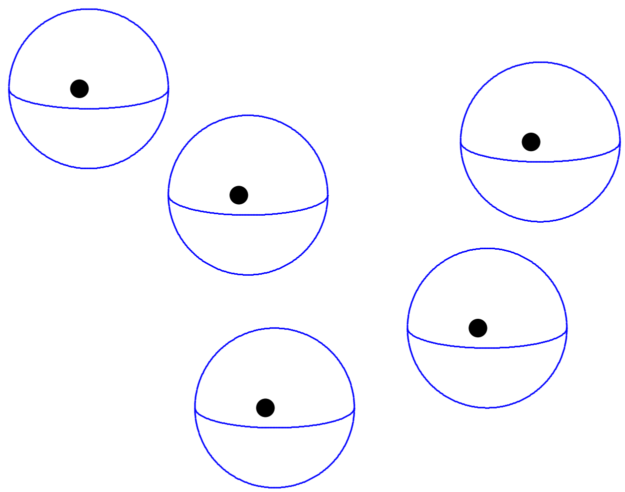
$F = \star dV$ curvature θ on $P \to \mathbb{R}^3 - \{\text{pts}\}$.



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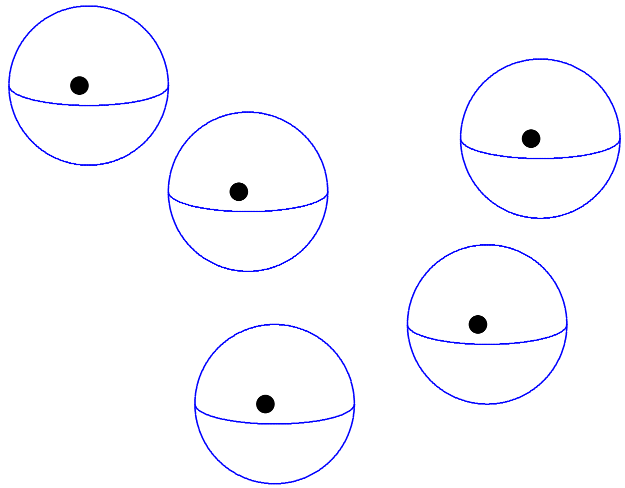

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$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

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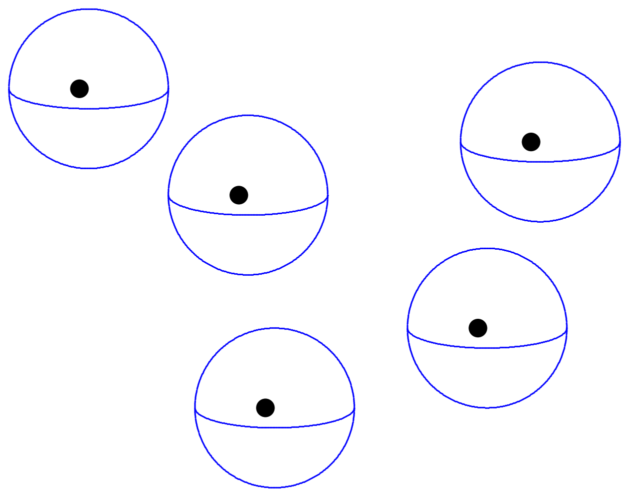

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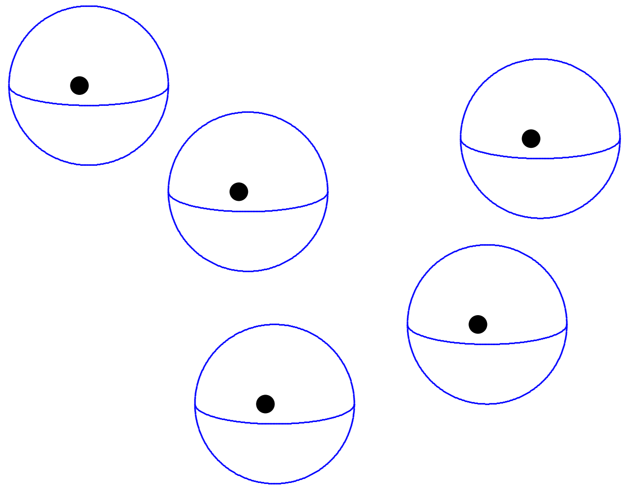

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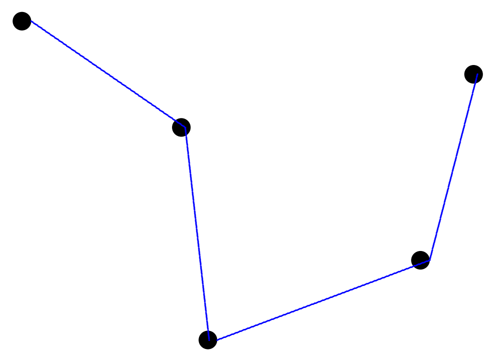

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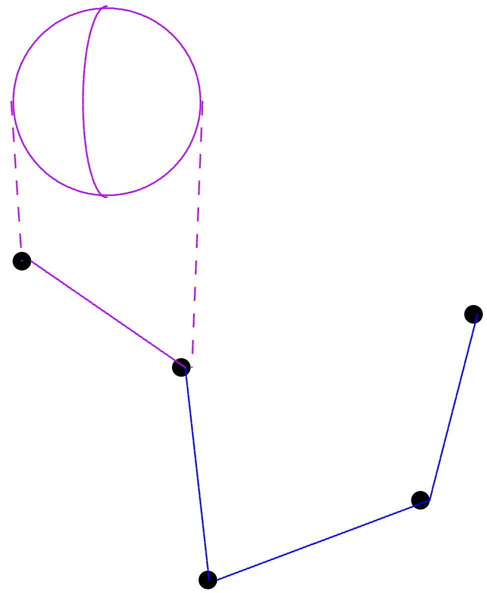

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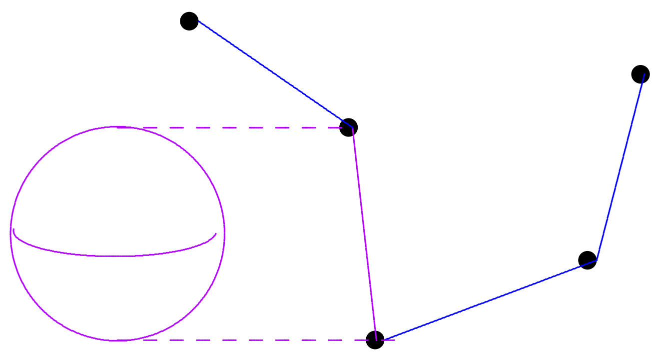

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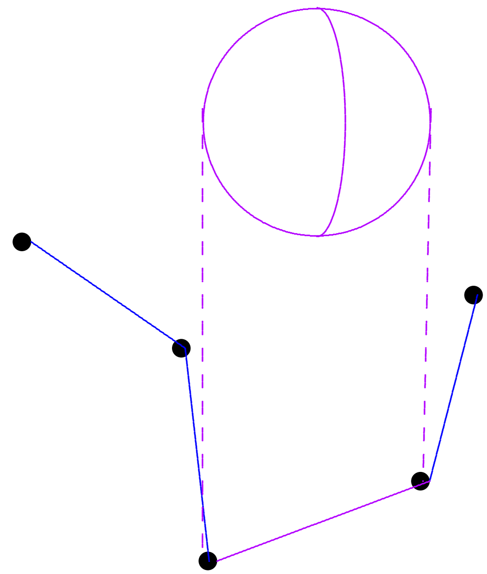

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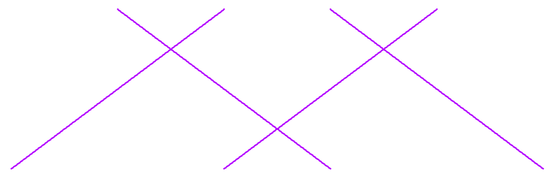

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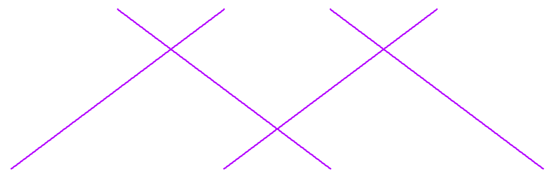

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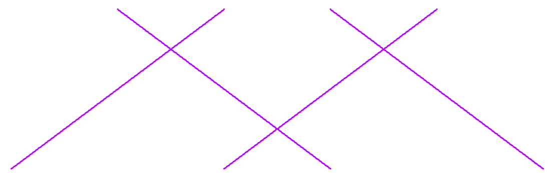

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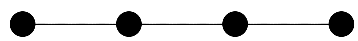

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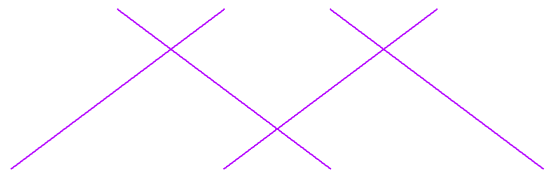

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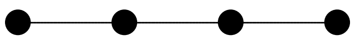

Diffeotype:

Deform retracts to $k = \ell - 1$ copies of S^2,
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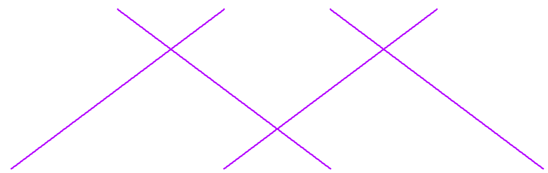

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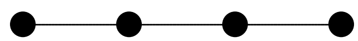

Diffeotype:

Plumb together k copies of T^*S^2
according to diagram.

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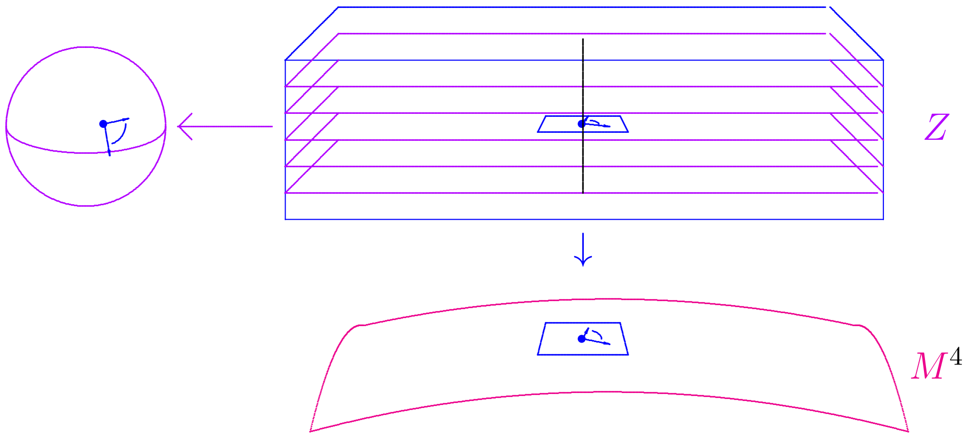

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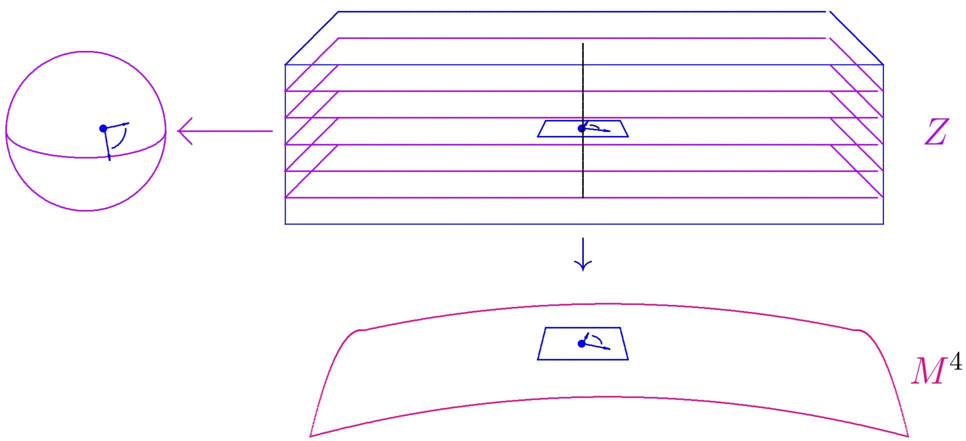

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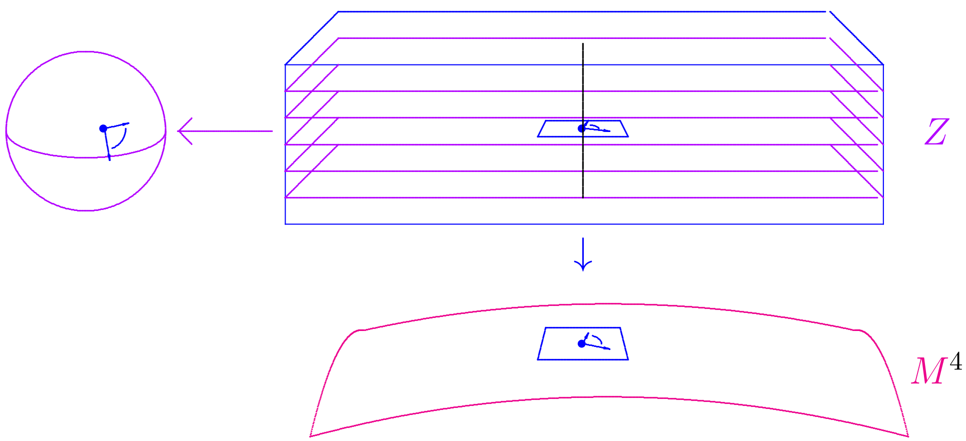

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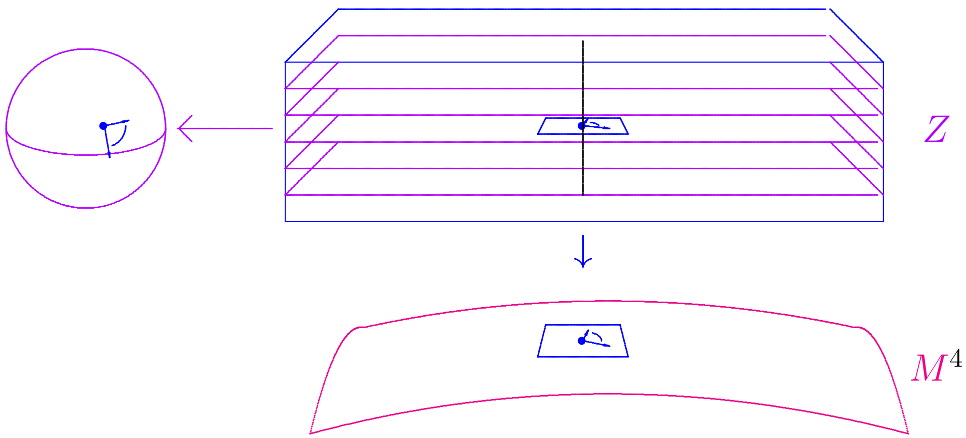

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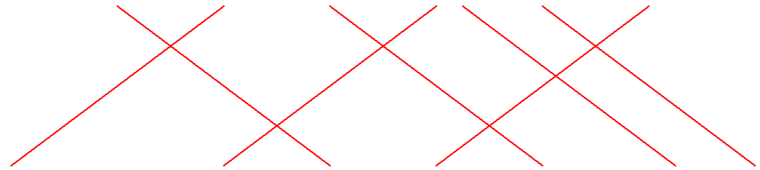

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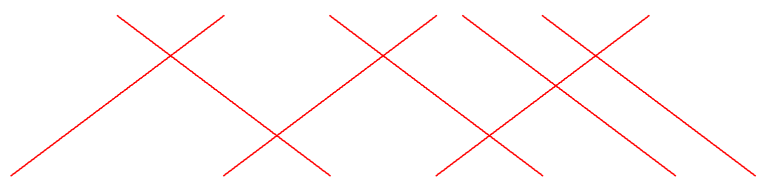

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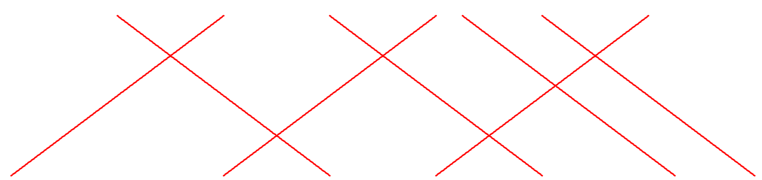

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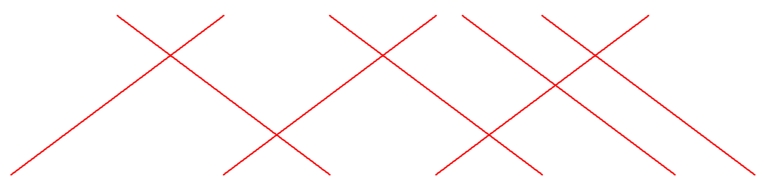

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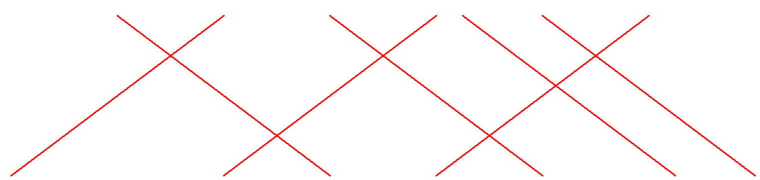
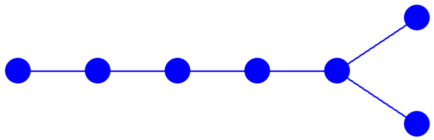


Intersection pattern dual to Dynkin diagram!



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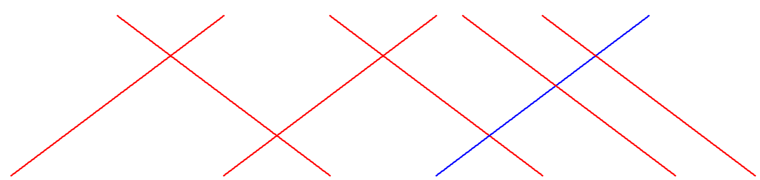

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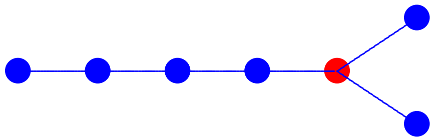

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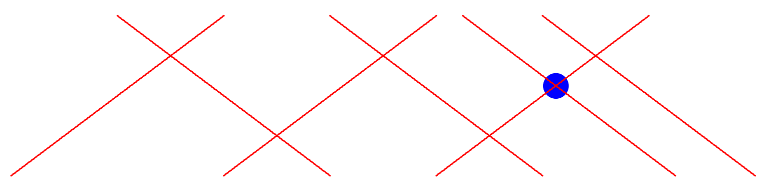

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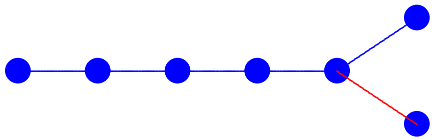

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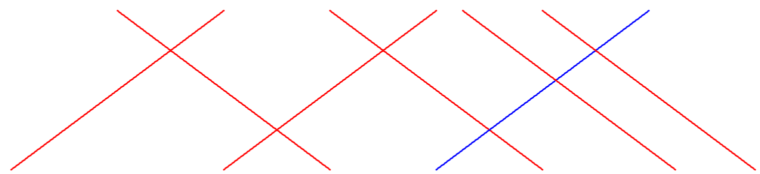

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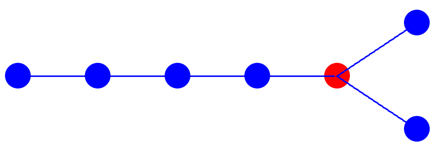

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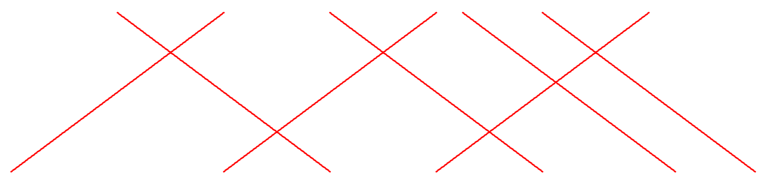

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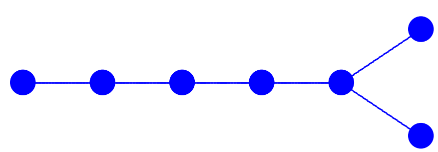

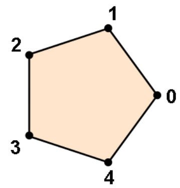

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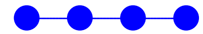

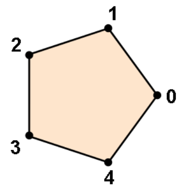

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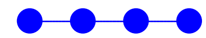

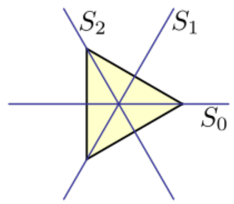

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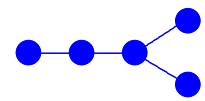

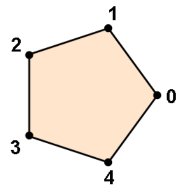

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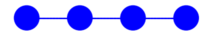

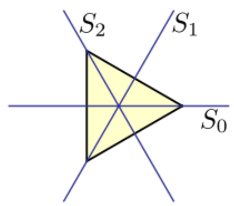

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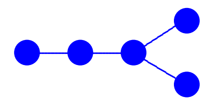

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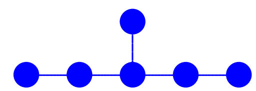

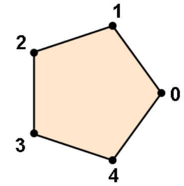

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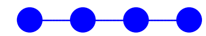

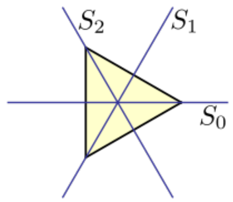

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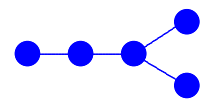

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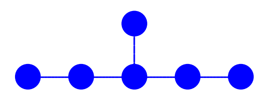

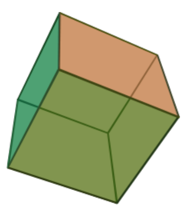

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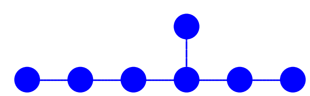

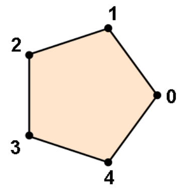

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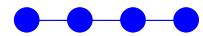

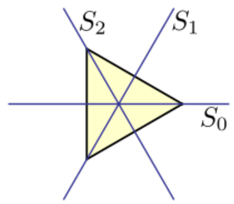

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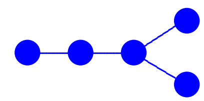

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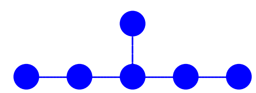

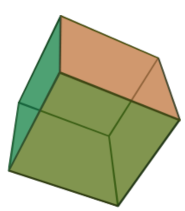

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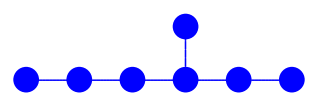

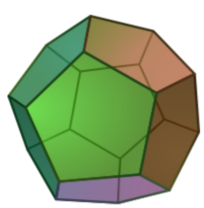

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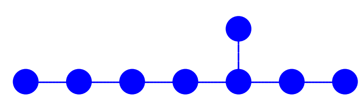

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Proved by Kronheimer, who also showed (1989) this gives complete classification of ALE hyper-Kählers.

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(L '91)

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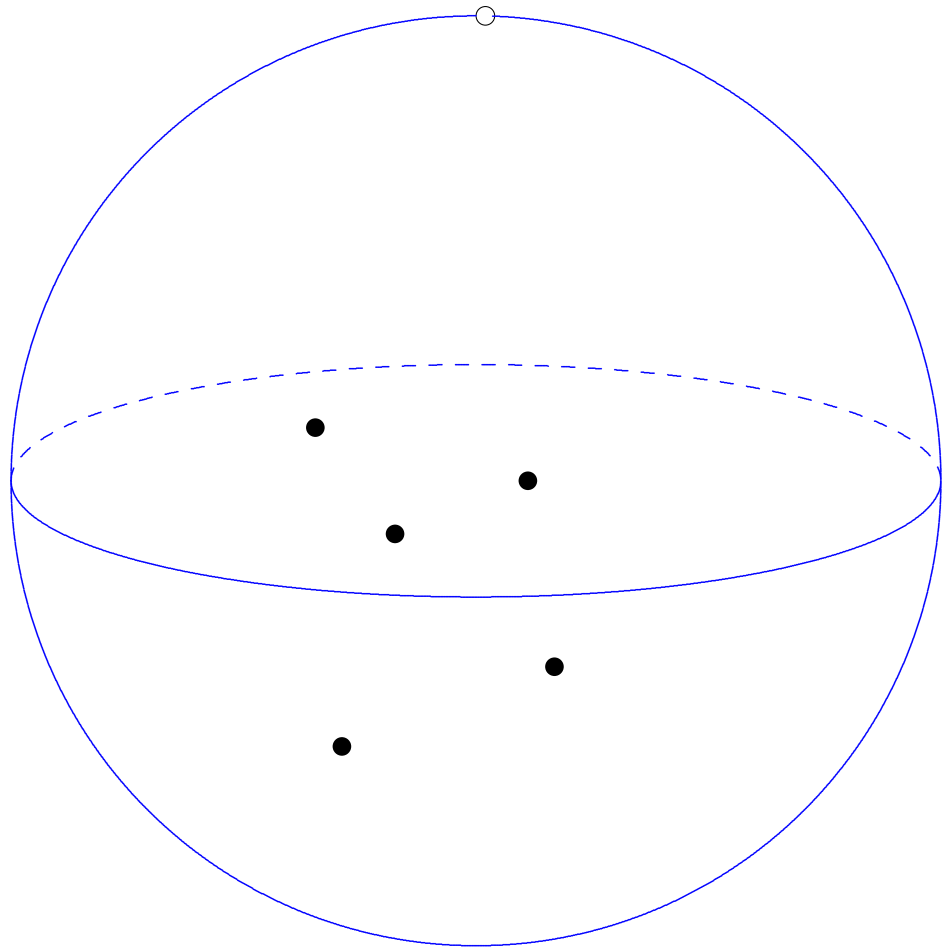

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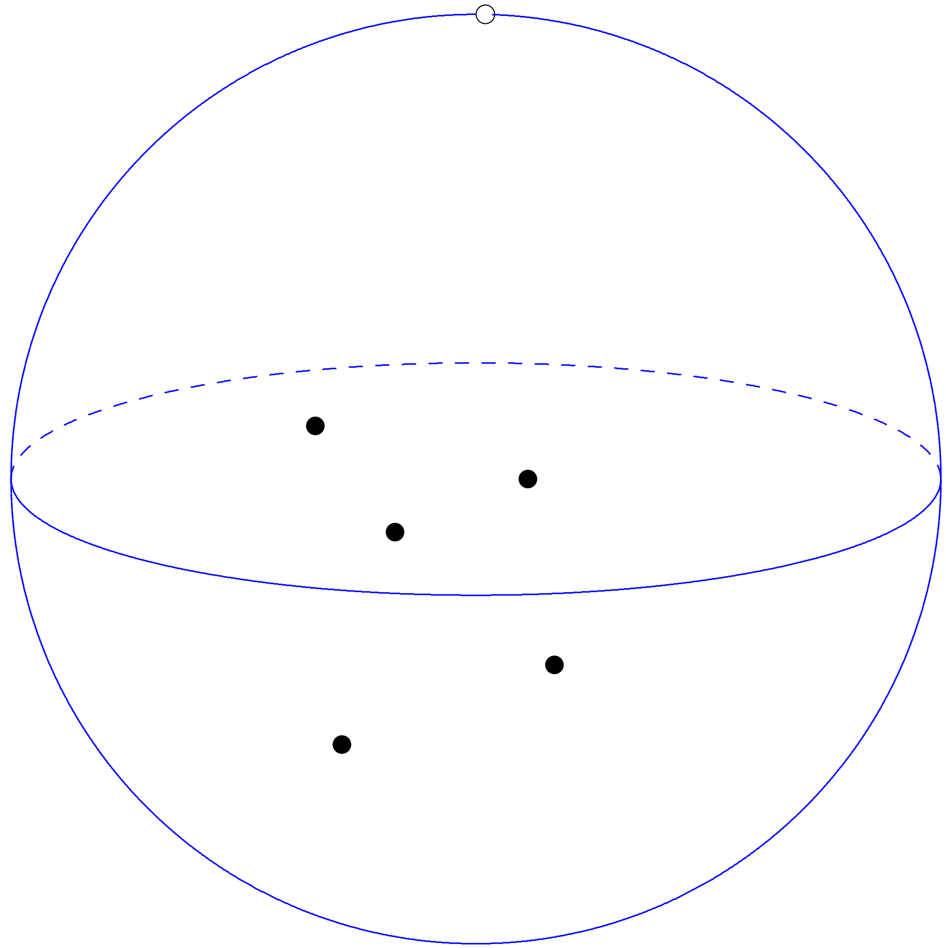

Data: k points in $\mathcal{H}^3$ and one point at infinity.

