On

Four-Dimensional

Optimal Metrics

Claude LeBrun
SUNY Stony Brook

"Does every smooth compact manifold admit a *best metric*?"

— René Thom, c. 1960

"Does every smooth compact manifold admit a *best metric*?"

— René Thom, c. 1960

"What is best?"

— Socrates, c. 400 BC

"Does every smooth compact manifold admit a *best metric*?"

— René Thom, c. 1960

"What is best?"

— Socrates, c. 400 BC

"An optimal metric would seem to mean the *least curved one*."

— Marcel Berger

"Does every smooth compact manifold admit a *best metric*?"

— René Thom, c. 1960

"What is best?"

— Socrates, c. 400 BC

"An optimal metric would seem to mean the *least curved one*."

— Marcel Berger

"Huh?"

— Anonymous

Least curved metric?

Least curved metric?

Given a smooth compact manifold M, is there a Riemannian metric on M which has curvature uniformly close to zero?

Least curved metric?

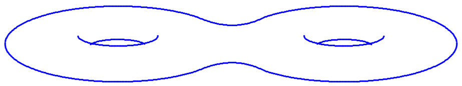
Given a smooth compact manifold M, is there a Riemannian metric on M which has curvature uniformly close to zero?

Of course! Rescale!

Least curved metric?

Given a smooth compact manifold M, is there a Riemannian metric on M which has curvature uniformly close to zero?