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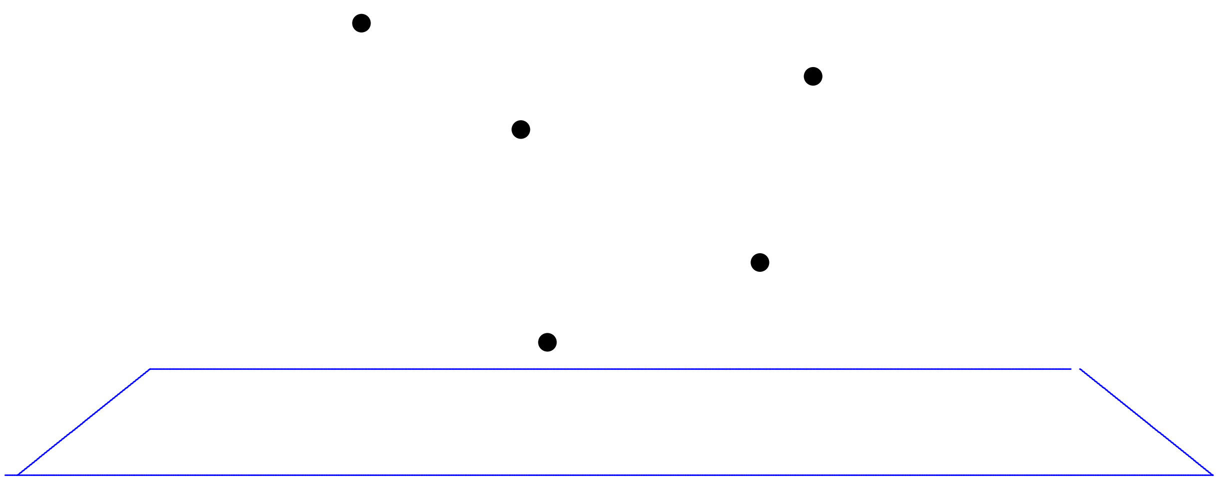

Data: k points in $\mathcal{H}^3$ = upper half-space model.

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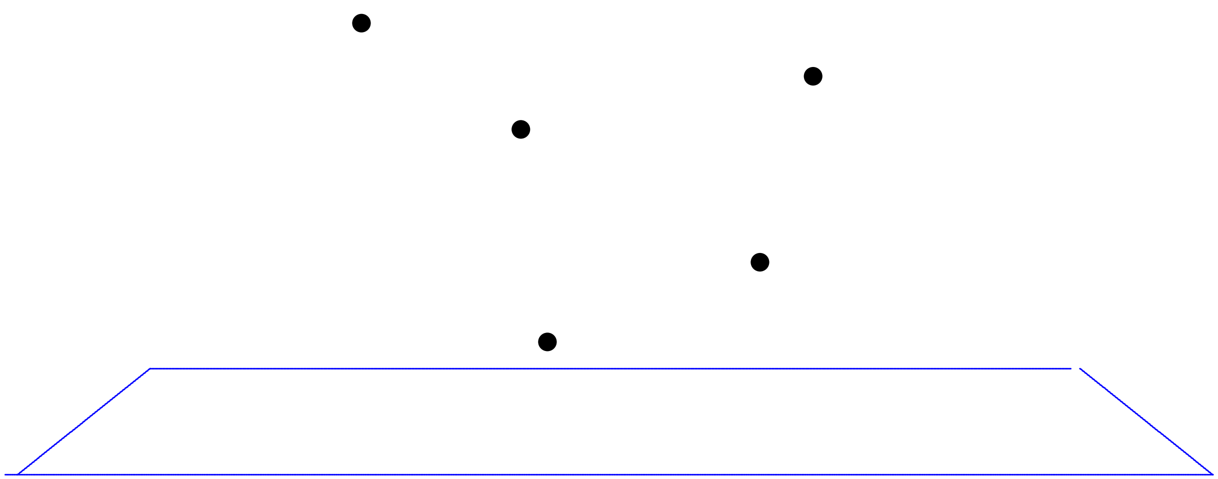

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$$V = 1 + \sum_{j=1}^{k} G_j$$

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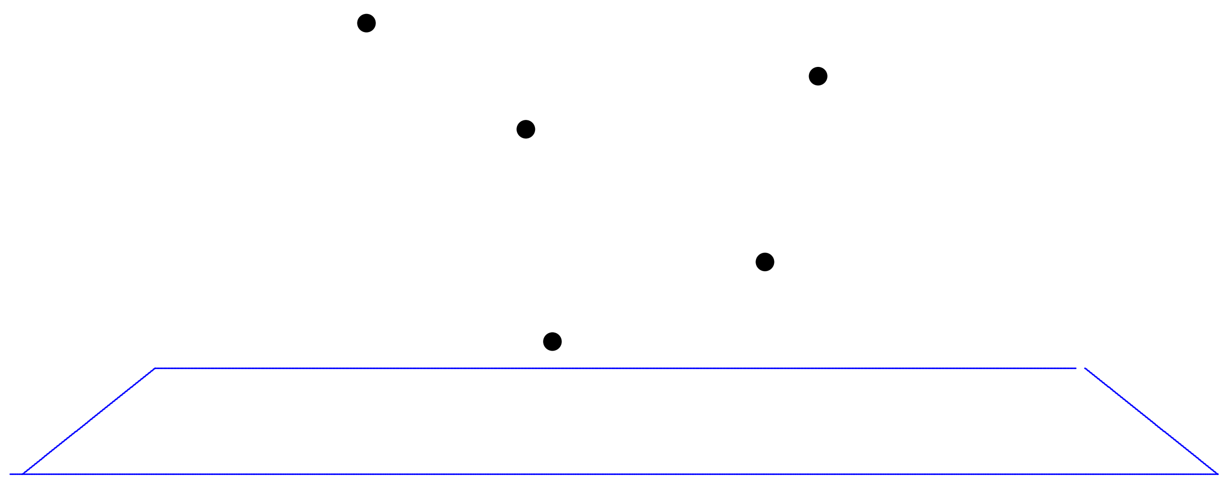

Data: k points in $\mathcal{H}^3$. $\Longrightarrow$ V with $\Delta V = 0$

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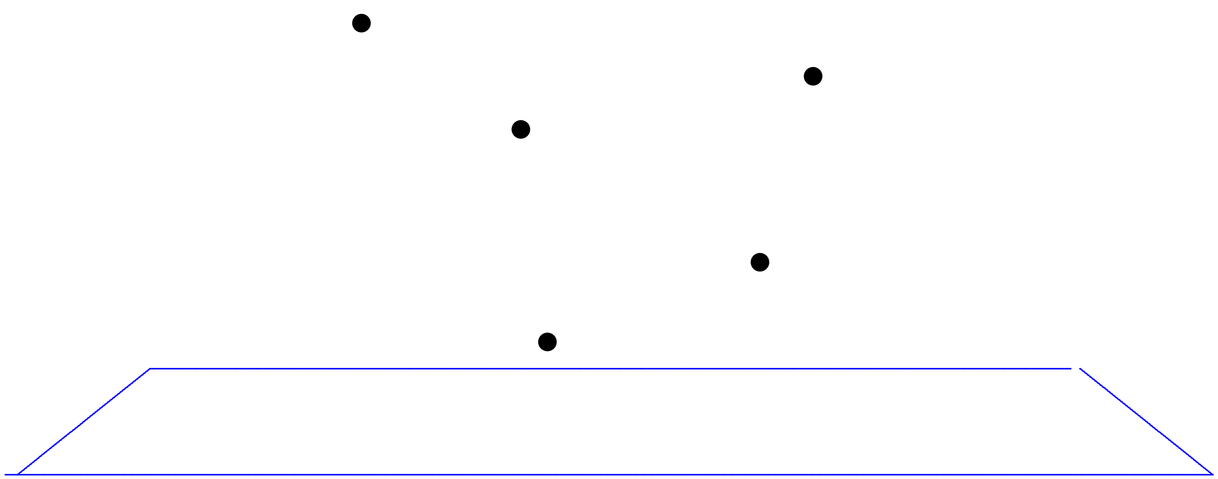

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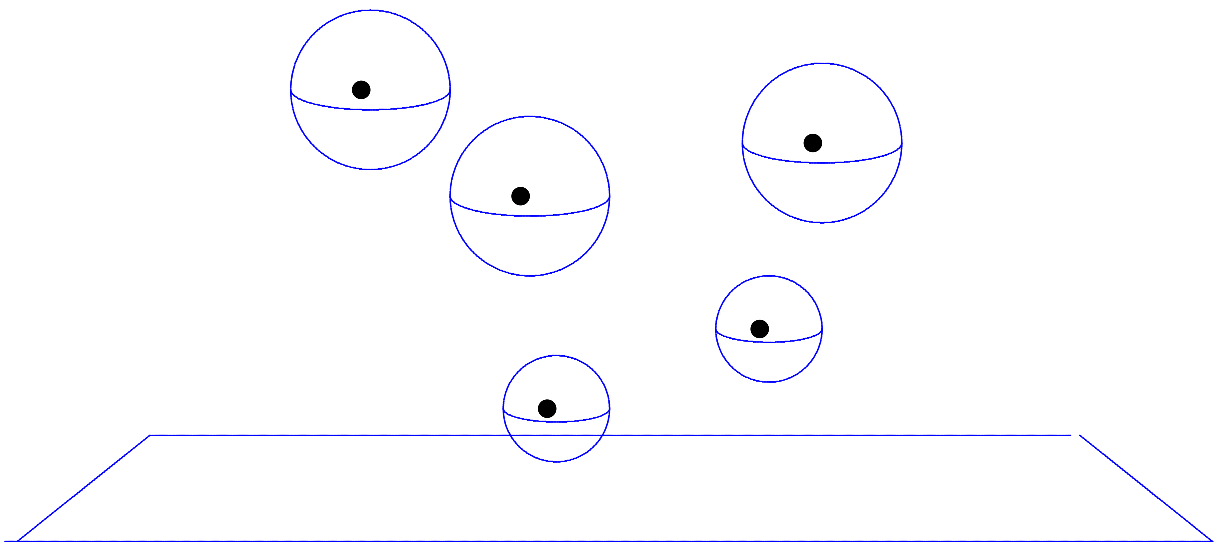

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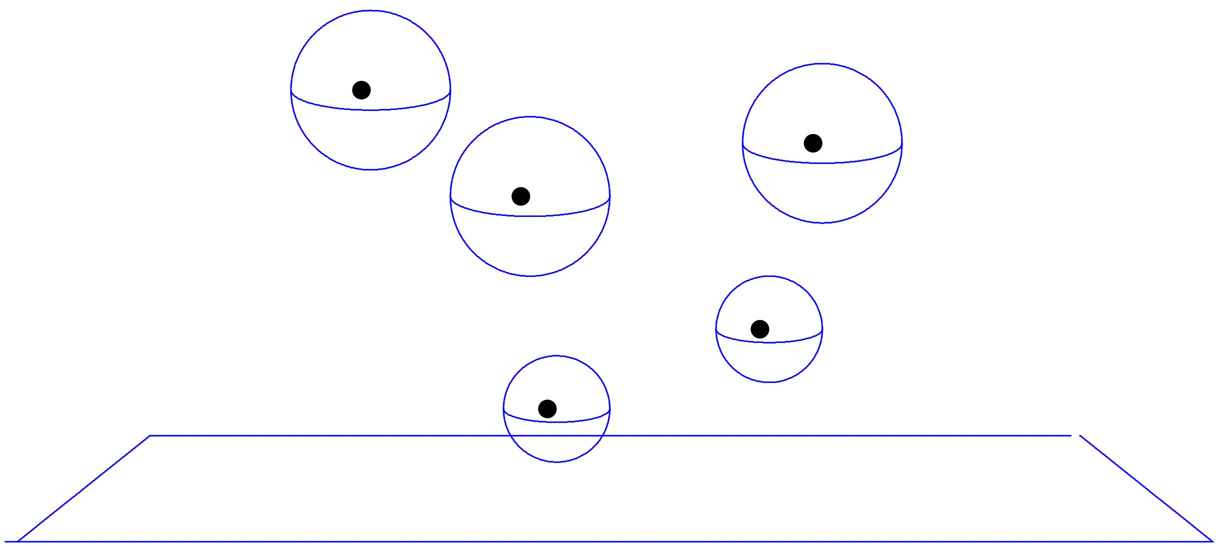

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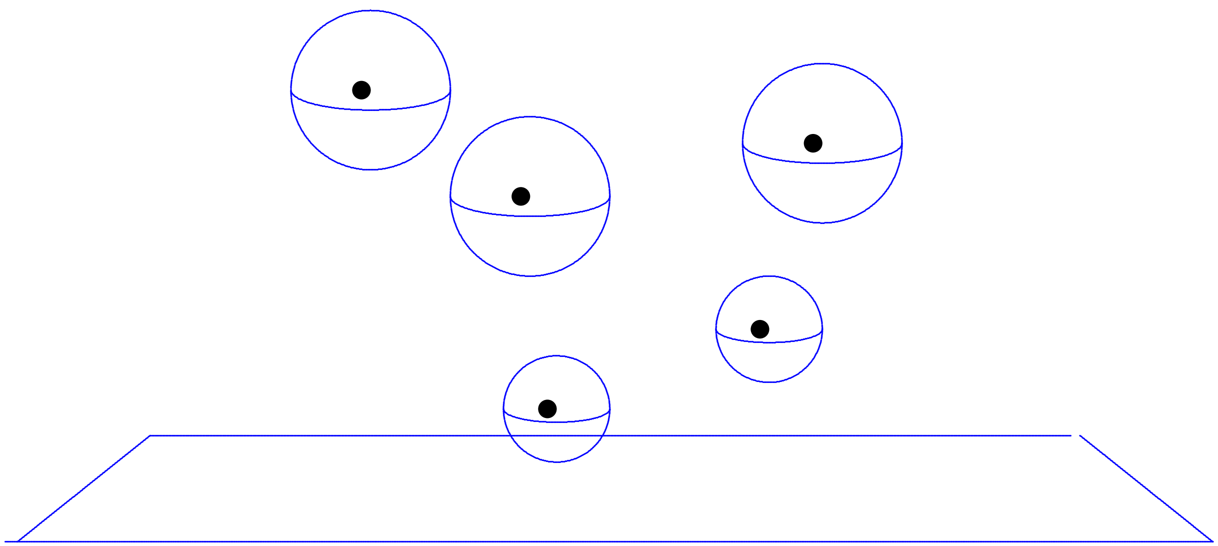

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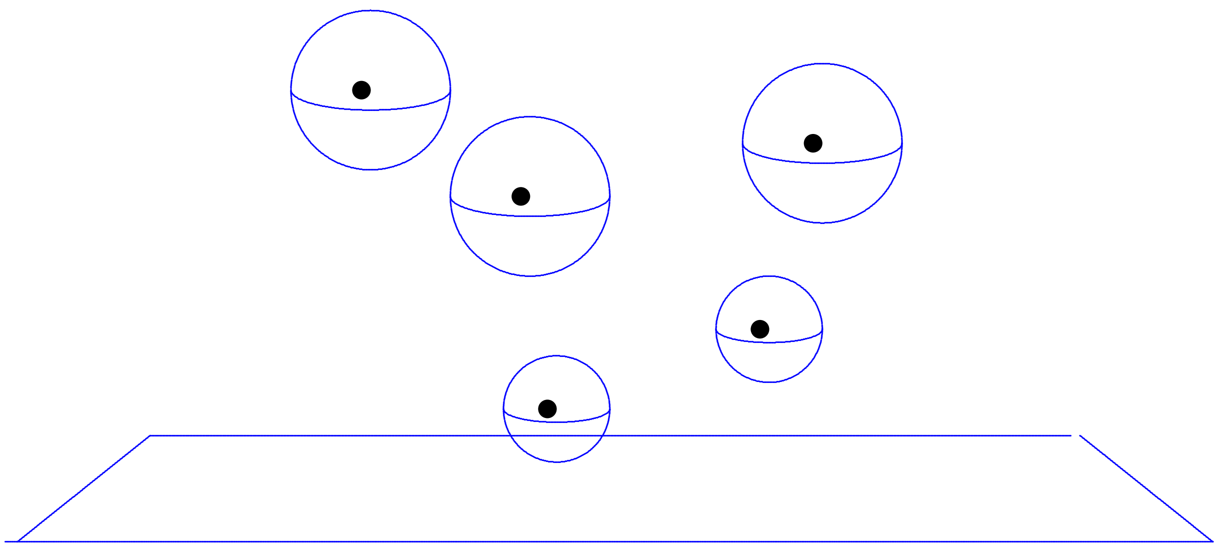

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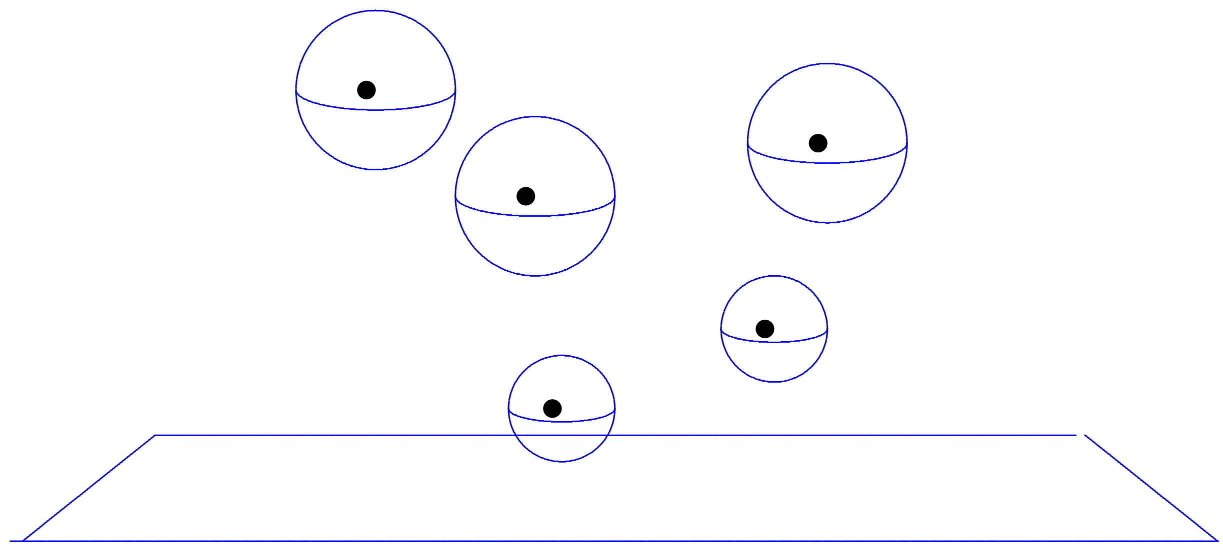

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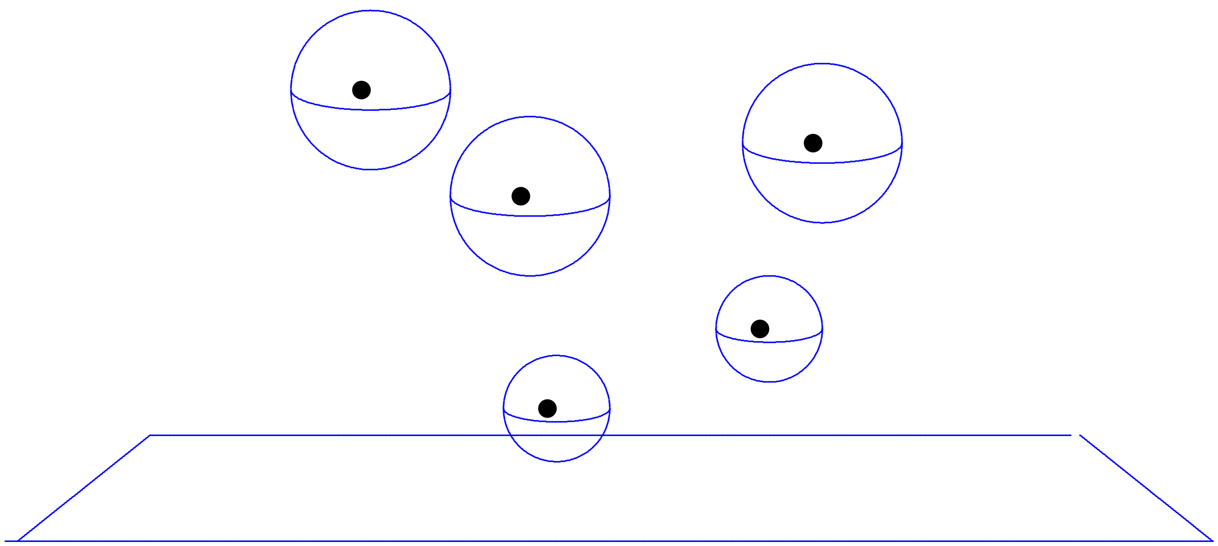

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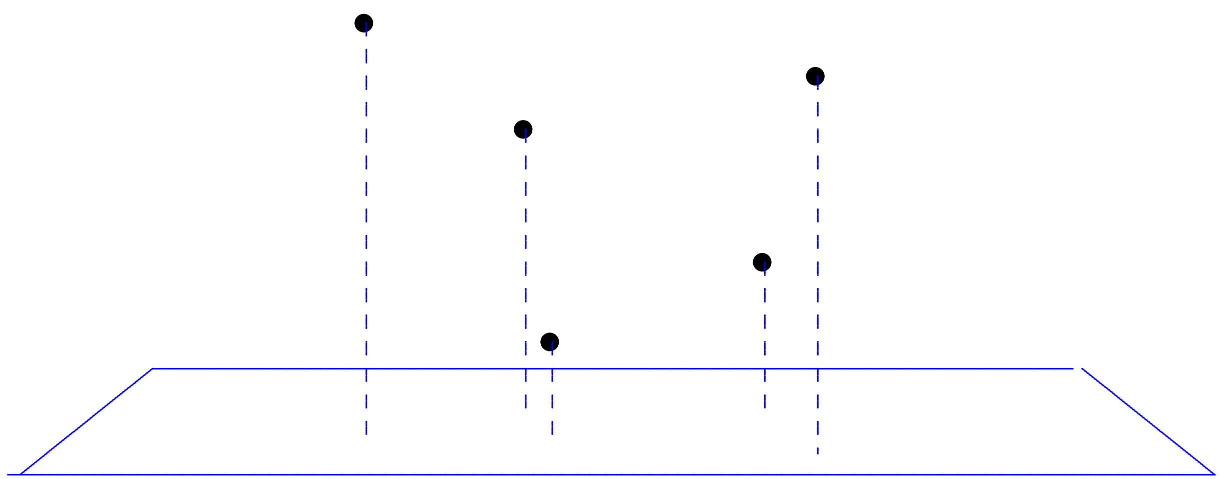

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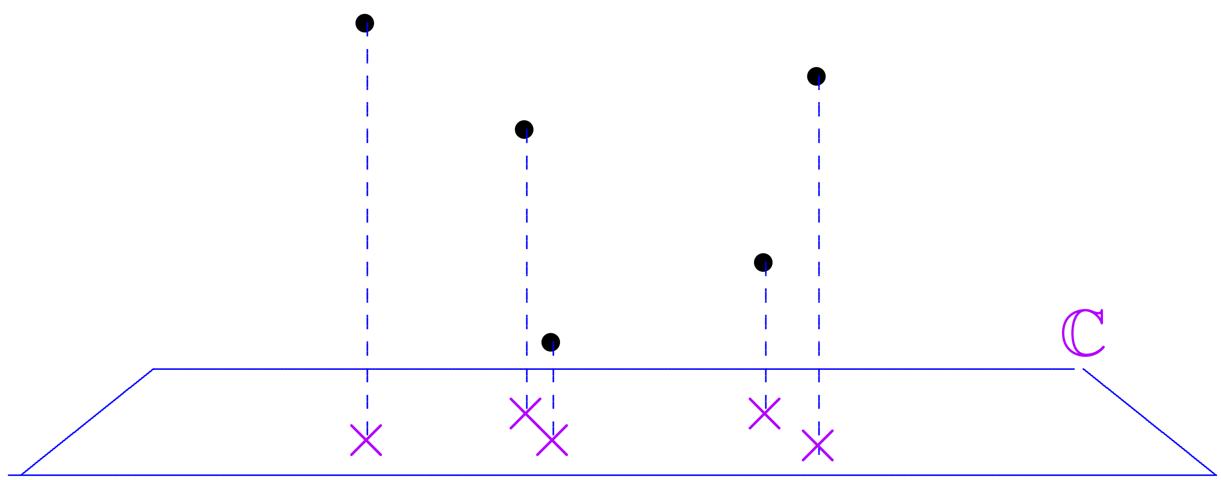

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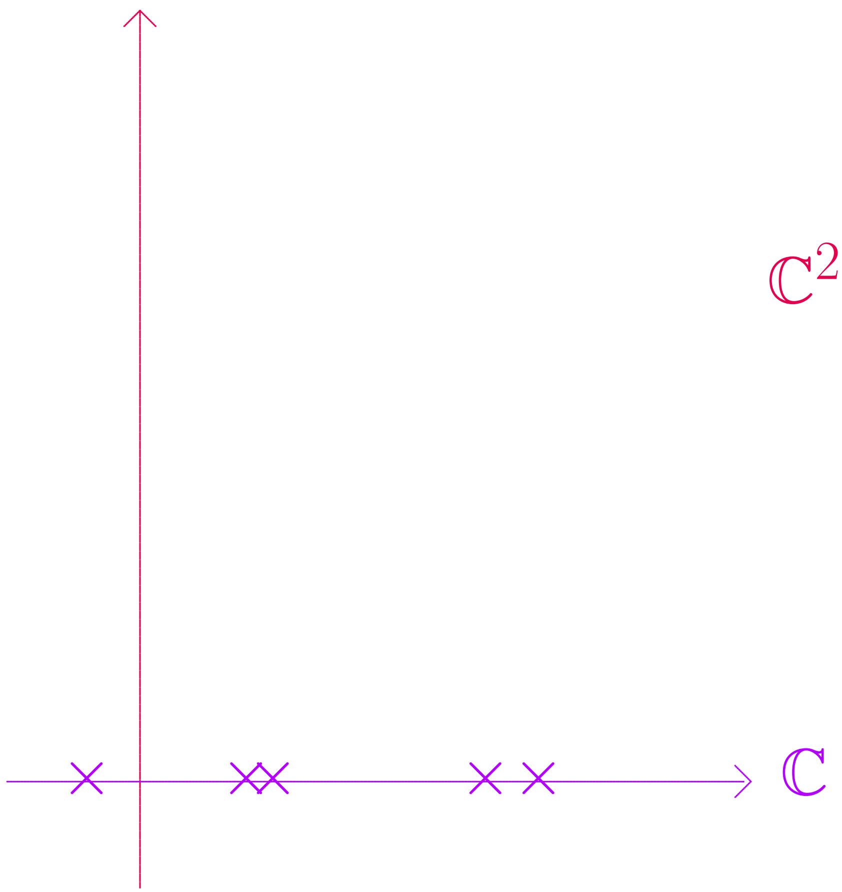

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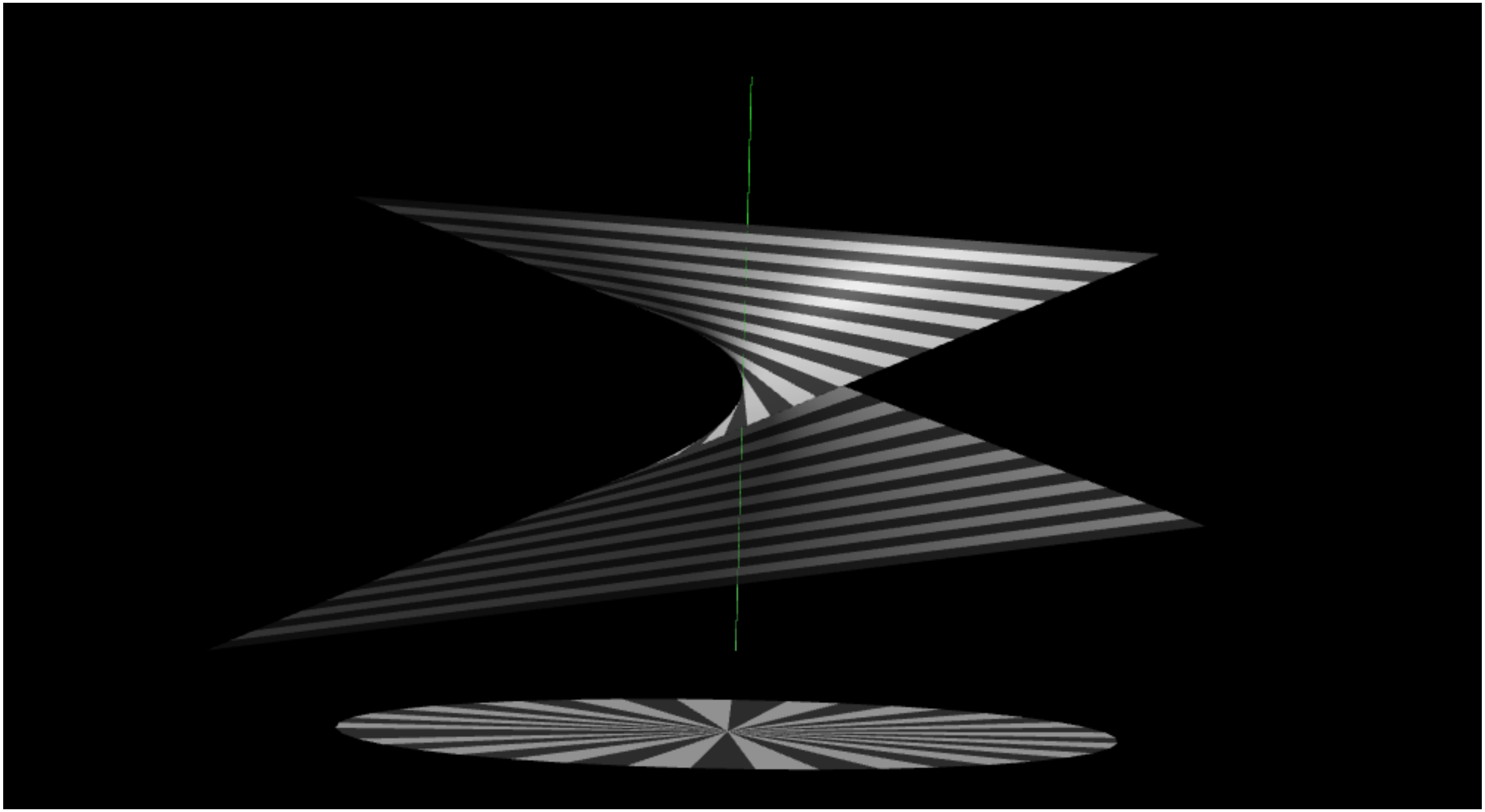

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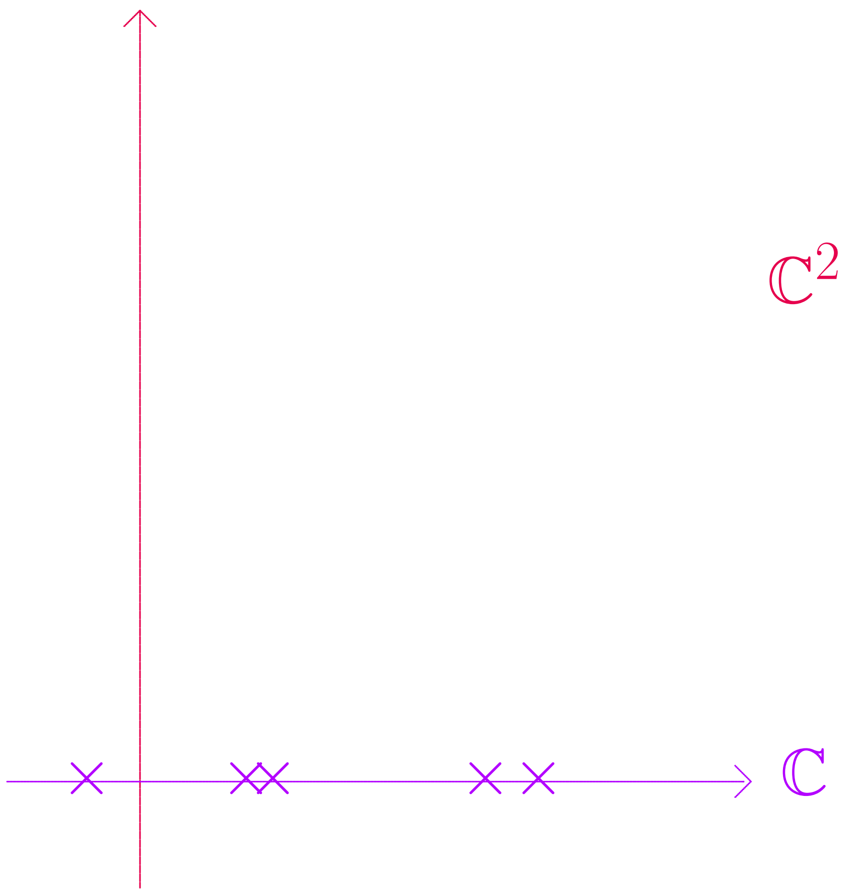

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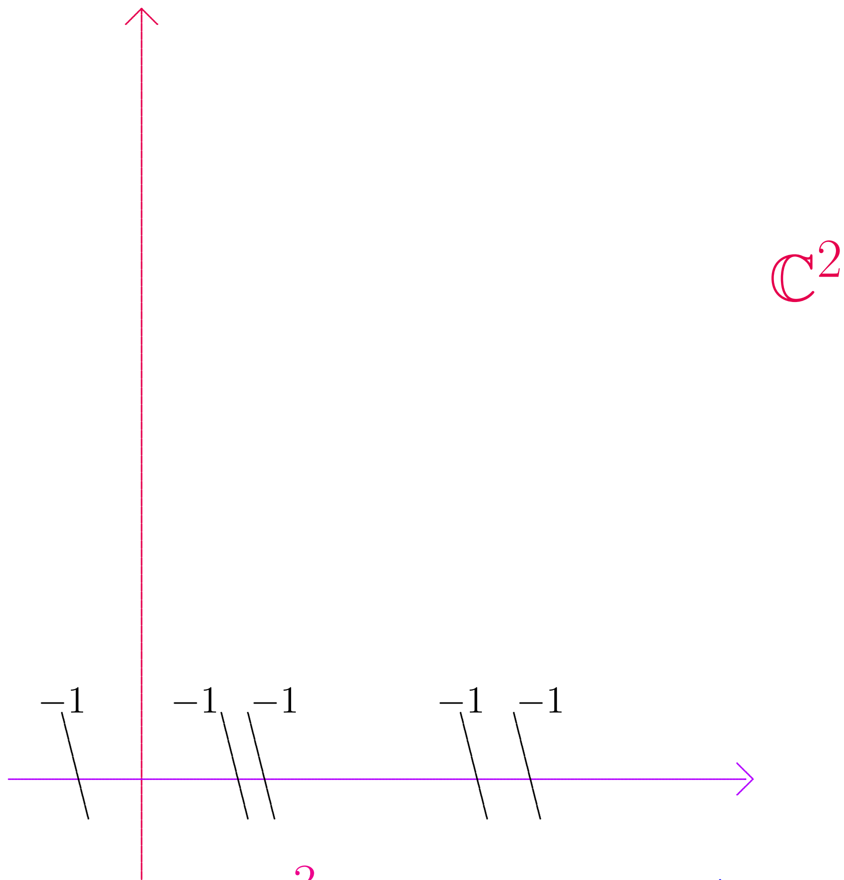

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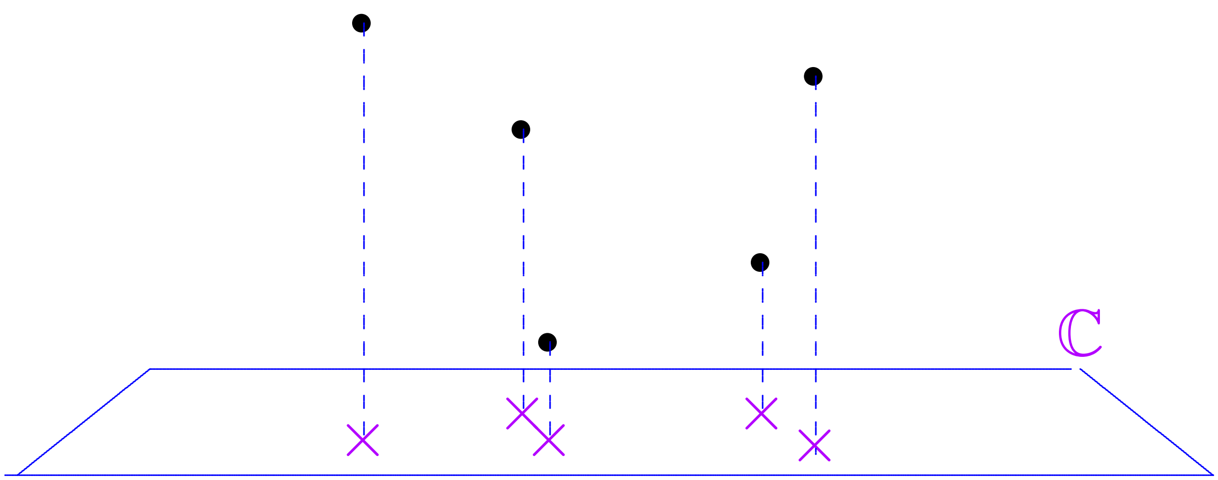

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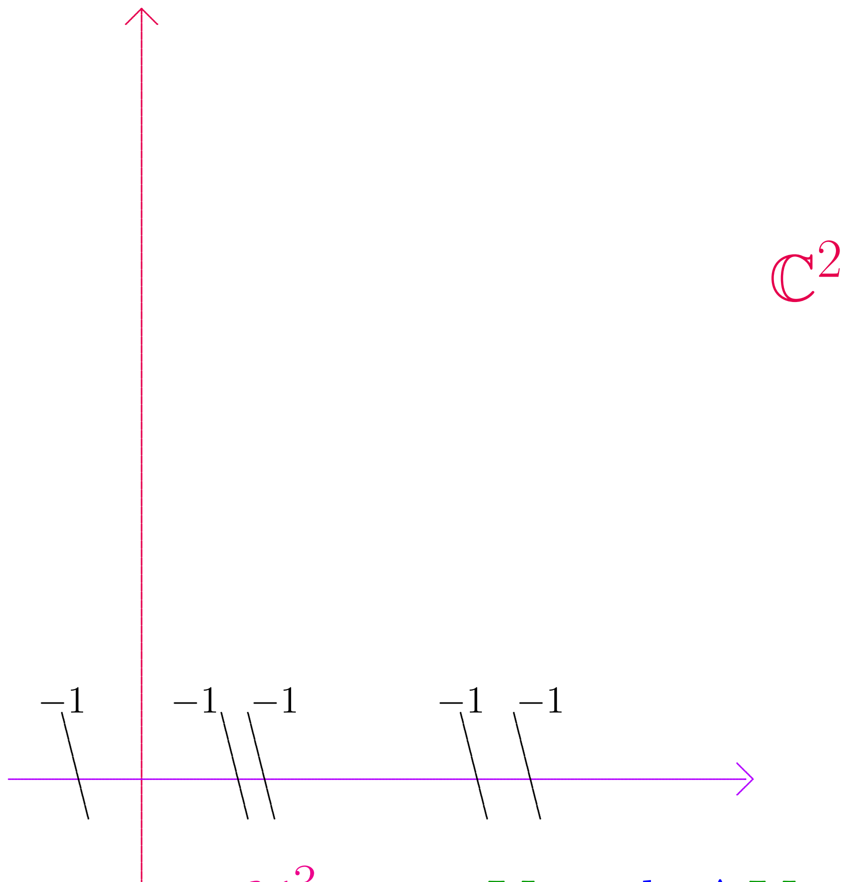

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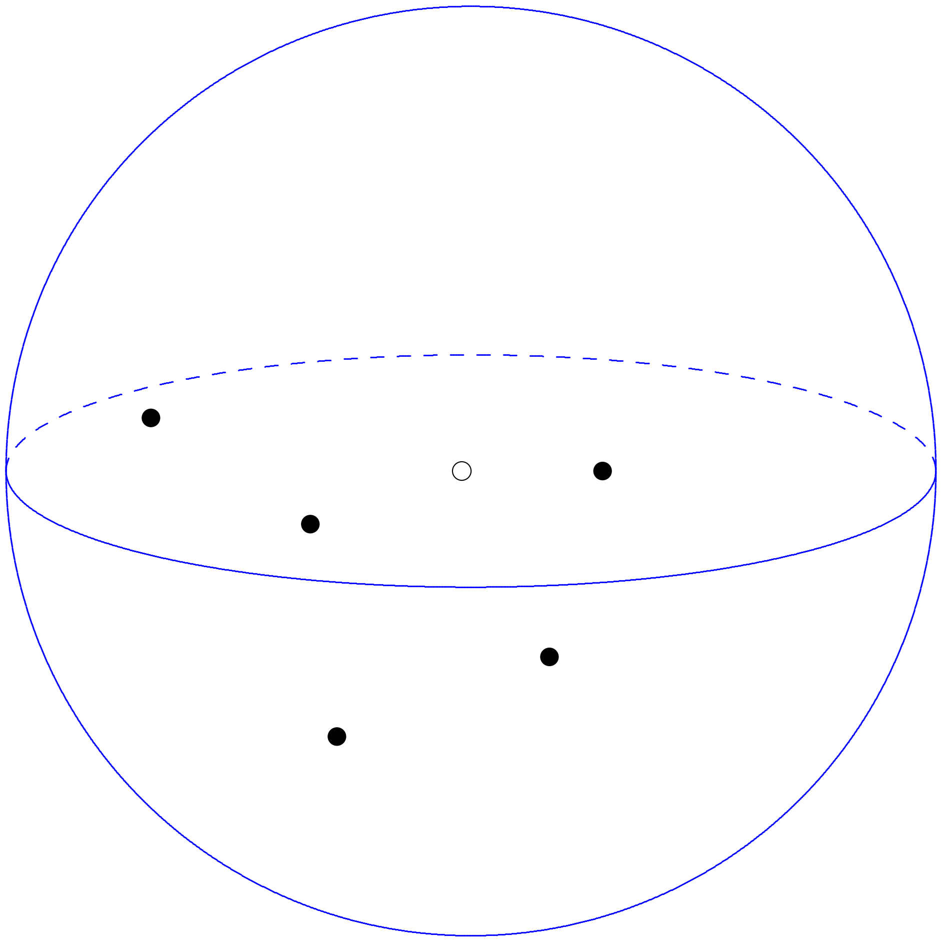

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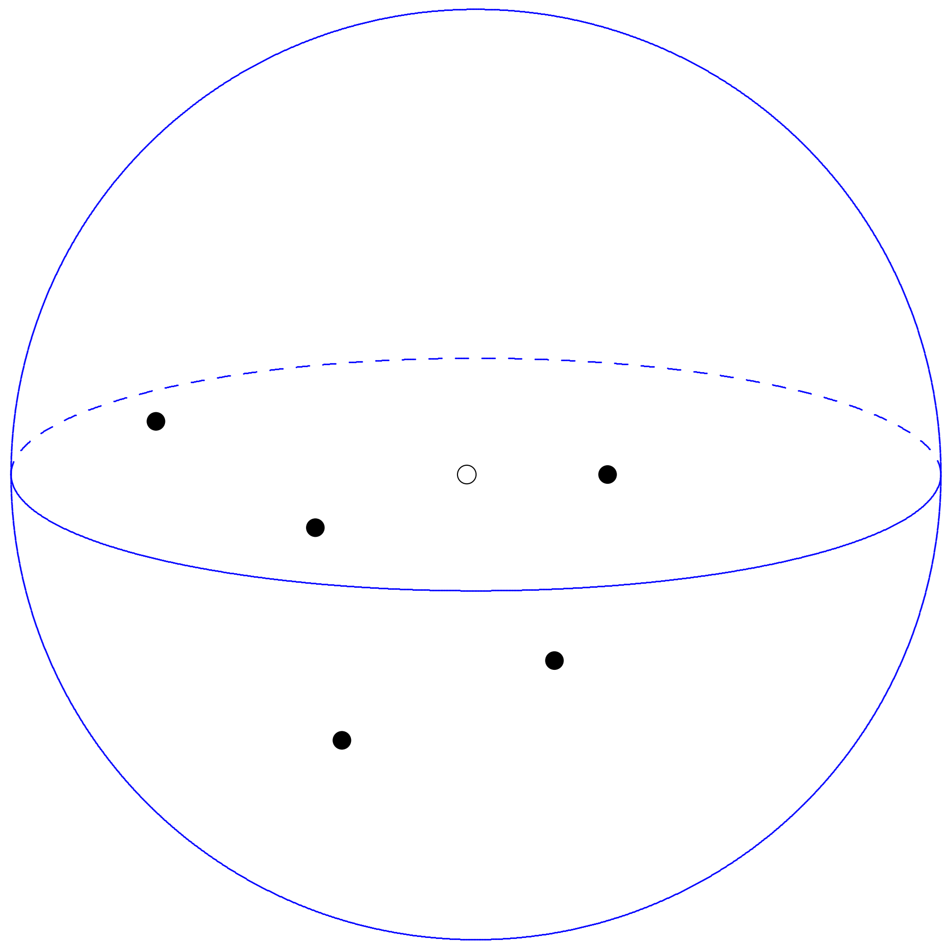

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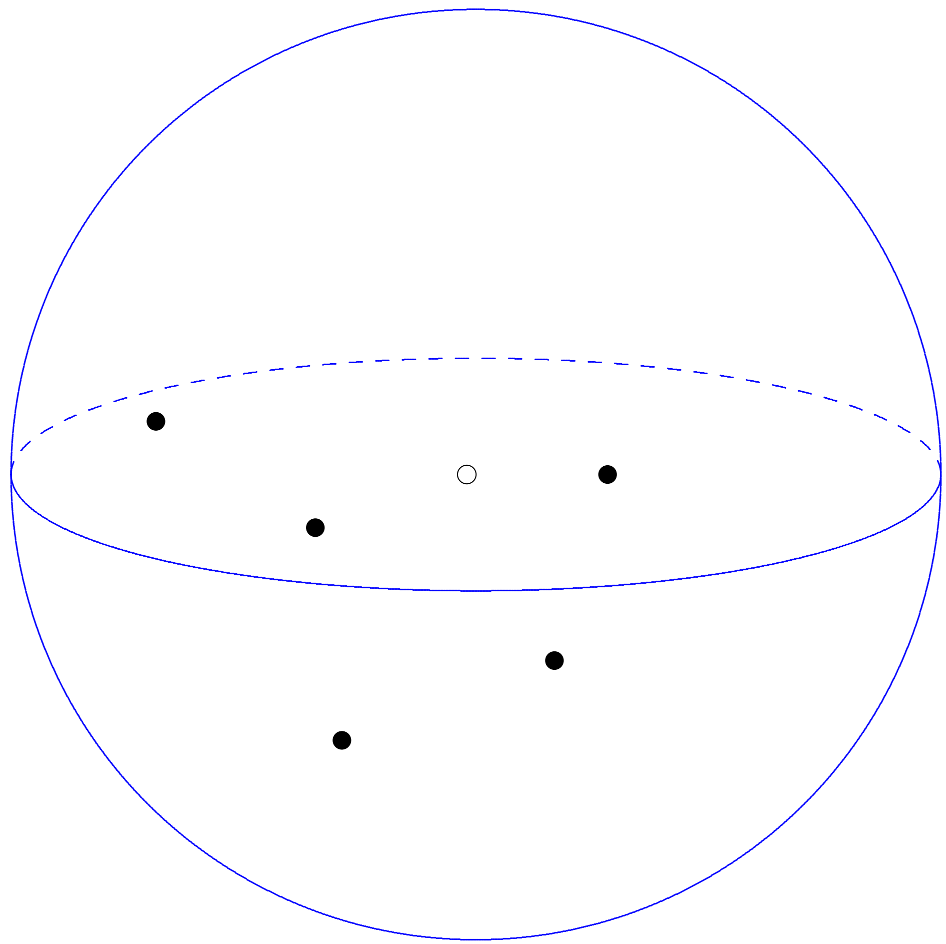

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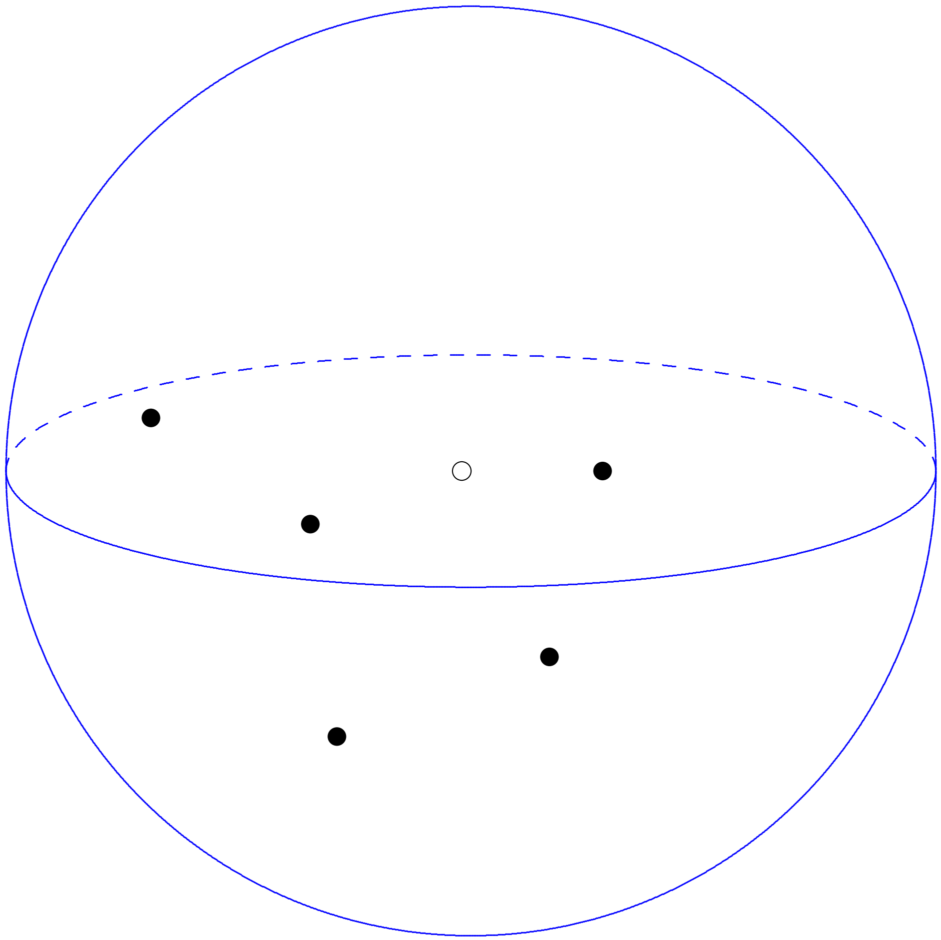

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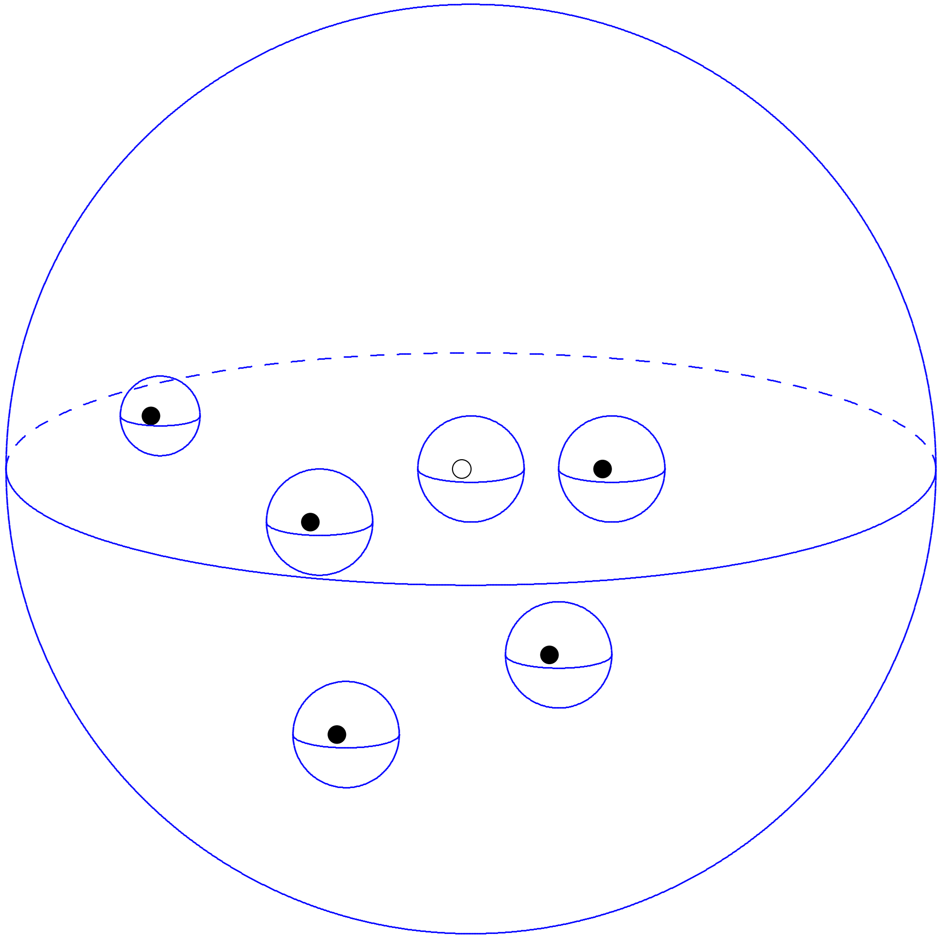

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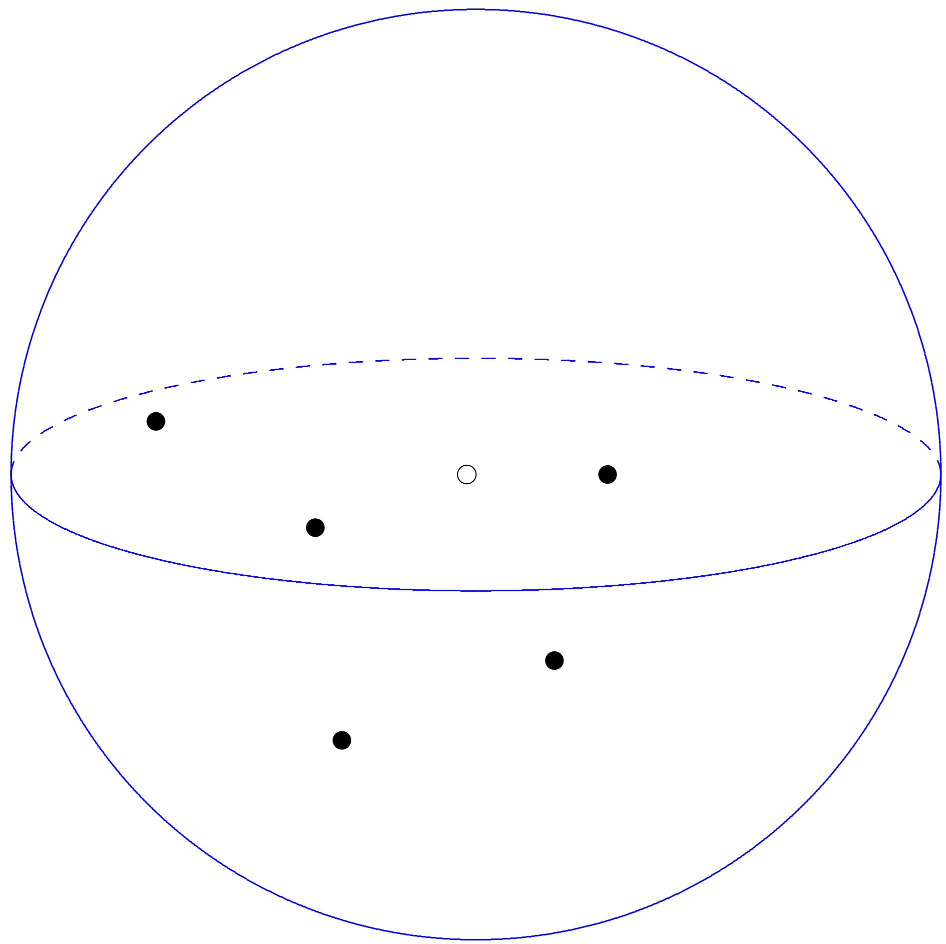

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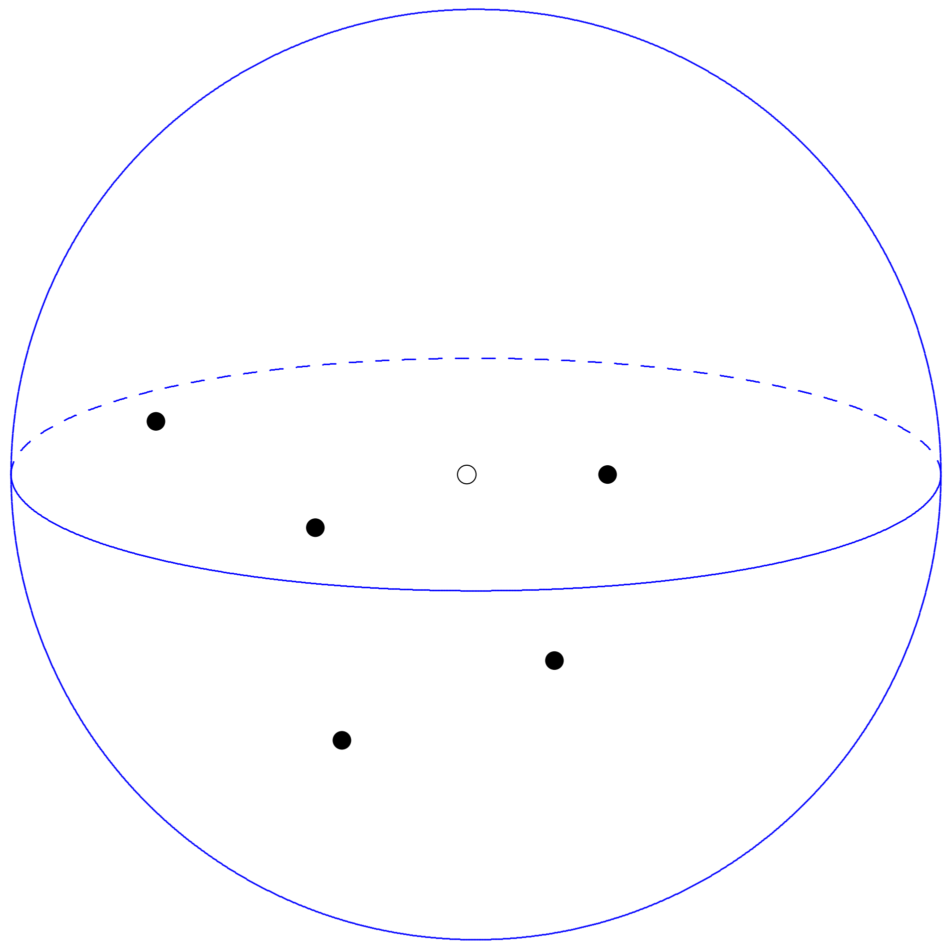

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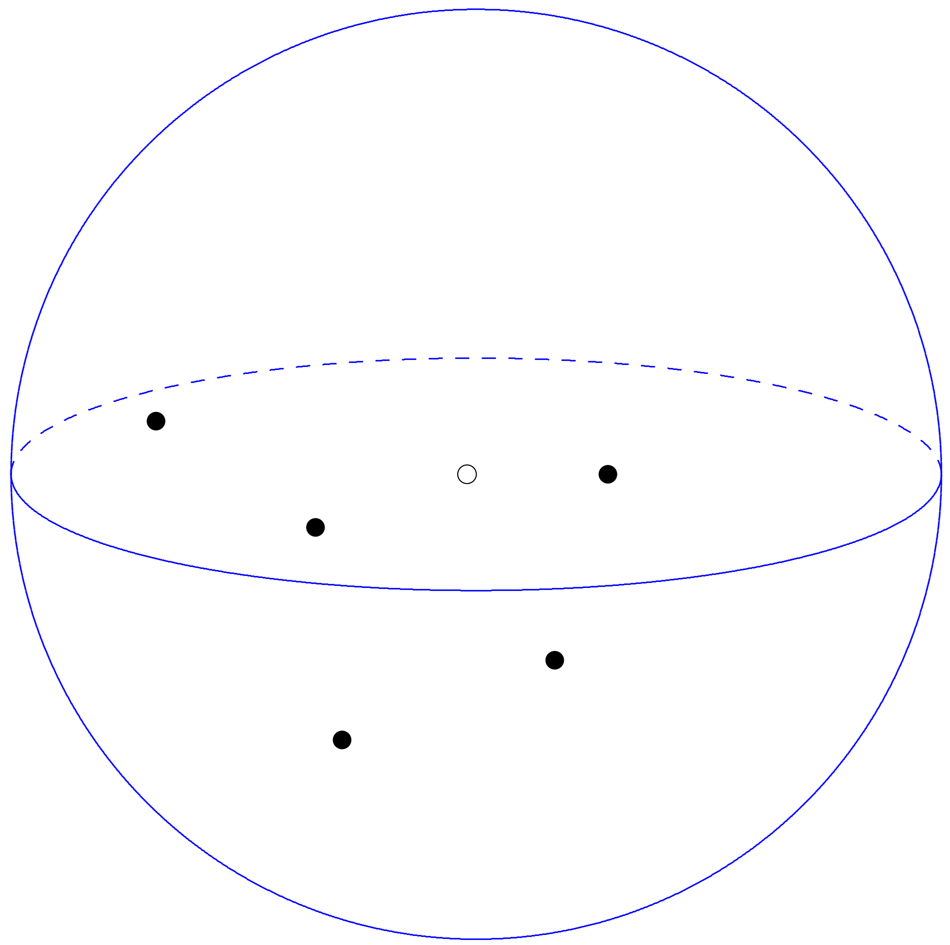

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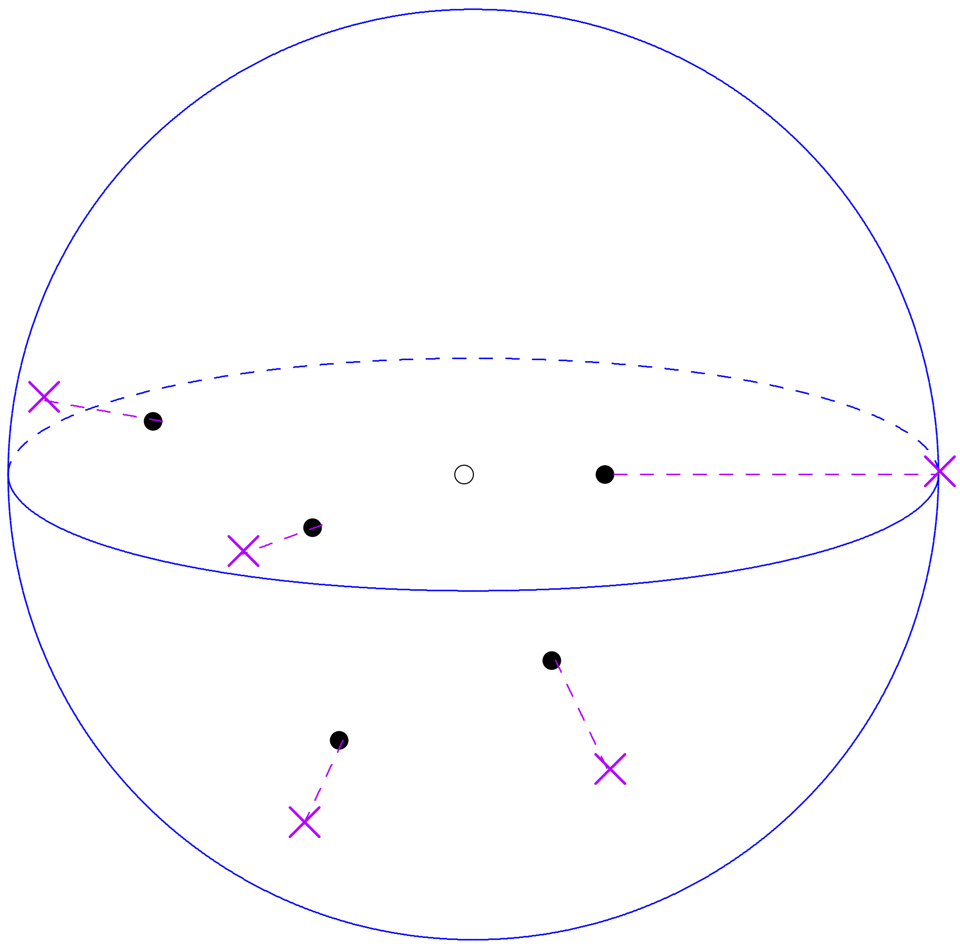

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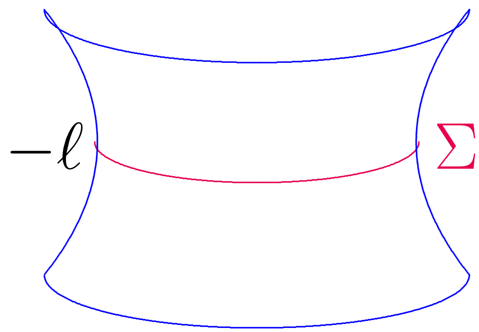

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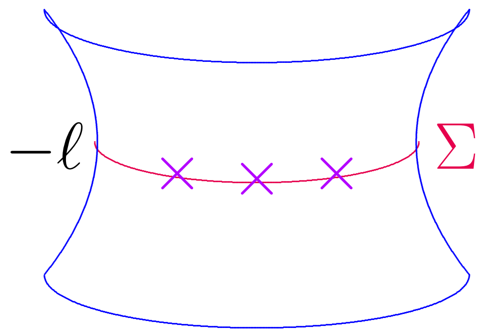

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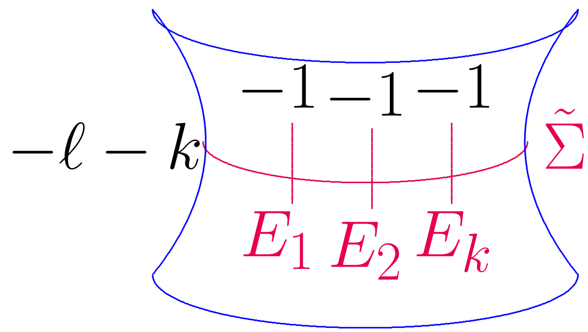

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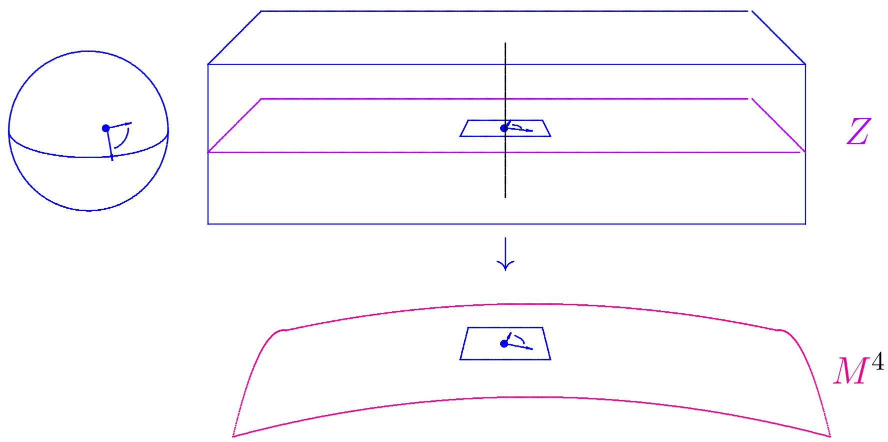
Any scalar-flat Kähler surface (M^4, g, J) has a

Penrose Twistor Space (Z, J),

which is once again a complex 3-manifold.



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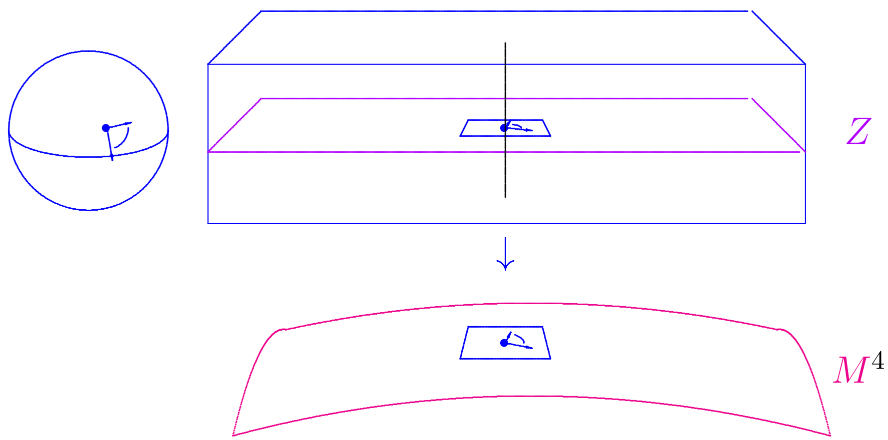

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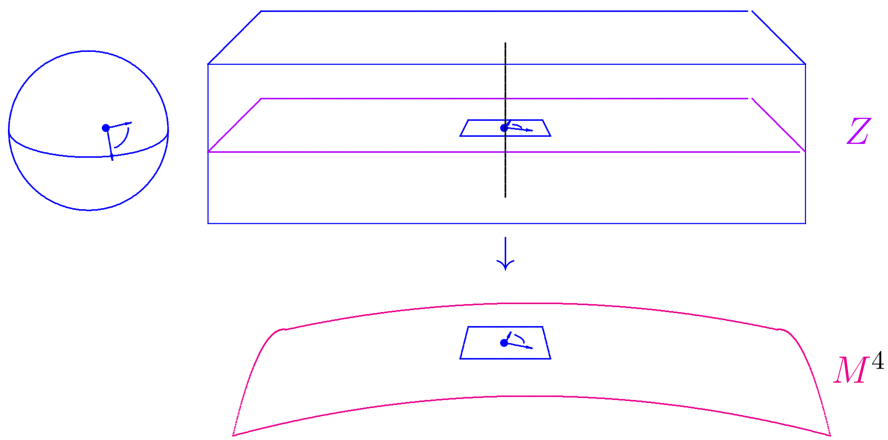

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But will not discuss this today for lack of time.

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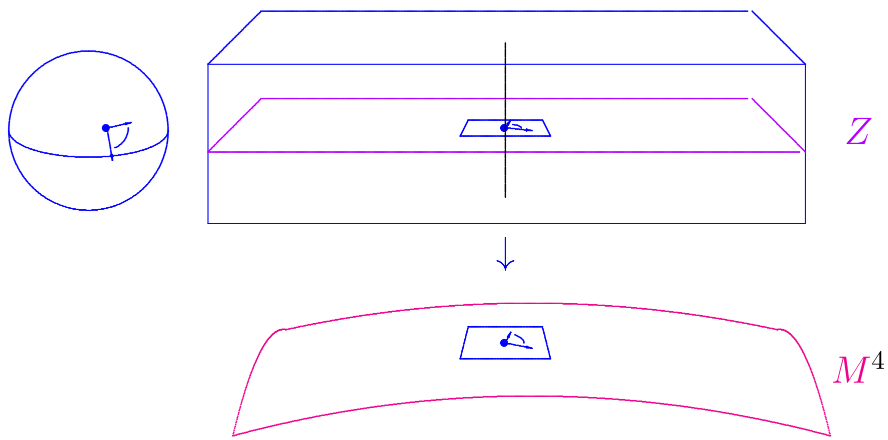

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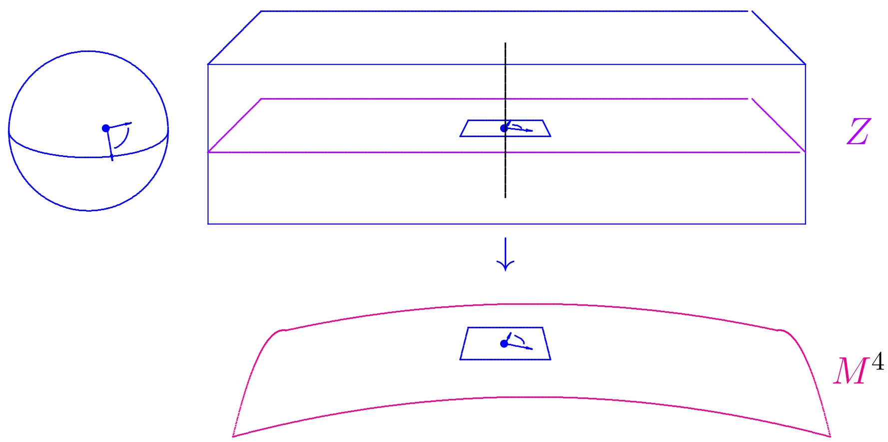

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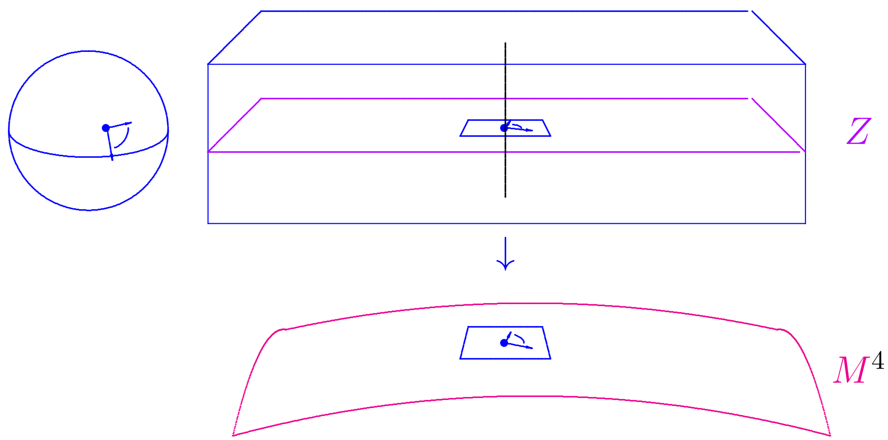

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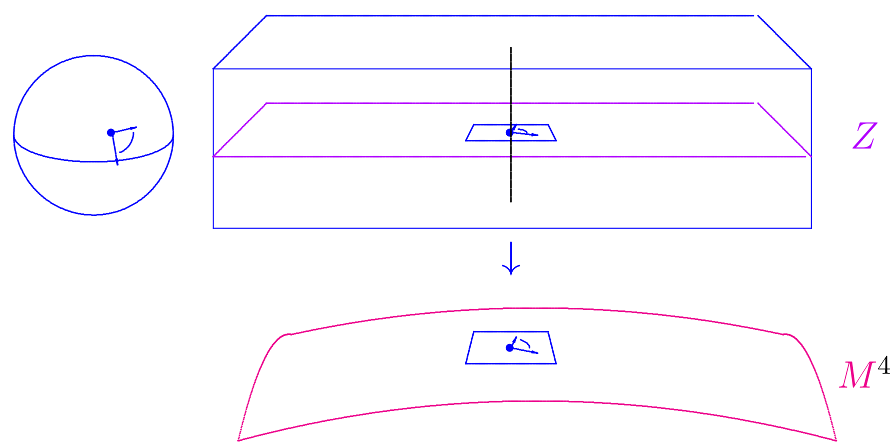

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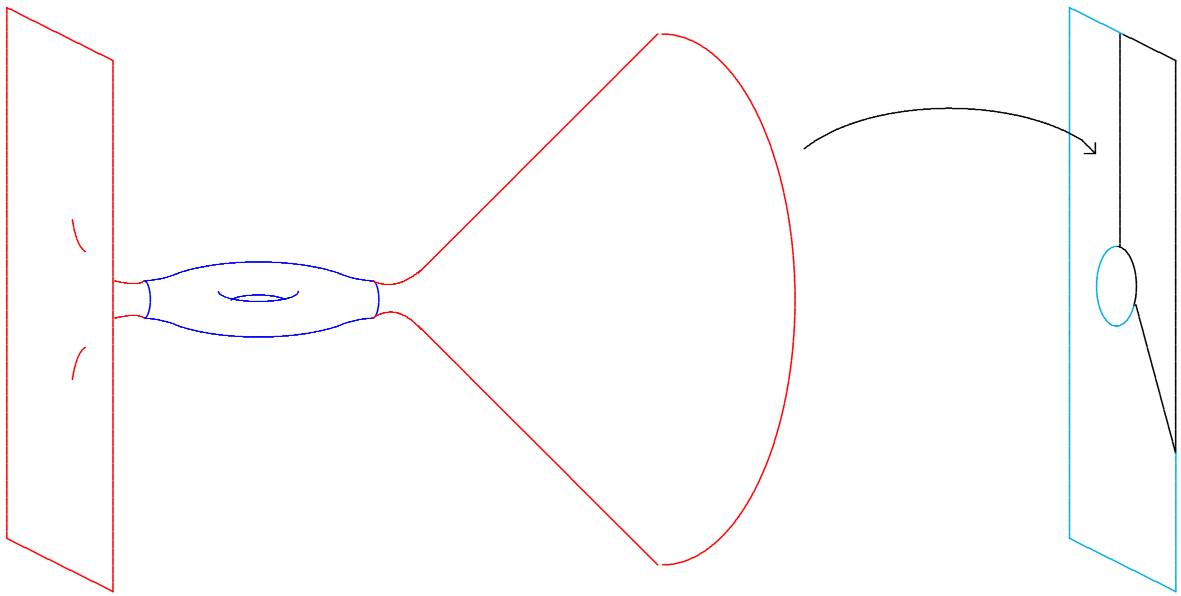
But full classification remains an open problem.

Definition. *Complete, non-compact* n*-manifold* (M^n, g) *is* asymptotically locally Euclidean *(ALE) if* $\exists$ *compact set* $K \subset M$ *such that* $M - K \approx \coprod_i (\mathbb{R}^n - D^n)/\mathit{\Gamma}_i$*, where* $\mathit{\Gamma}_i \subset \mathbf{O(n)}$*, such that*



$$g_{jk} = \delta_{jk} + O(|x|^{1-\frac{n}{2}-\varepsilon})$$

$$g_{jk,\ell} = O(|x|^{-\frac{n}{2}-\varepsilon}), \quad s \in L^1$$

Definition. *The* mass *(at a given end) of an* *ALE* n*-manifold is defined to be*

$$\mathfrak{m}(M,g) := \qquad\qquad\qquad\qquad \left[g_{ij,i} - g_{ii,j}\right]$$

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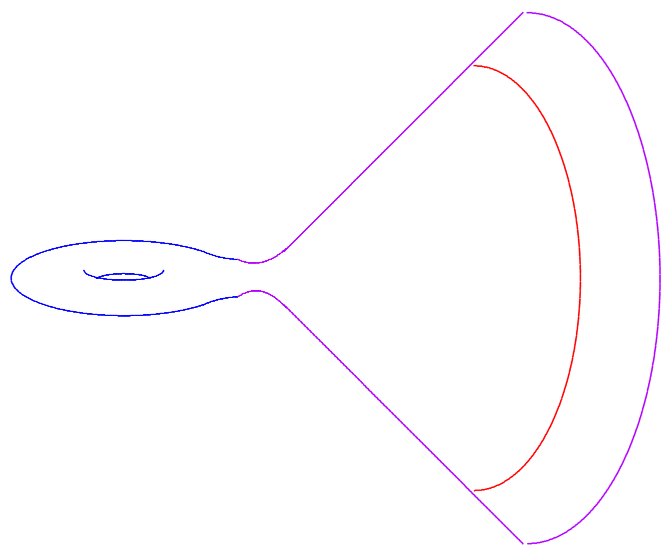

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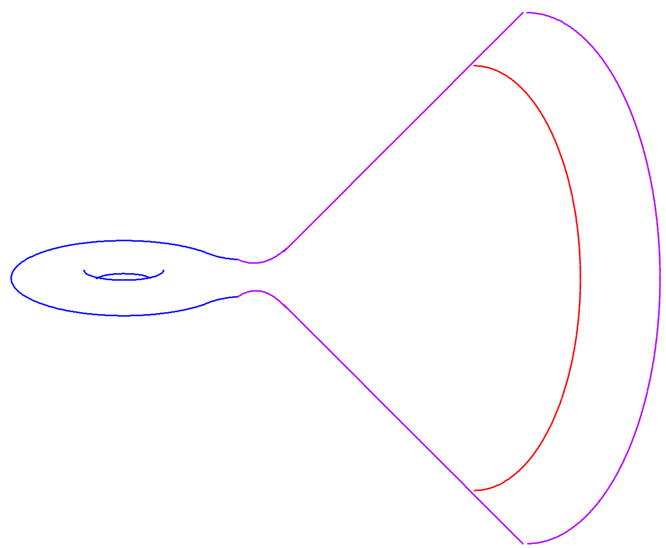

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Bartnik/Chruściel (1986): With weak fall-off conditions, the mass is well-defined & coordinate independent.

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$$m(M,g) := \lim_{\varrho\to\infty} \frac{\Gamma(\frac{n}{2})}{4(n-1)\pi^{n/2}} \int_{\Sigma(\varrho)} \left[g_{ij,i} - g_{ii,j}\right] \nu^j \alpha_E$$

where

- $\Sigma(\varrho) \approx S^{n-1}/\mathit{\Gamma}_i$ *is given by* $|\vec{x}| = \varrho$;
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$$g_{jk} = \delta_{jk} + O(|x|^{1-\frac{n}{2}-\varepsilon})$$

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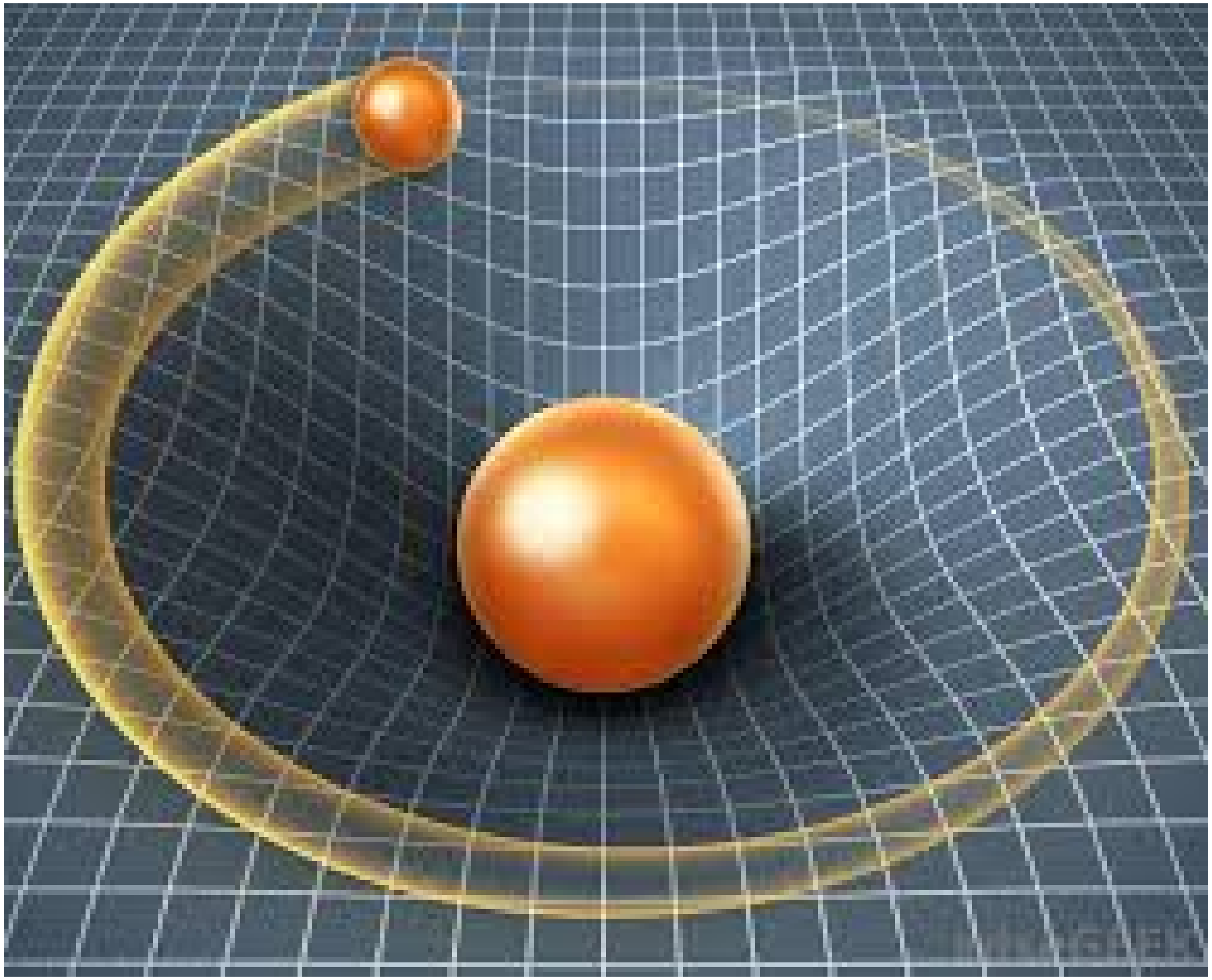

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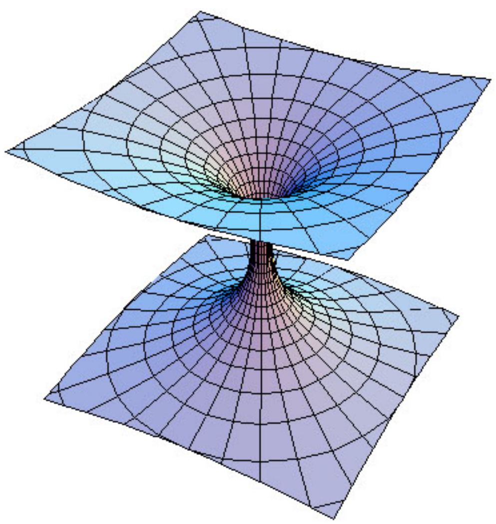

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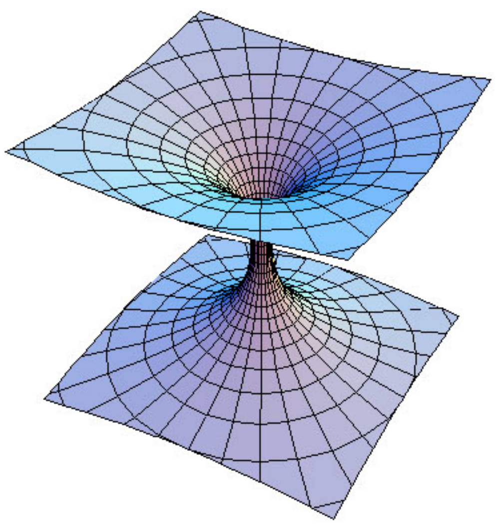

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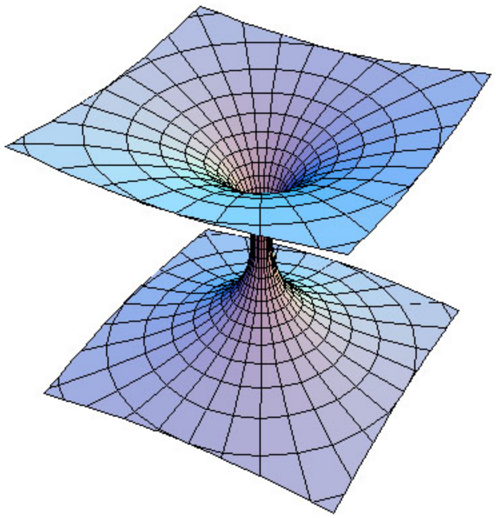

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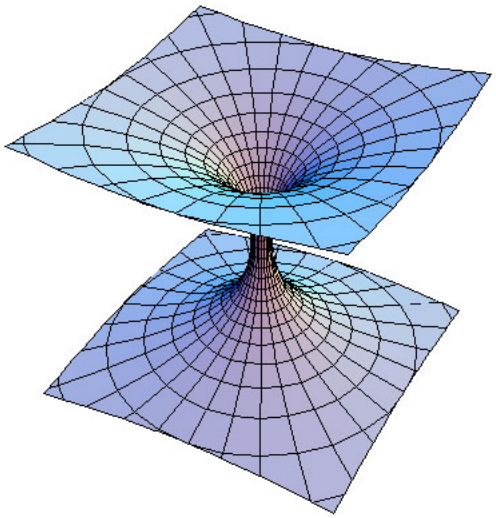

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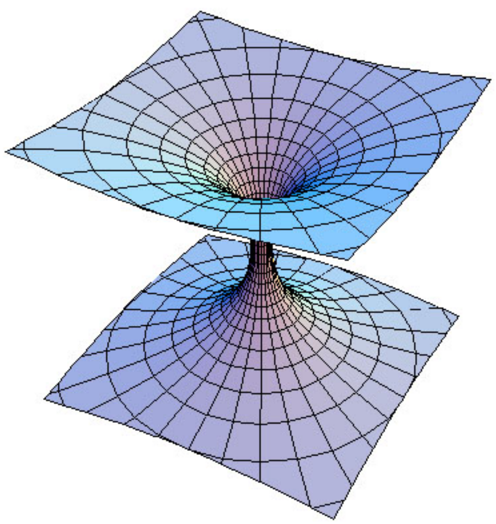

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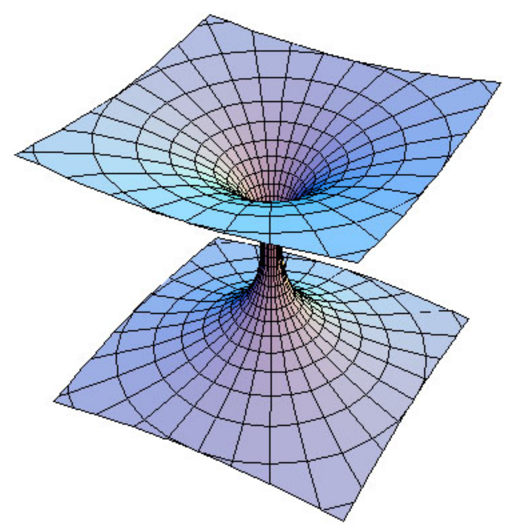

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But notice that this crude model for the mass in particular assumes faster metric fall-off!

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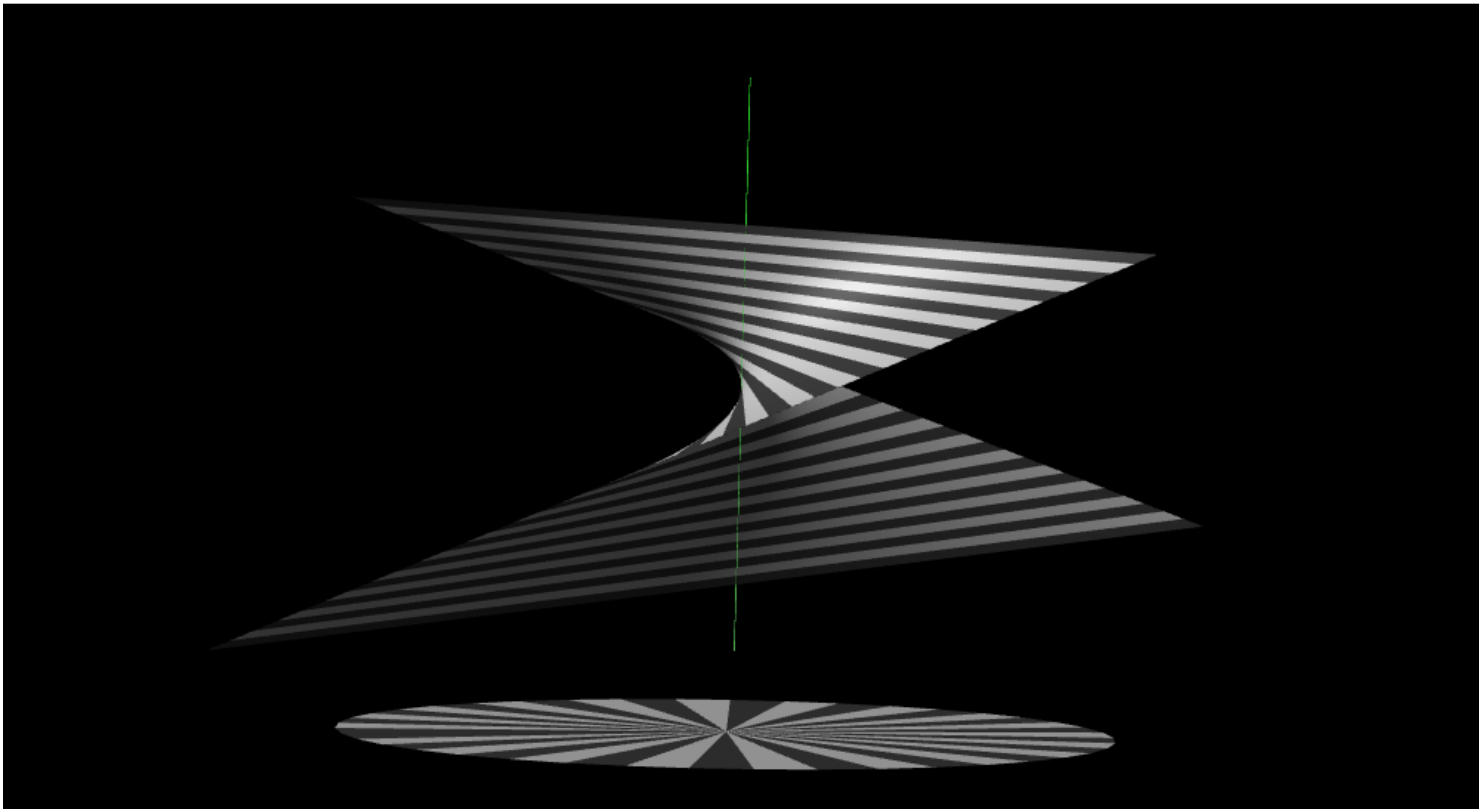

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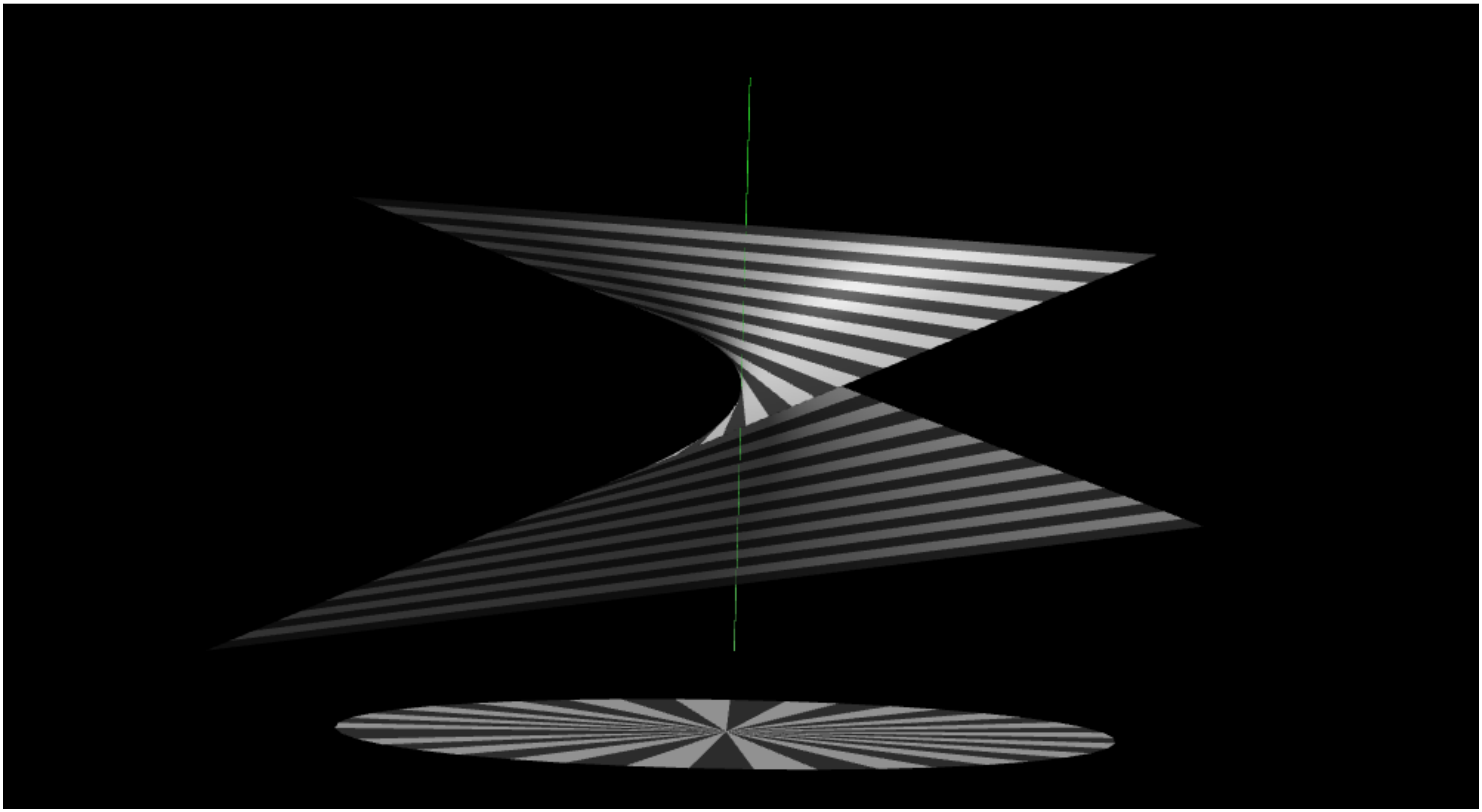

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Comm. Math. Phys. 347 (2016) 621–653.

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unless $\varepsilon > \frac{1}{2}$, when Chruściel fall-off sufficed.

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In particular, Chruściel fall-off suffices to imply all the following results:

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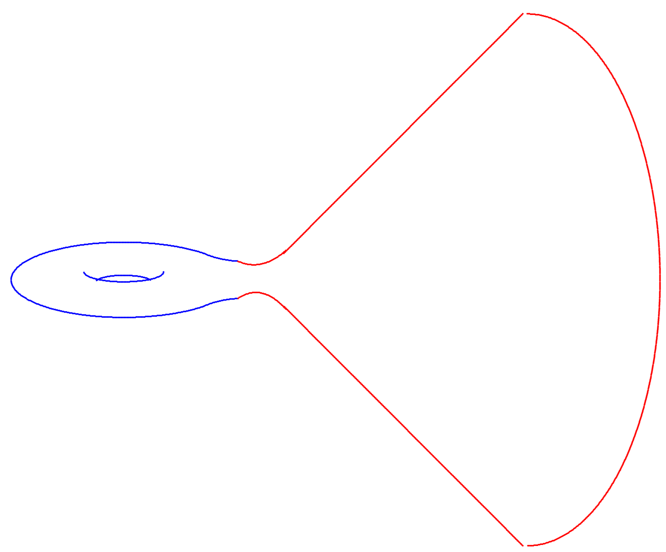

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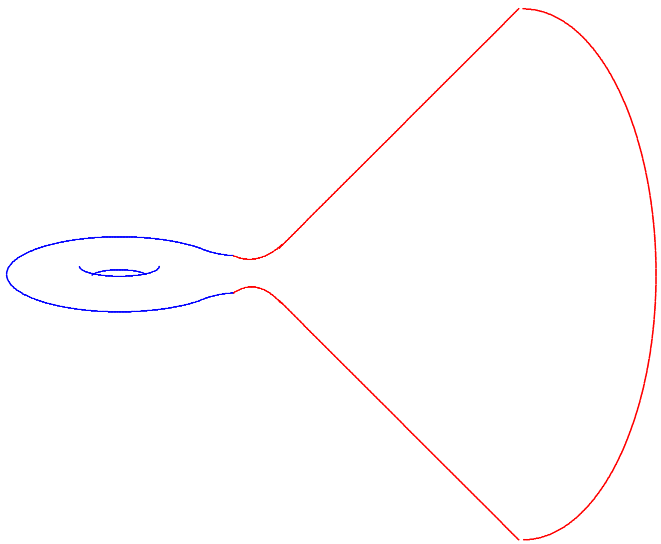

$$\boxed{n = 2m \geq 4}$$

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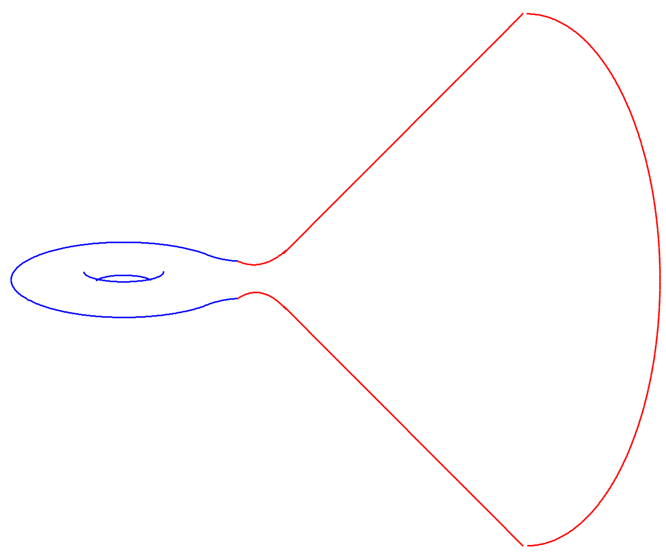

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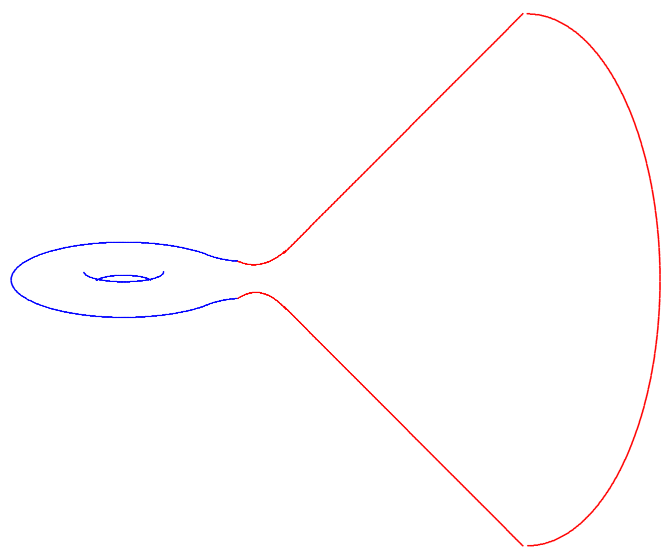

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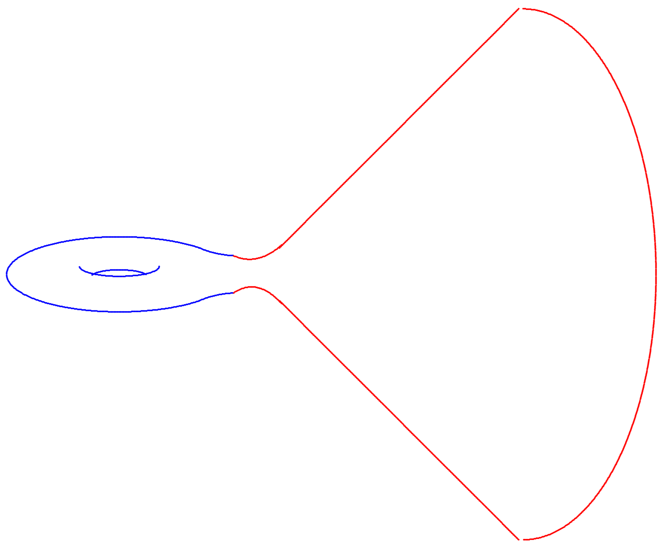

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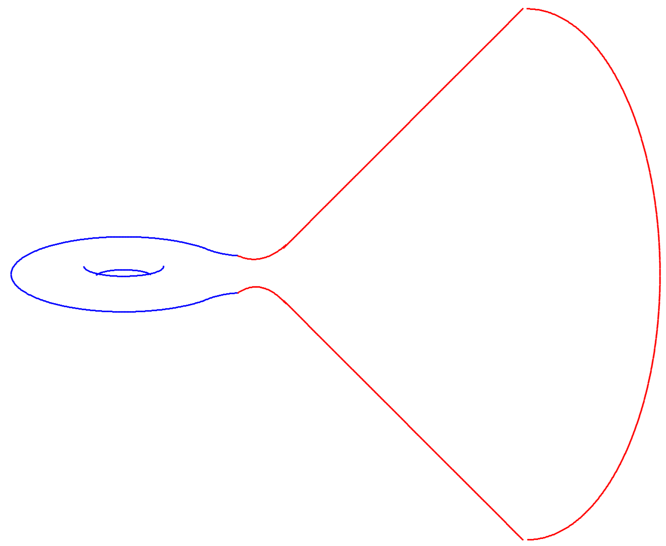

Upshot:

Mass of an ALE Kähler manifold is unambiguous.

Does not depend on the choice of an end!

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New proof shows this follows from Chruściel fall-off.

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Theorems A & B are corollaries concerning scalar-flat Kähler metrics.

Another key consequence. . .

Theorem E (Penrose Inequality).

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This has an interesting corollary…

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Now use Bishop-Gromov inequality.

Some applications . . .

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Corollary. *Any ALE scalar-flat Kähler manifold* (M, g, J) *of complex dimension* m *has mass given by*

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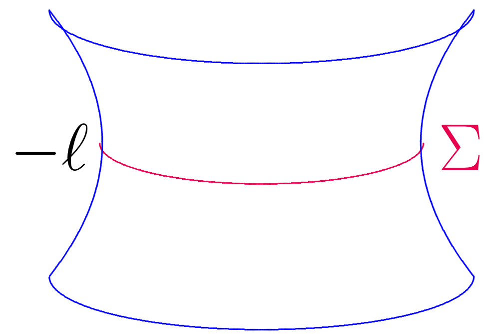
Example. Blow up Chern class $-\ell$ line bundle over $\mathbb{CP}_1$ at k points on zero section Σ.



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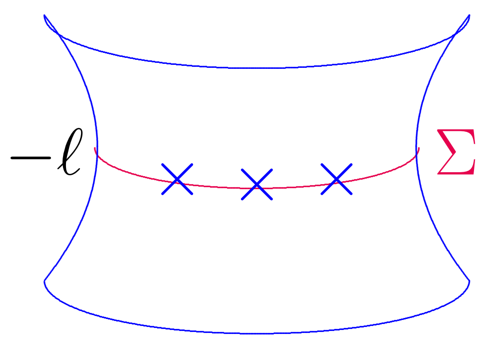

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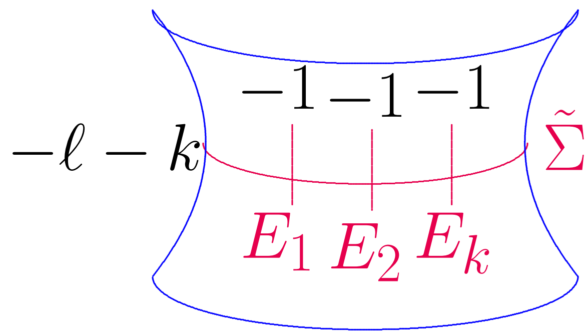

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L, Calderbank-Singer, Lock-Viaclovsky...

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That is, $\mathfrak{m}(M, g, J)$ *is completely determined by*

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Theorems A & B are corollaries concerning scalar-flat Kähler metrics.

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Theorems A & B are corollaries concerning scalar-flat Kähler metrics. But for such metrics, faster fall-off is guaranteed, so new proof is not actually needed!

How does one prove main results?

In high dimensions, the complex structure J of an ALE Kähler manifold is always standard at infinity.

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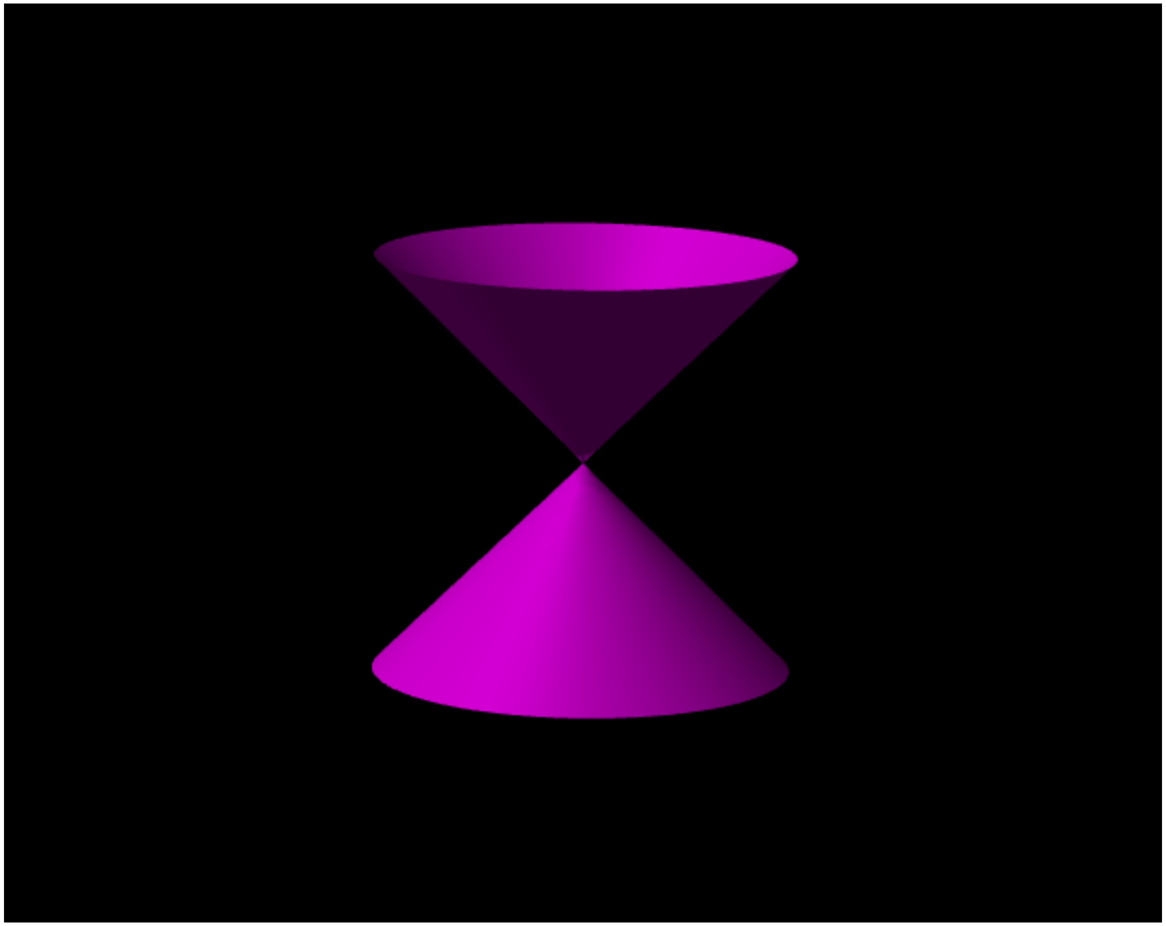

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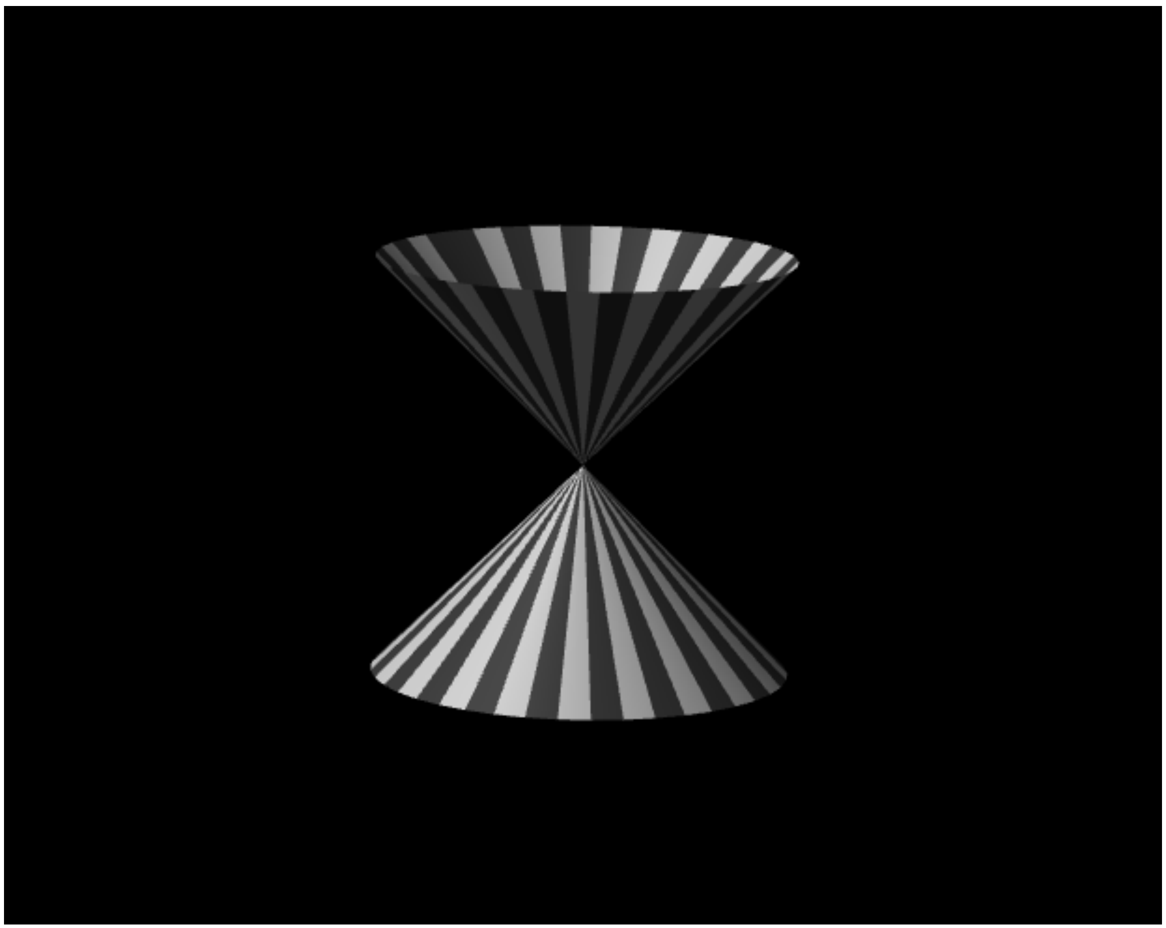

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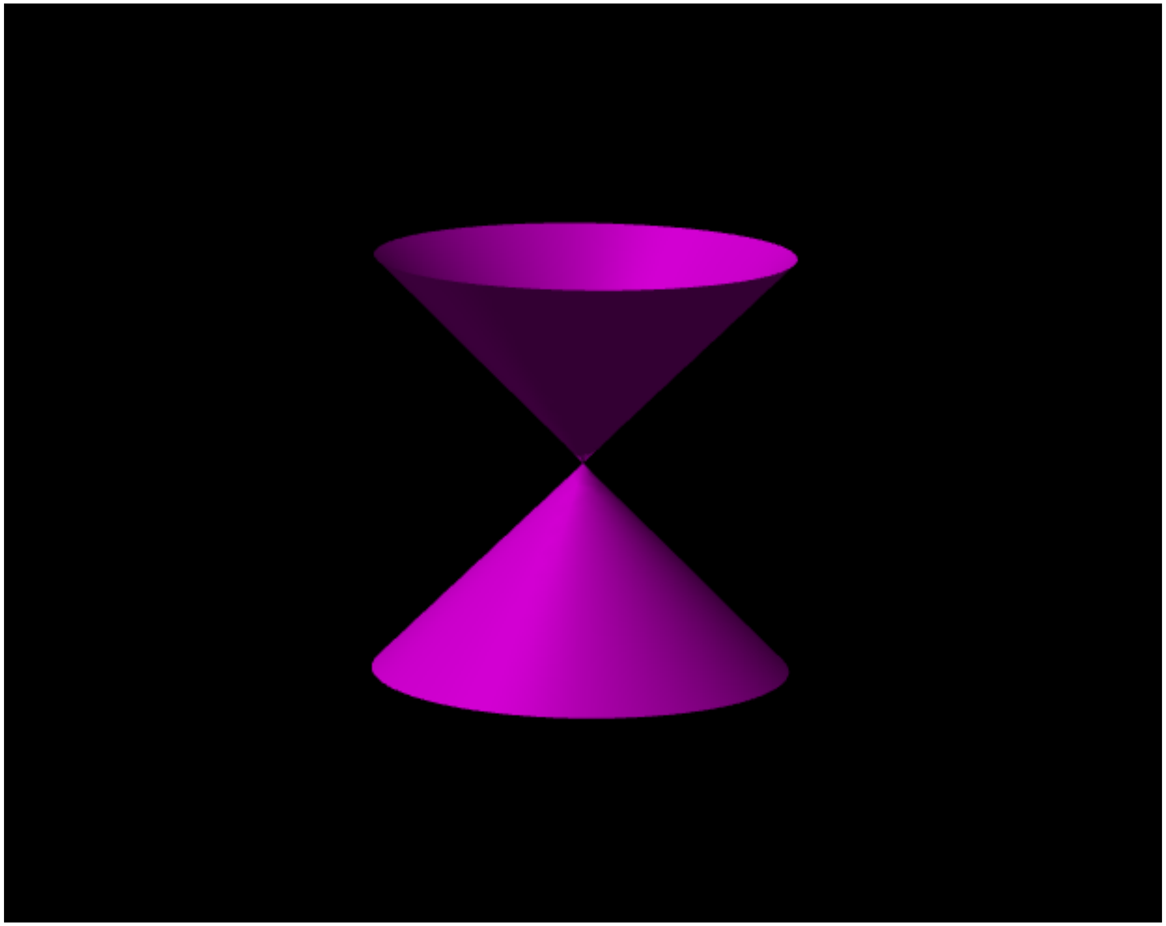

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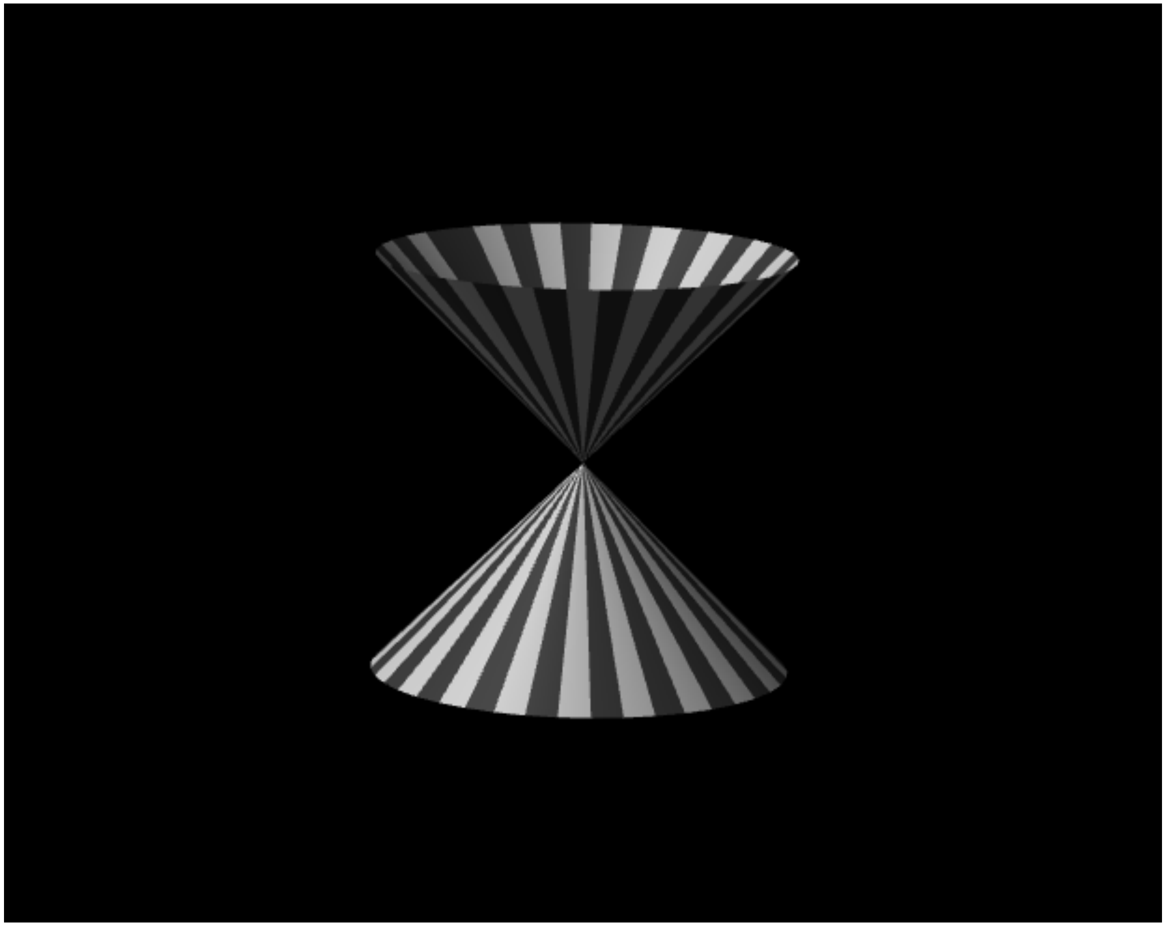

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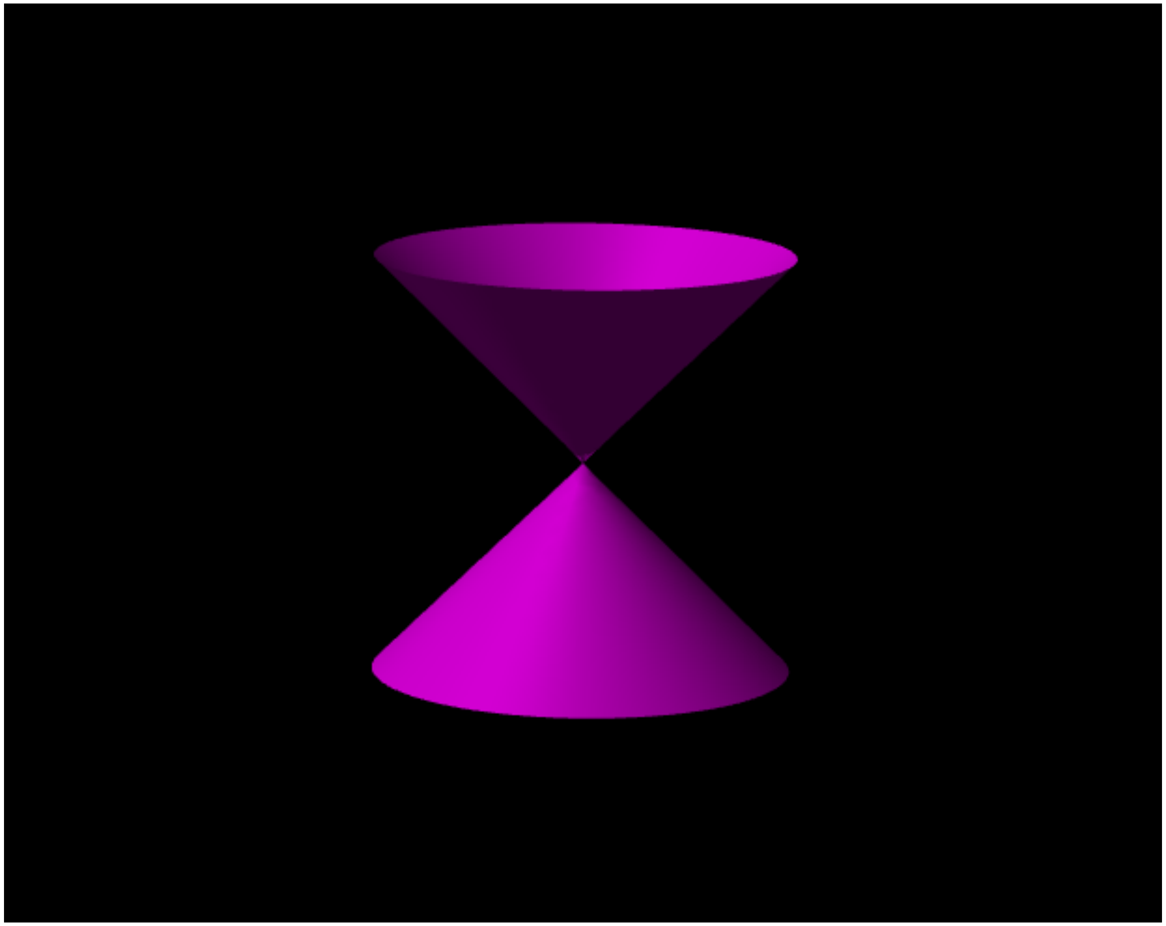

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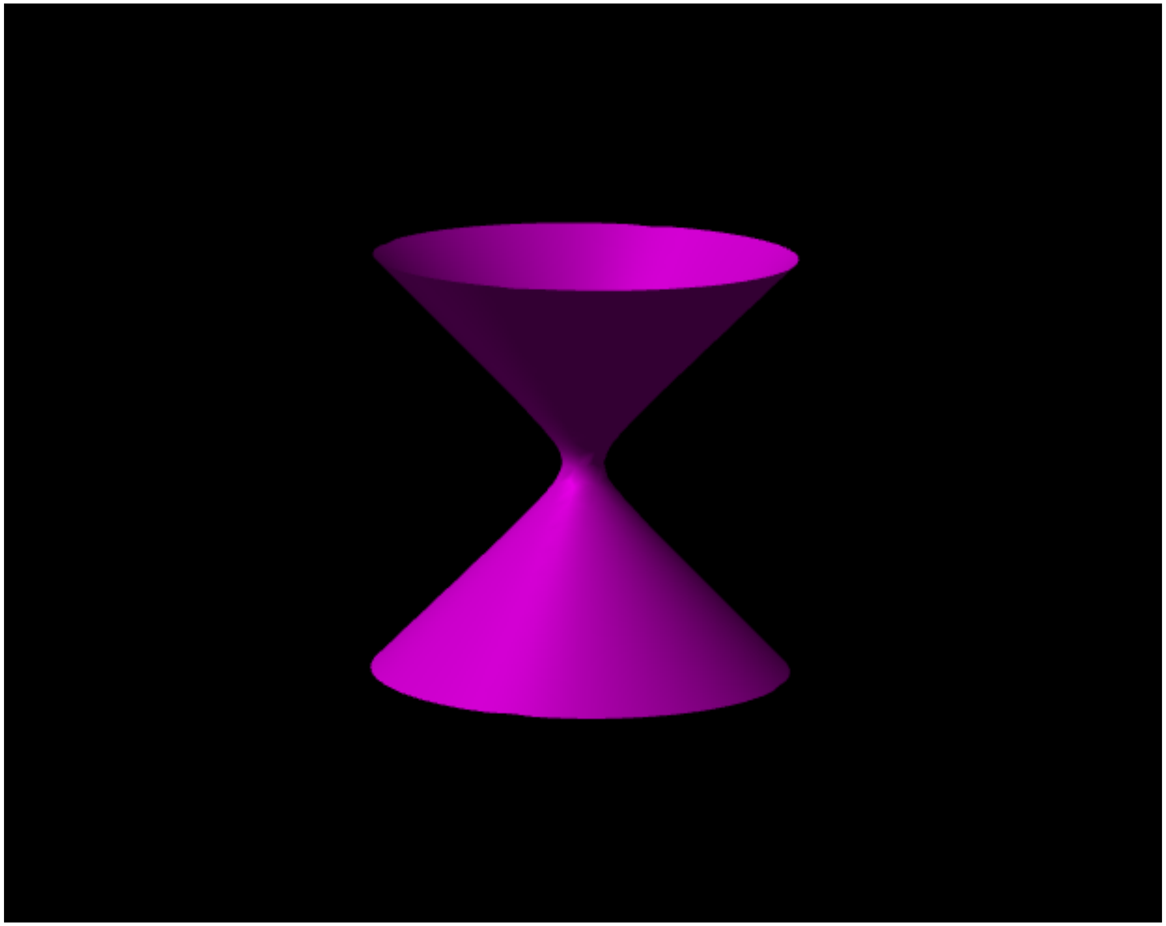

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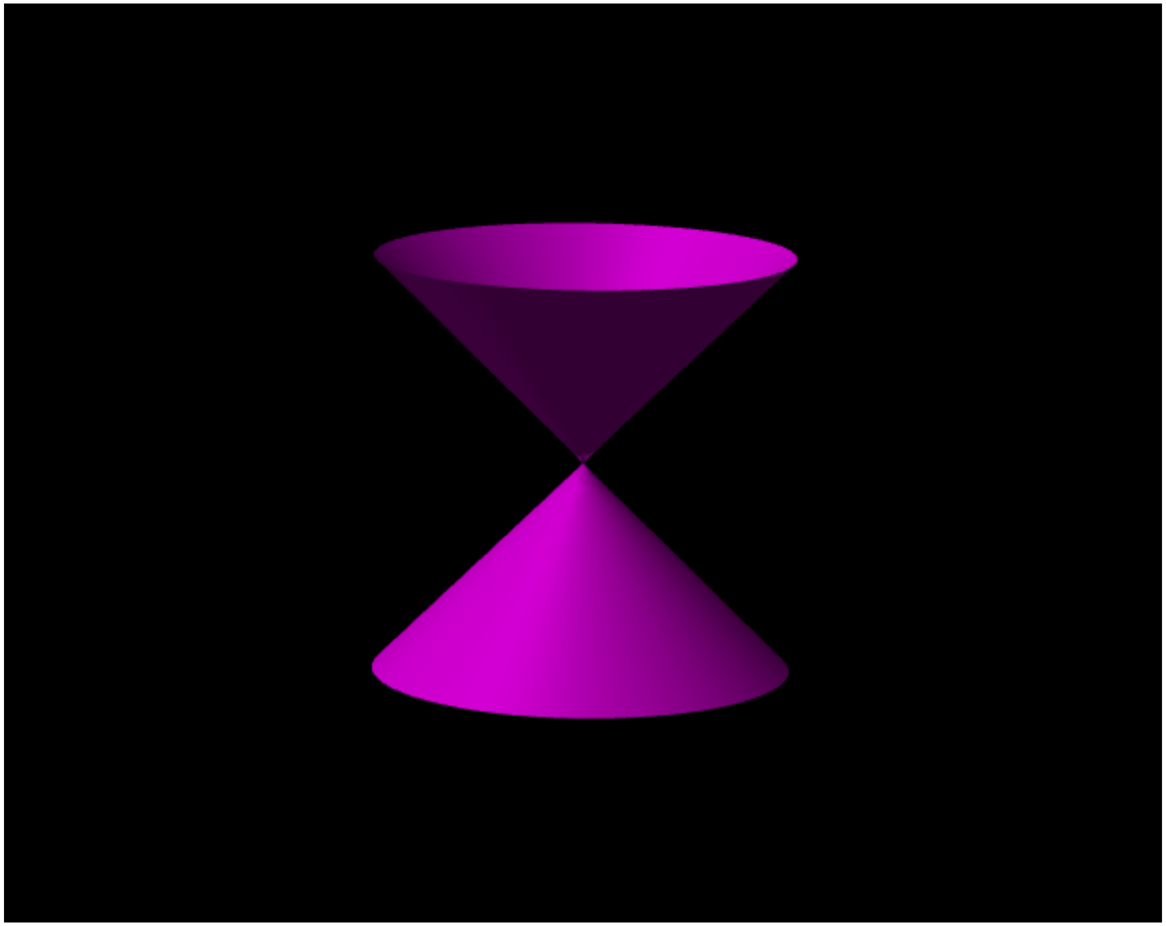

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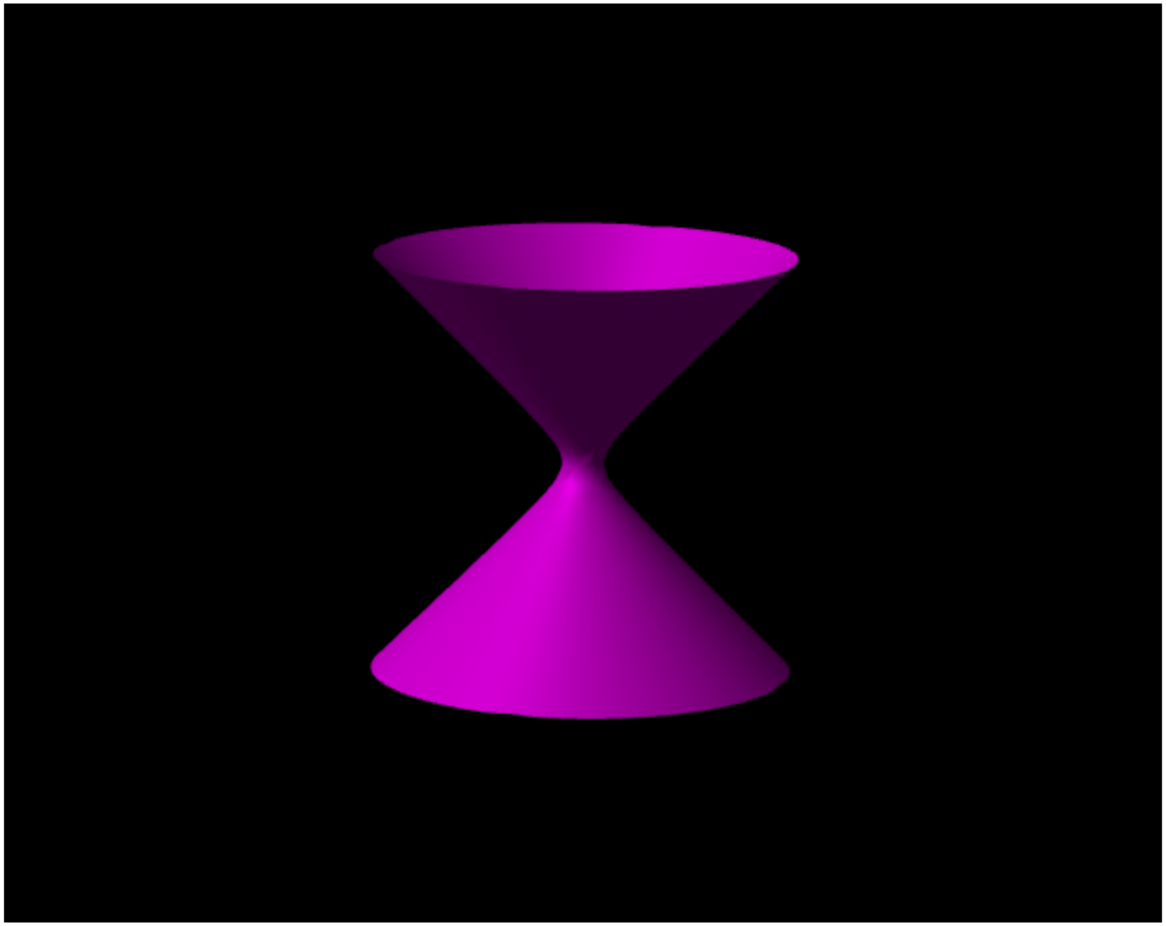

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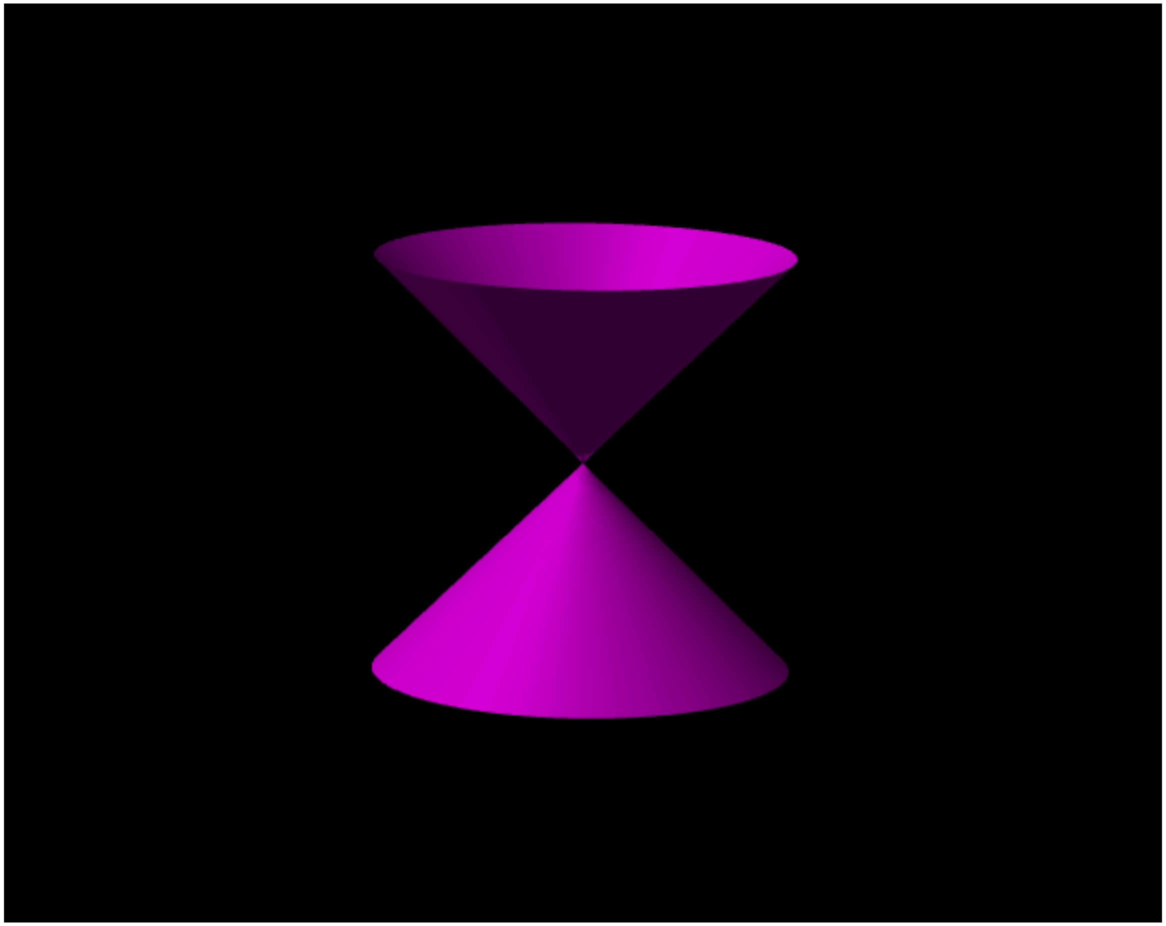

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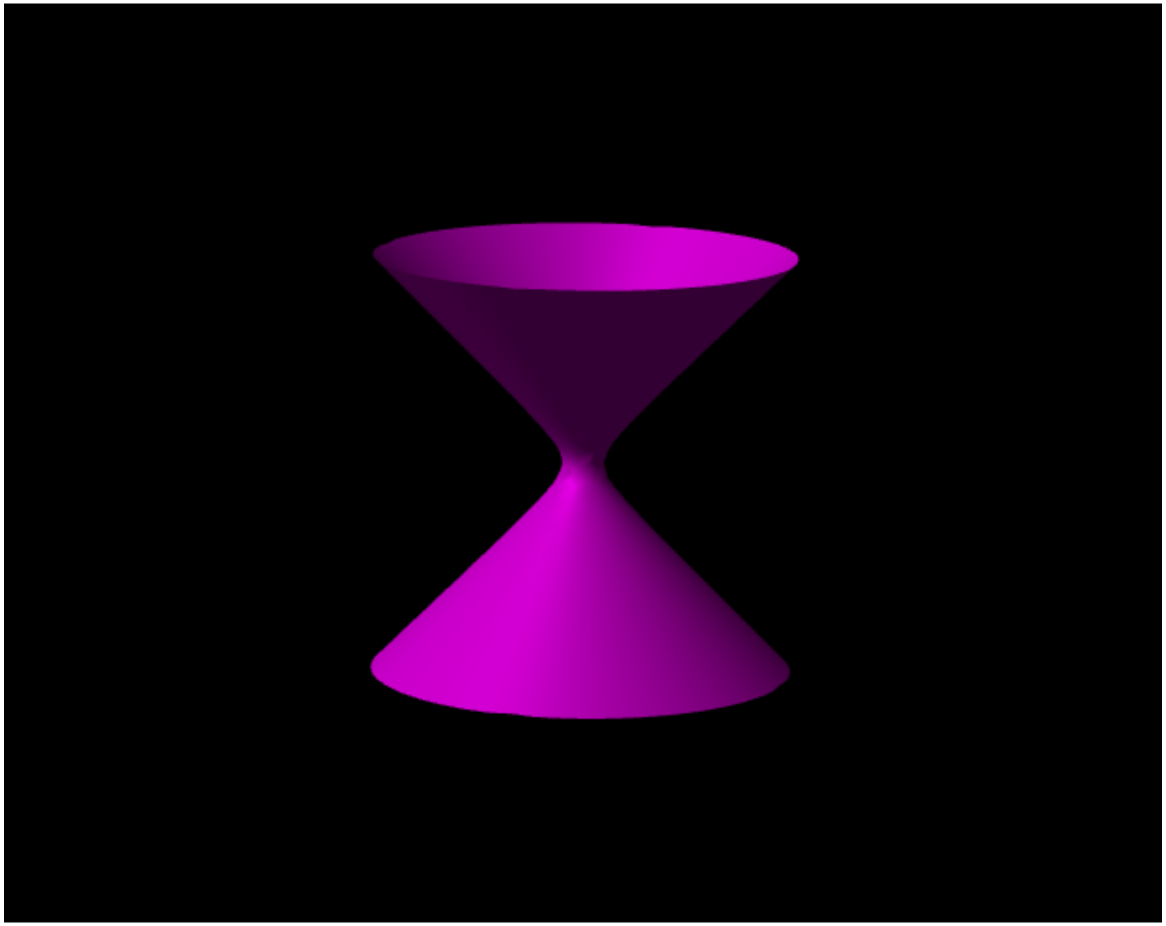

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Fortunately, however, the symplectic structure is always is standard at infinity!

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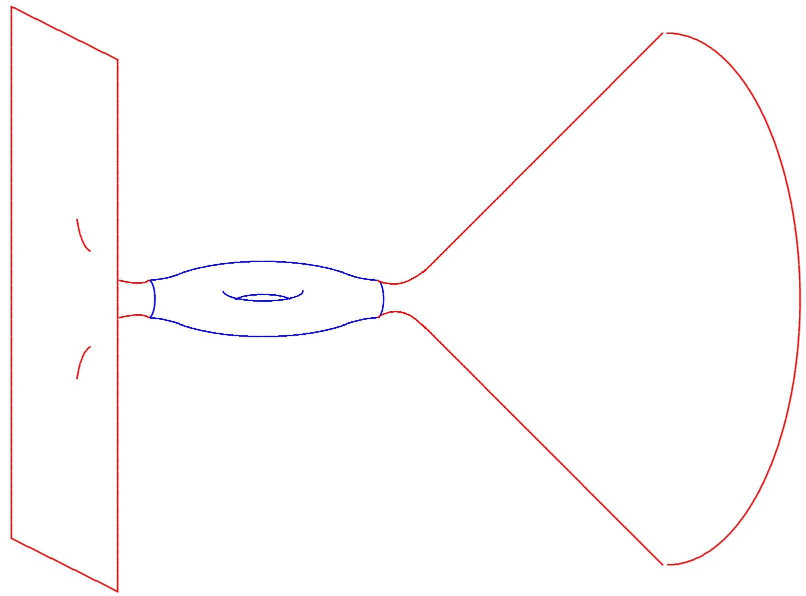
Quantitative version of Moser stability argument. . .

Symplectically compactify any ALE Kähler surface:



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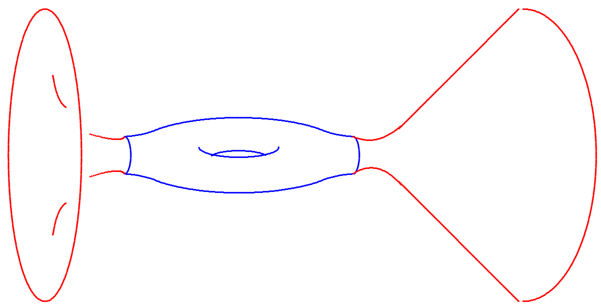

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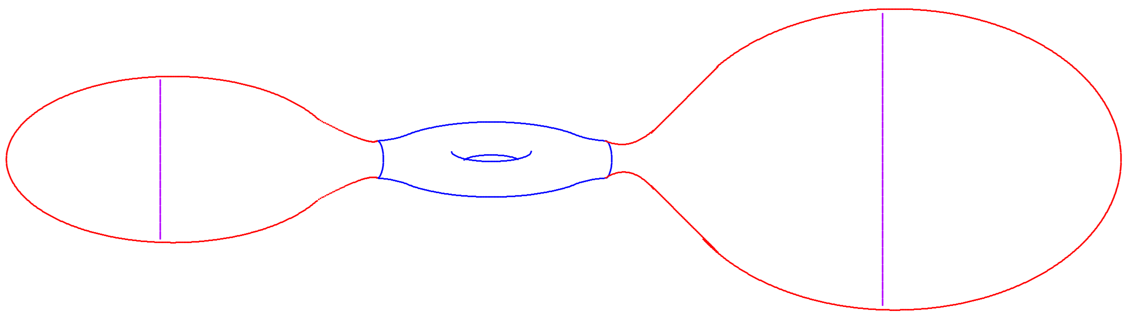

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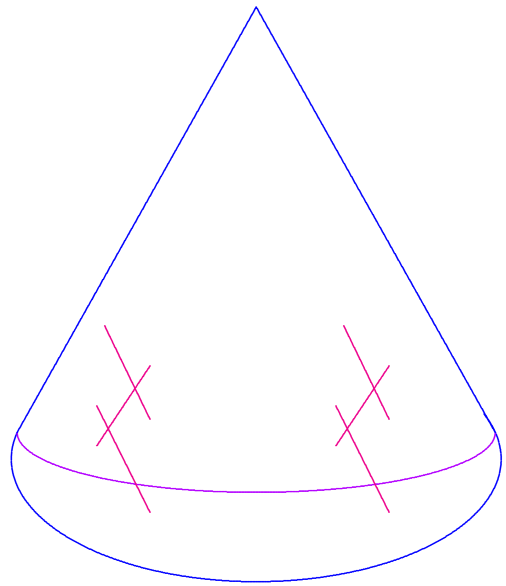

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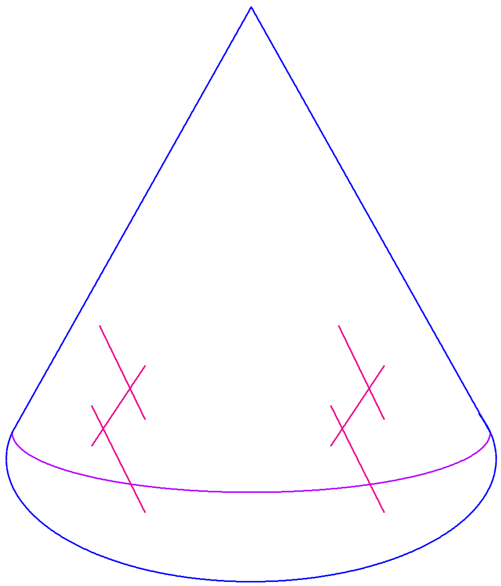

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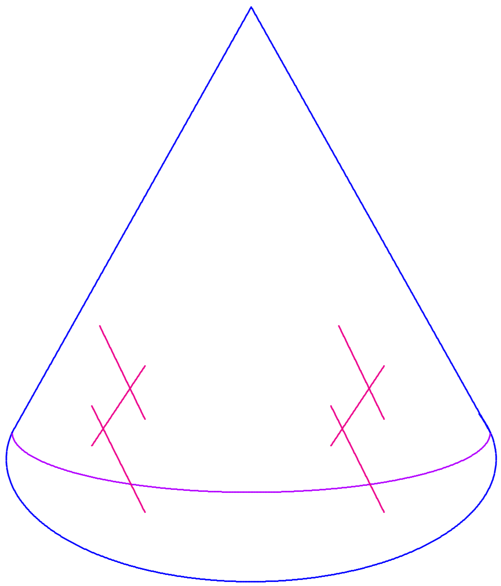

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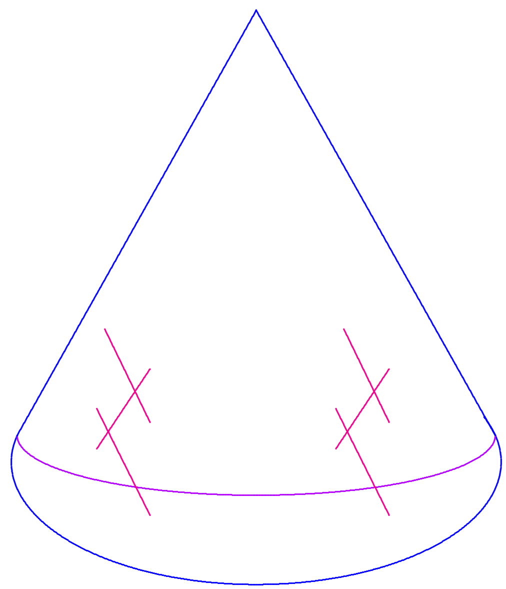

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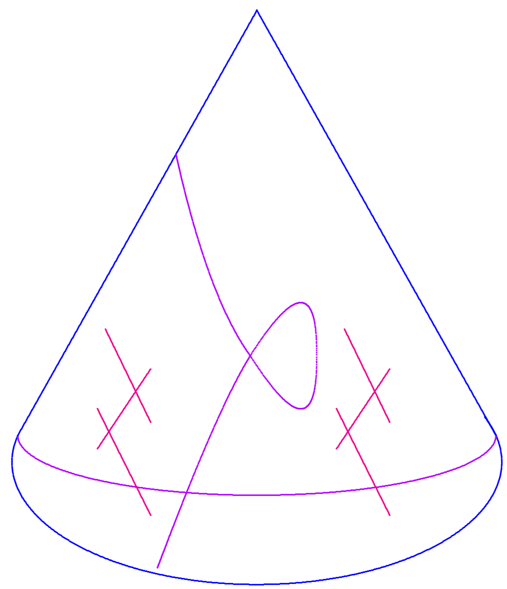

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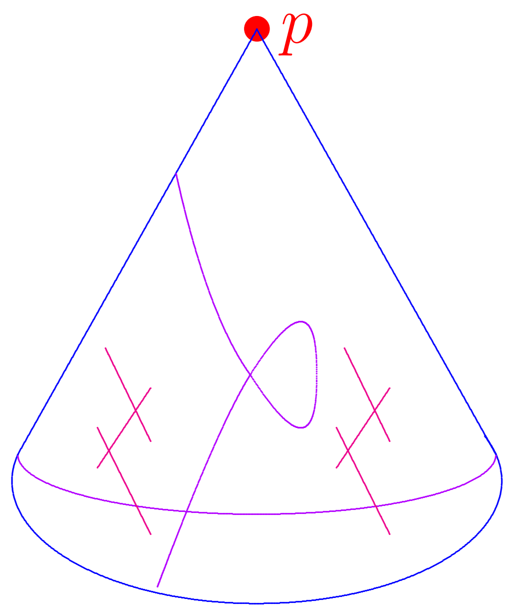

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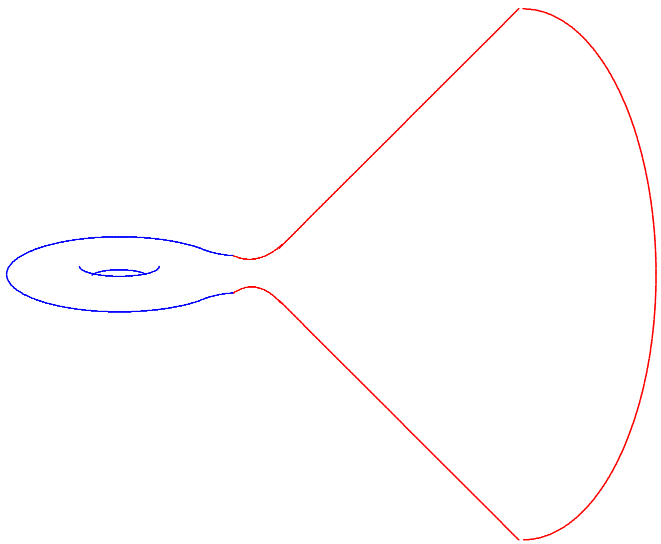
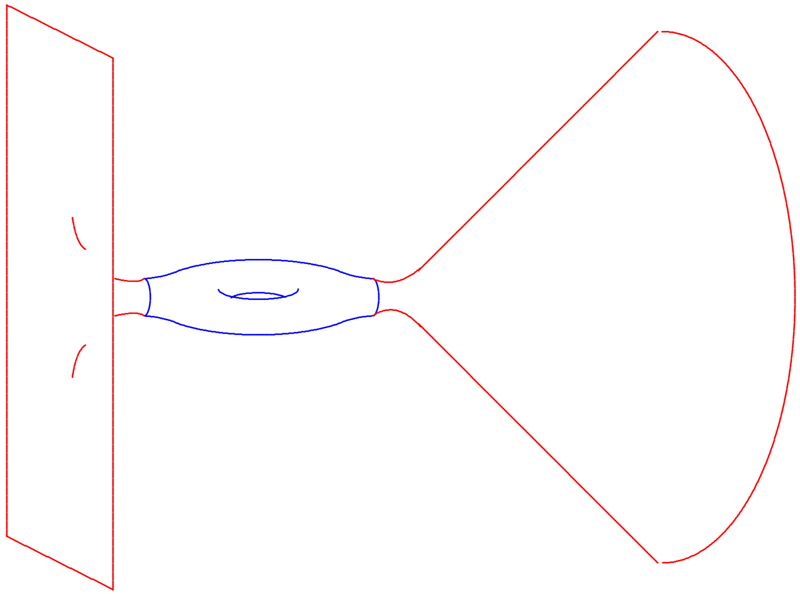
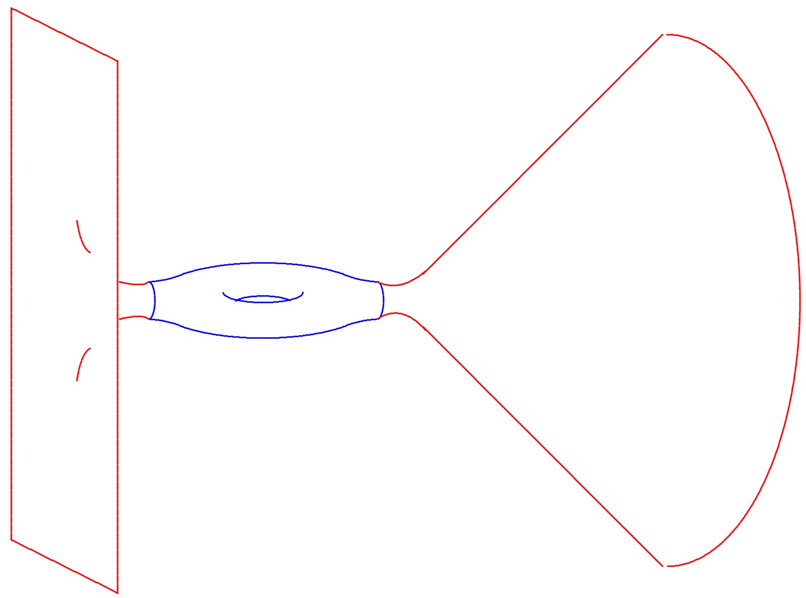
Lemma. *Any ALE Kähler manifold has only one end.*

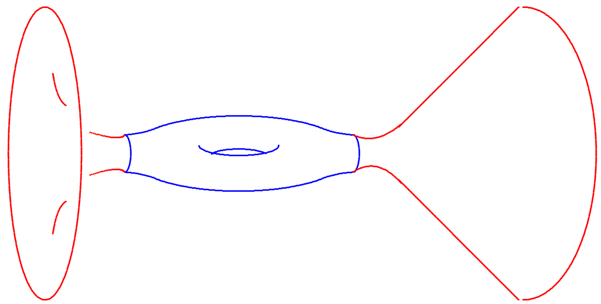




What if M^4 has more than one end?



Compactify M as symplectic 4-manifold $\widehat{M}$.



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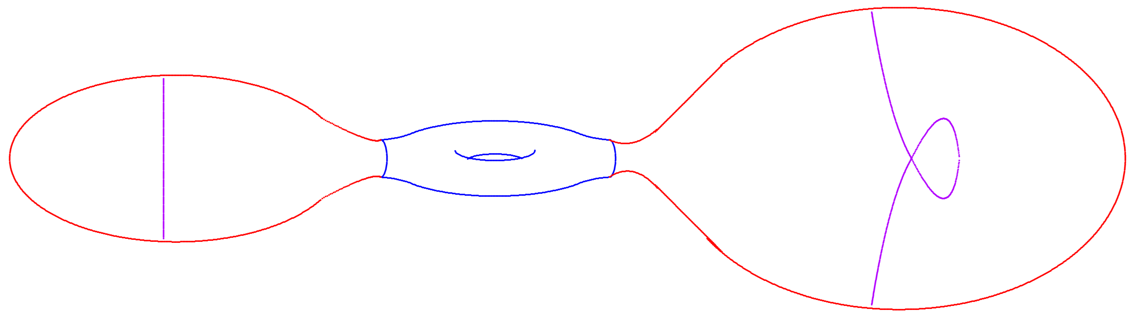

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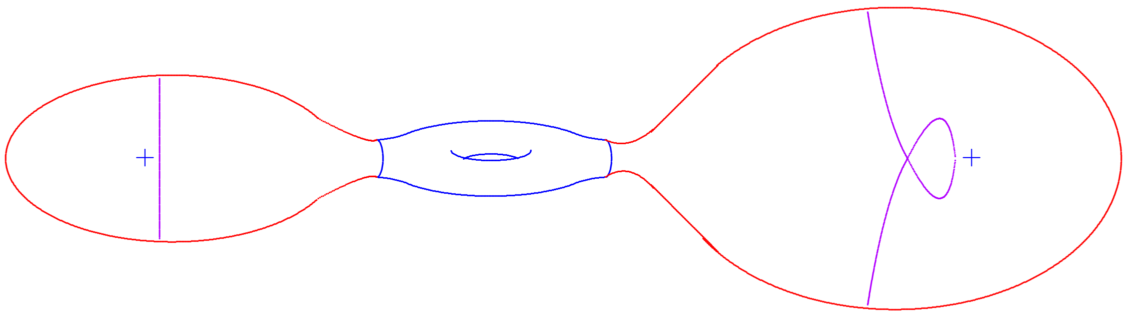

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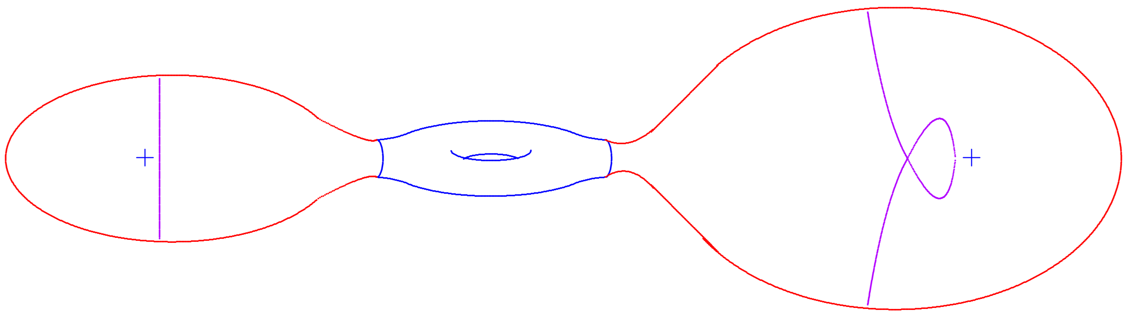

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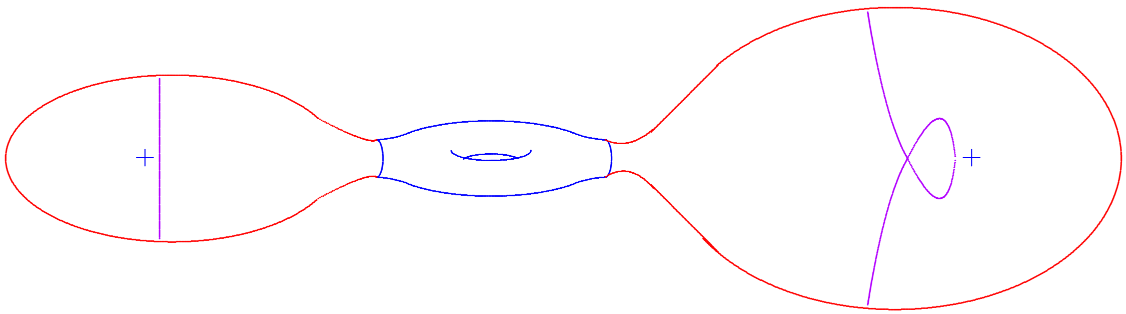
McDuff $\Longrightarrow$ $\widehat{M} \approx$ rational complex surface.



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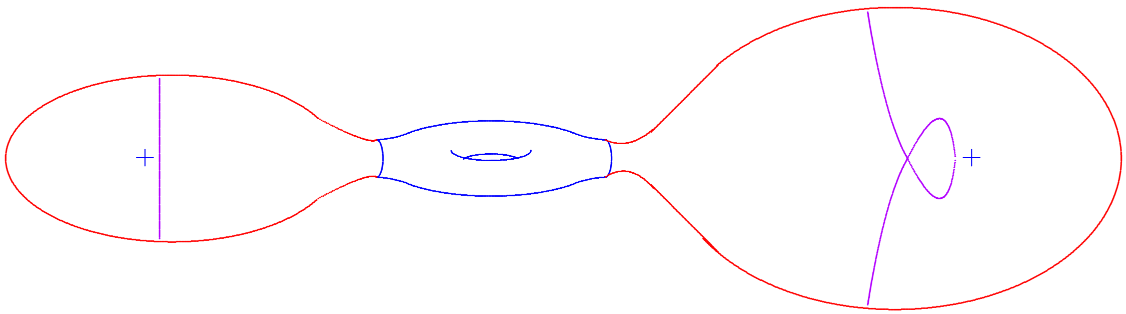
McDuff $\Longrightarrow$ intersection form $(+- \cdots -)$.



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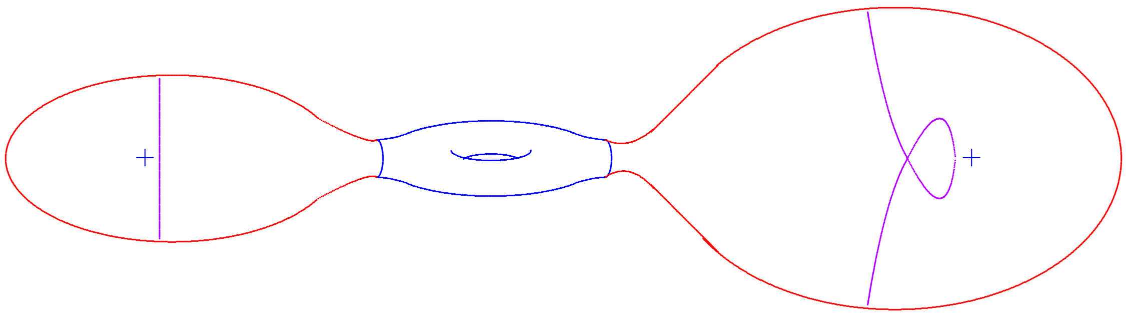

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Since each end contributes positive direction. . .

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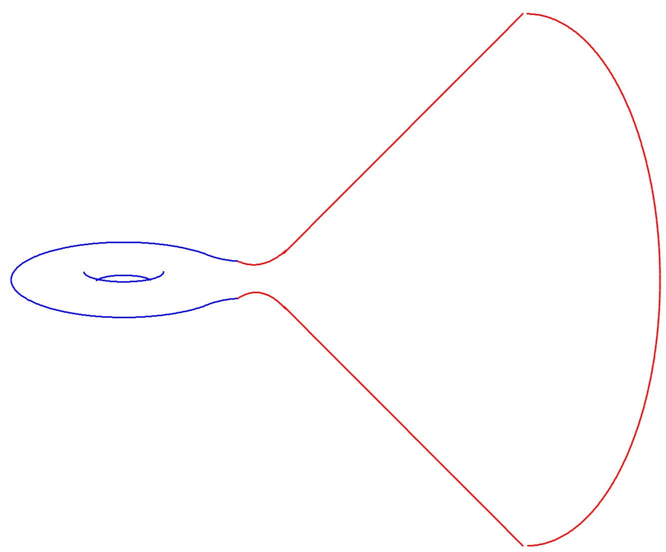

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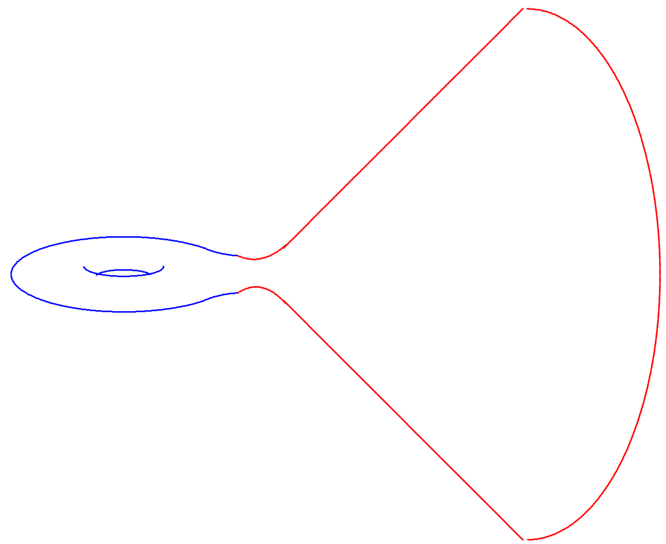

[In higher dimensions, one similarly shows that (M, J) can be compactified as Kähler orbifold. The Hodge theorem on intersection form instead tells one that form on $H^{1,1}(\widehat{M}, \mathbb{R})$ is of type $(+- \cdots -)$.]

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The following result provides the key…

Proposition. *Let g be a C^2 Kähler metric on $(\mathbb{R}^{2m} - \mathbf{D}^{2m})/\mathit{\Gamma}$, $m \geq 2$, where $\mathit{\Gamma} \subset \mathbf{SO}(2m)$ is some finite group that acts without fixed-points on S^{2m-1}.*

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We'll now deduce the mass formula...

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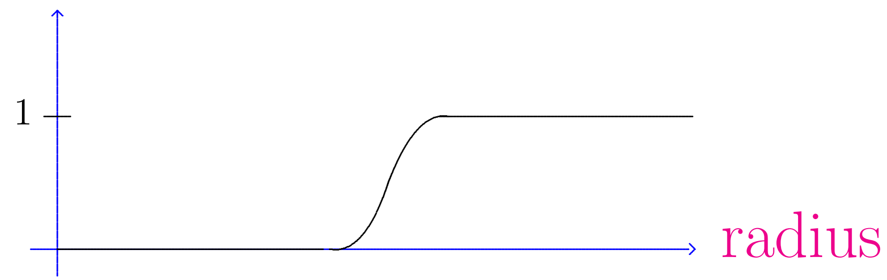

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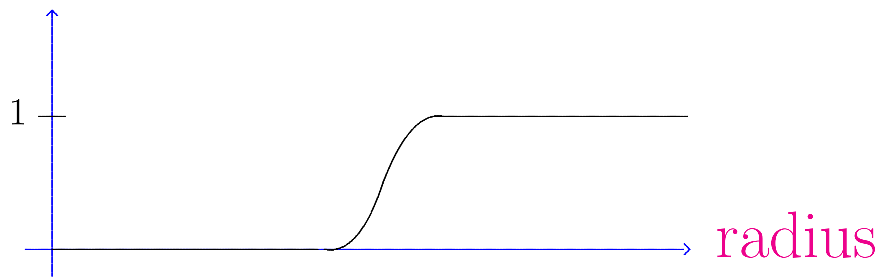

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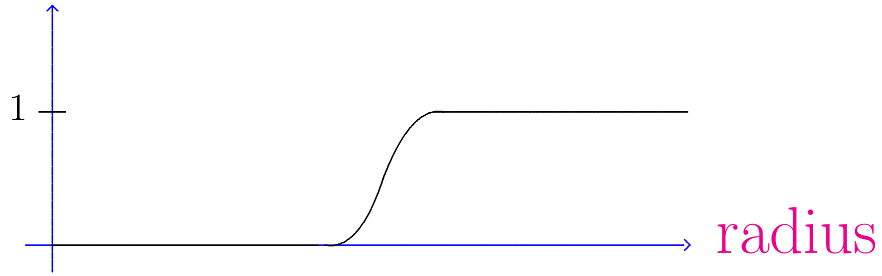

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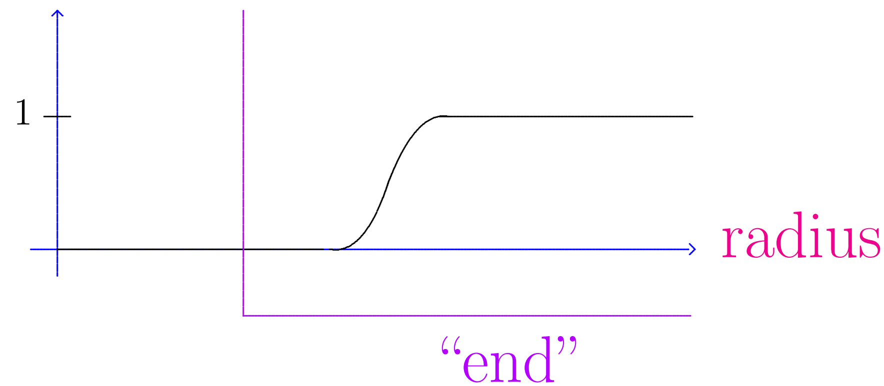

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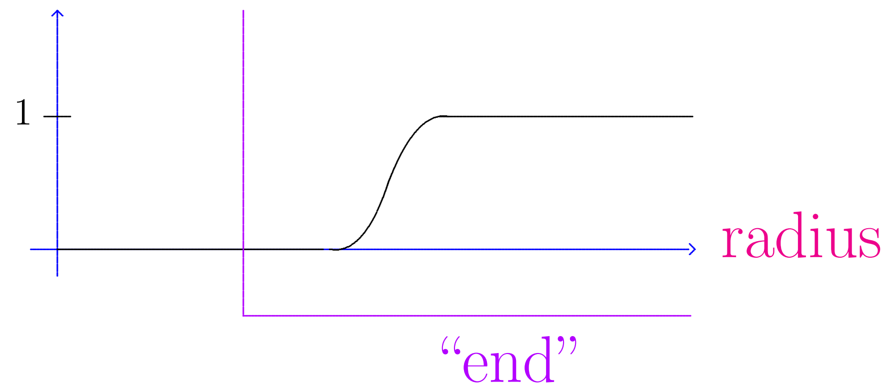

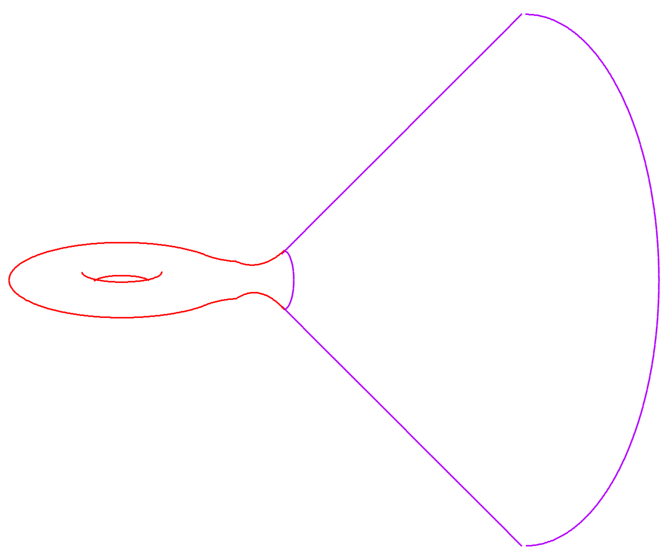

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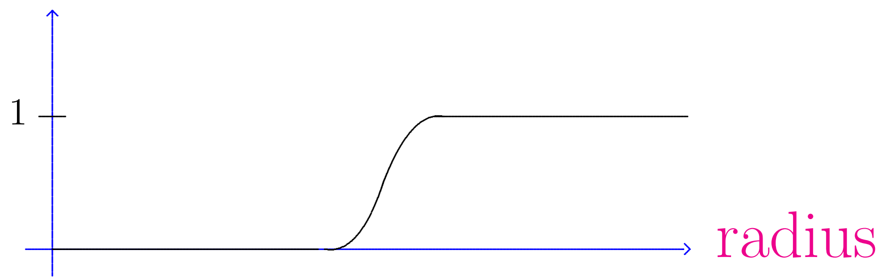

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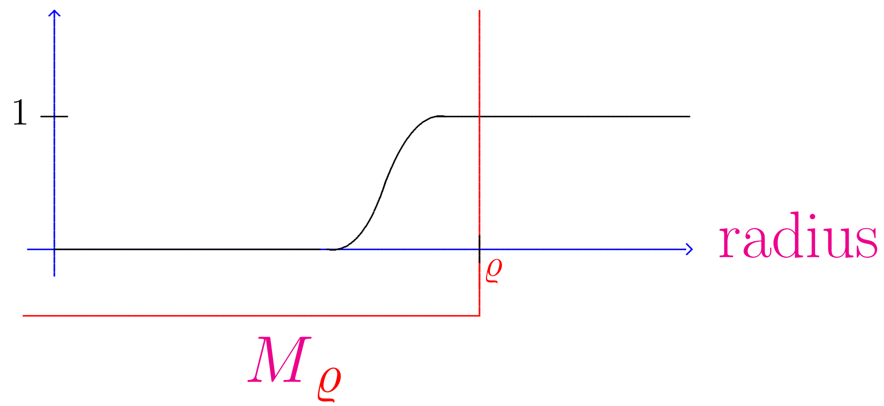

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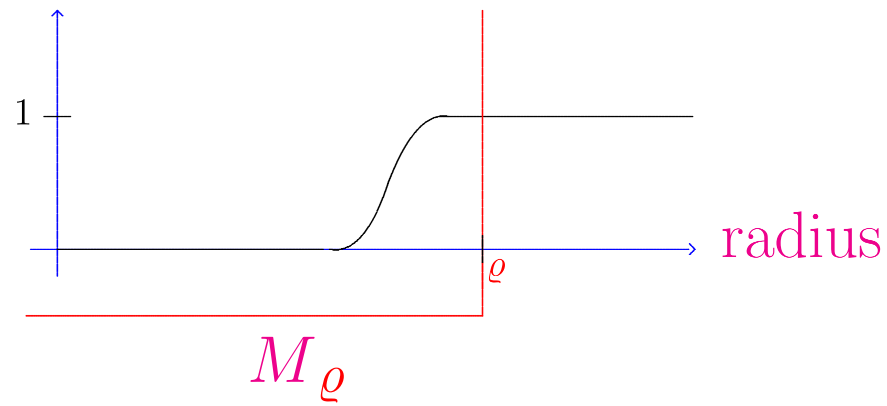

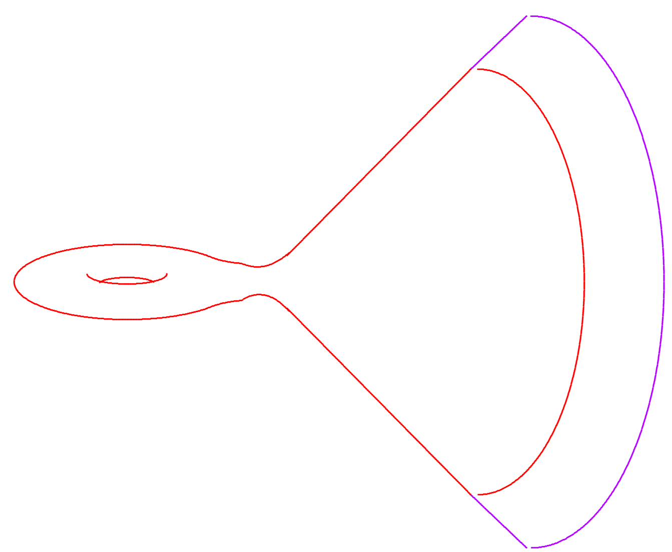

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Compactly supported, because $d\theta = \rho$ near infinity.

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where M_ϱ defined by radius $\leq \varrho$.

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because there is only one end!

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Let $f : M \to \mathbb{R}$ be smooth cut-off function:

$\equiv 0$ away from end,

$\equiv 1$ near infinity.

Set

$$\psi := \rho - d(f\theta)$$

$$[\psi] = \clubsuit([\rho]) = 2\pi\clubsuit(c_1) \in H^2_c(M)$$

$$\frac{2}{(m-1)!}\int_{S_\varrho/\Gamma} \theta\wedge\omega^{m-1} = -\frac{4\pi\langle\clubsuit(c_1), [\omega]^{m-1}\rangle}{(m-1)!} + \int_{M_\varrho} s\, d\mu$$

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Limit as $\varrho \to \infty$ now yields the mass formula.

$$\frac{4\pi^{m}(2m-1)}{(m-1)!}\mathfrak{m}(M,g) = -\frac{4\pi}{(m-1)!}\langle \clubsuit(c_1), [\omega]^{m-1}\rangle + \int_M s_g d\mu_g$$

$$\mathfrak{m}(M, g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1}\rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m}\int_M s_g \, d\mu_g$$

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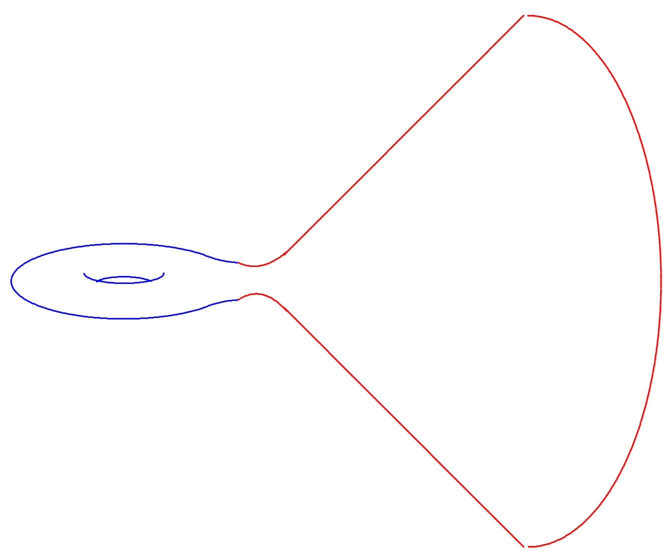

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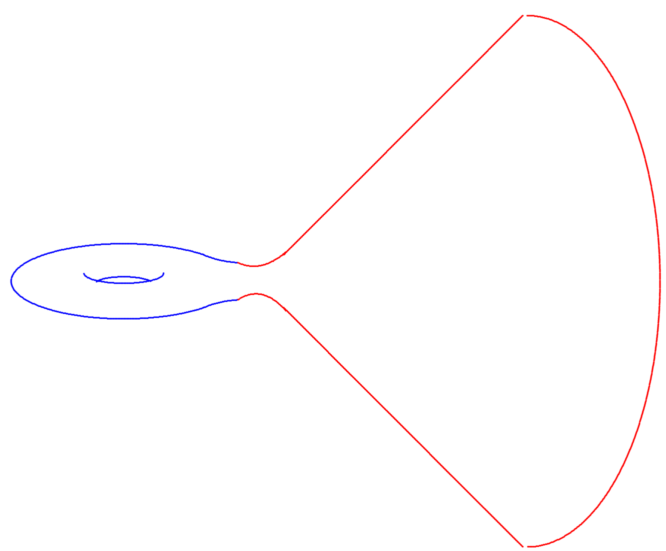

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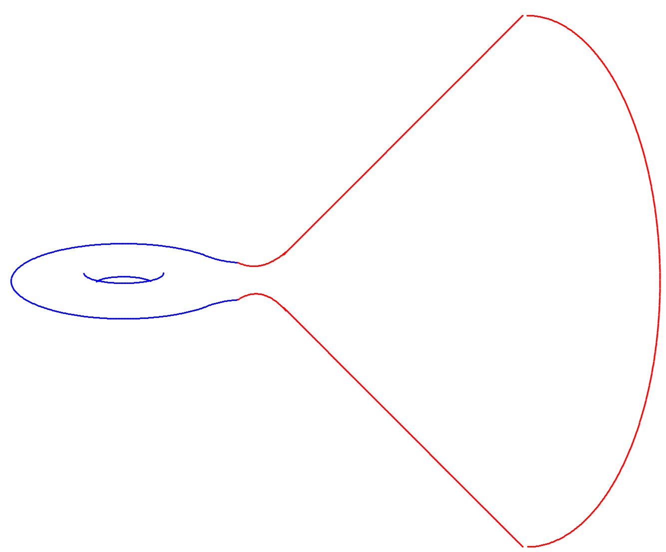
But the Penrose-type inequality is more subtle.

Theorem E (Penrose Inequality). *Let* (M^{2m}, g, J) *be an AE Kähler manifold with scalar curvature* $s \geq 0$. *Then* (M, J) *carries a canonical divisor* D *that is expressed as a sum* $\sum_j \mathbf{n}_j D_j$ *of compact complex hypersurfaces with positive integer coefficients, with the property that* $\bigcup_j D_j \neq \varnothing$ *whenever* $M \not\approx \mathbb{R}^{2m}$. *In terms of this divisor, we then have*

$$\mathfrak{m}(M, g) \geq \frac{(m-1)!}{(2m-1)\pi^{m-1}} \sum \mathbf{n}_j \, Vol(D_j)$$

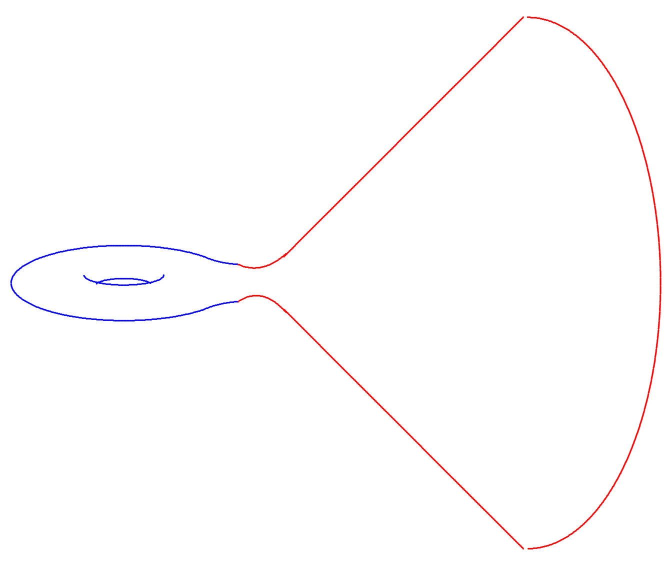
with $= \iff (M, g, J)$ *is scalar-flat Kähler.*

$$\mathfrak{m}(M,g) = \frac{\langle \clubsuit(-c_1), [\omega]^{m-1}\rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m}\int_M s_g d\mu_g$$



$m = 2:$

$$\mathscr{m}(M, g) = \frac{\langle \clubsuit(-c_1), [\omega] \rangle}{3\pi} + \frac{1}{12\pi} \int_M s_g d\mu_g$$



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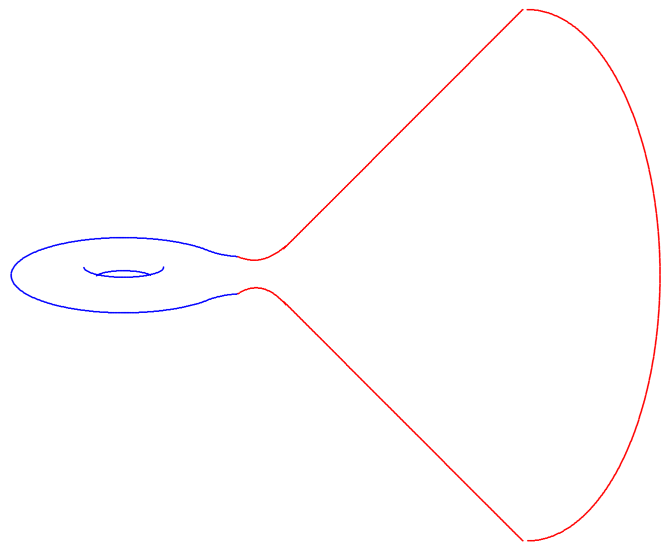
Technical challenge: Loss of control of derivatives!

Distance-decreasing map:



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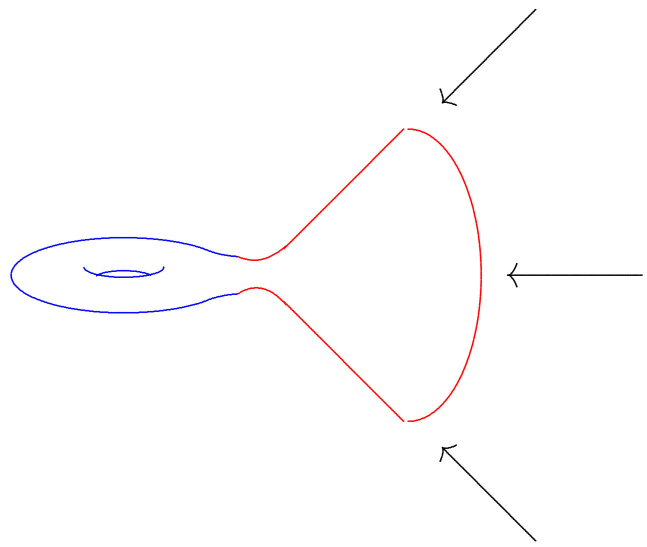

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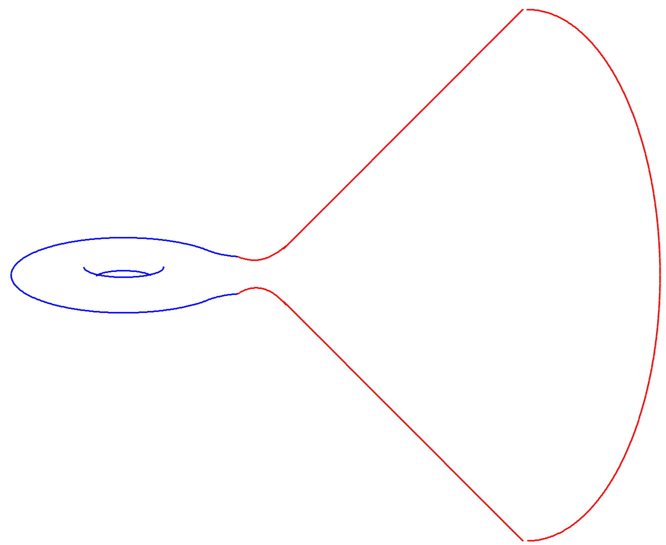

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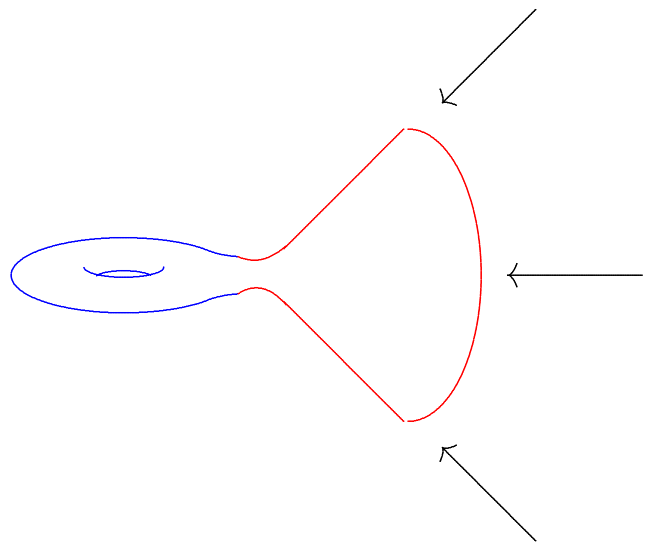

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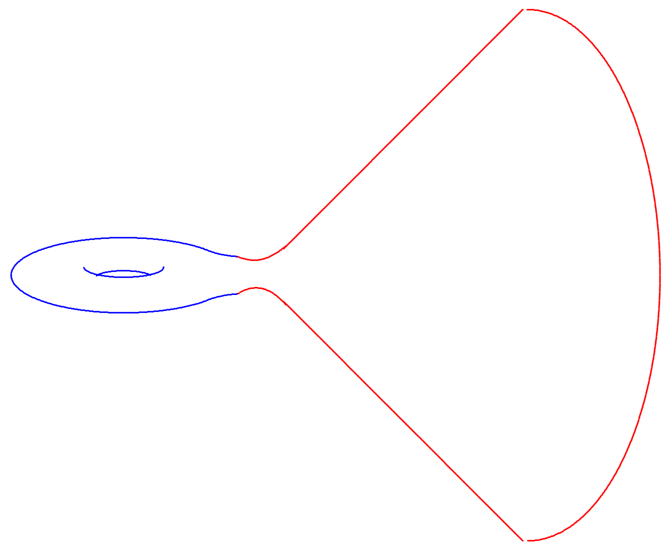

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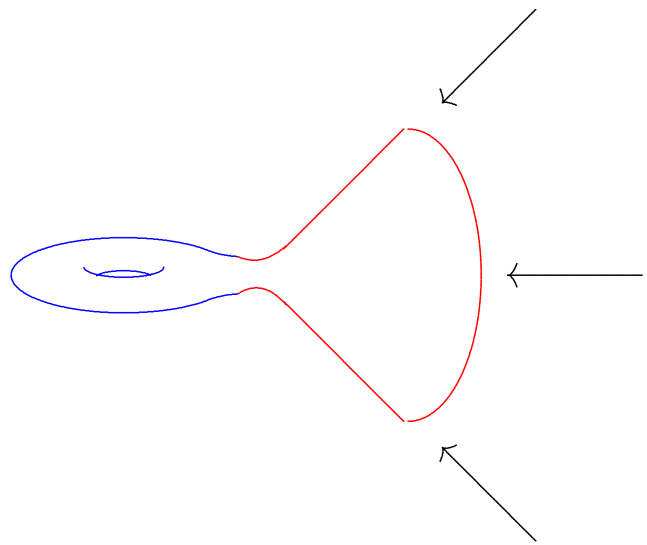

Robust under distortion of metric in outer region.

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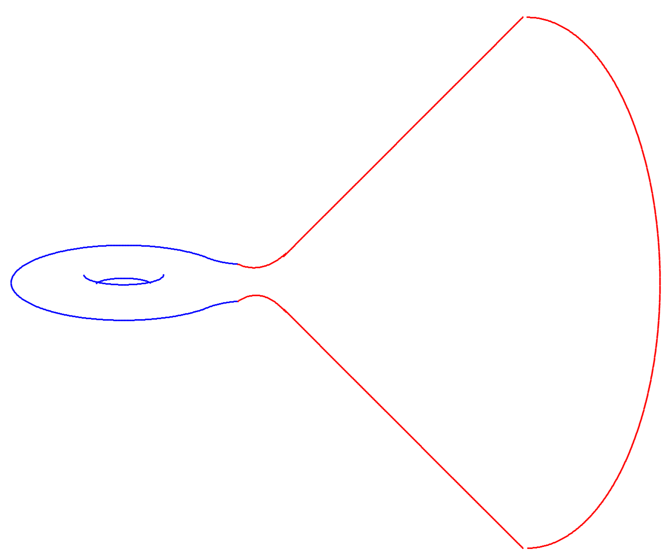
In (M, J), this gives desired Poincaré dual of $-c_1$.

$m = 2:$

$$\mathfrak{m}(M, g) = \frac{\langle \clubsuit(-c_1), [\omega] \rangle}{3\pi} + \frac{1}{12\pi} \int_M s_g d\mu_g$$



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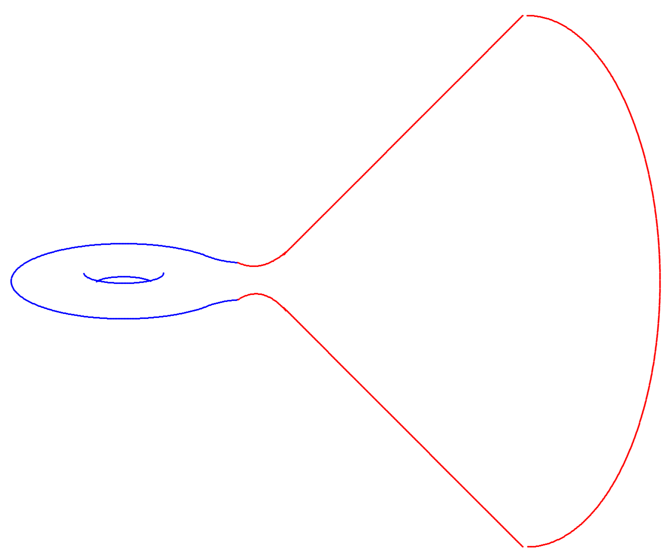

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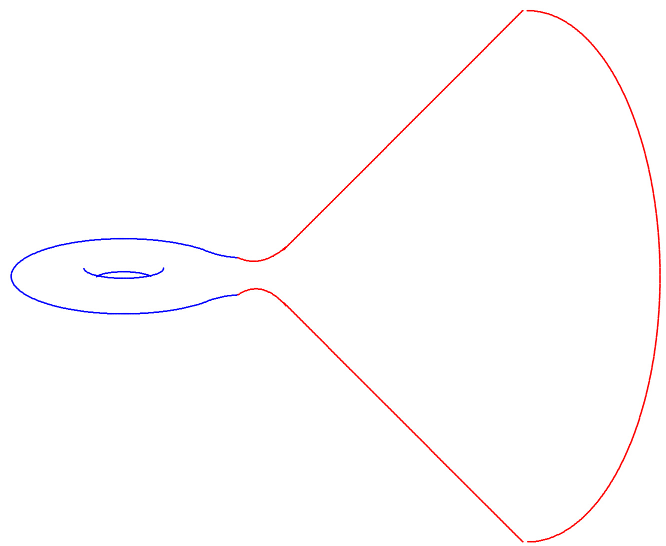

Thanks for Your Attention!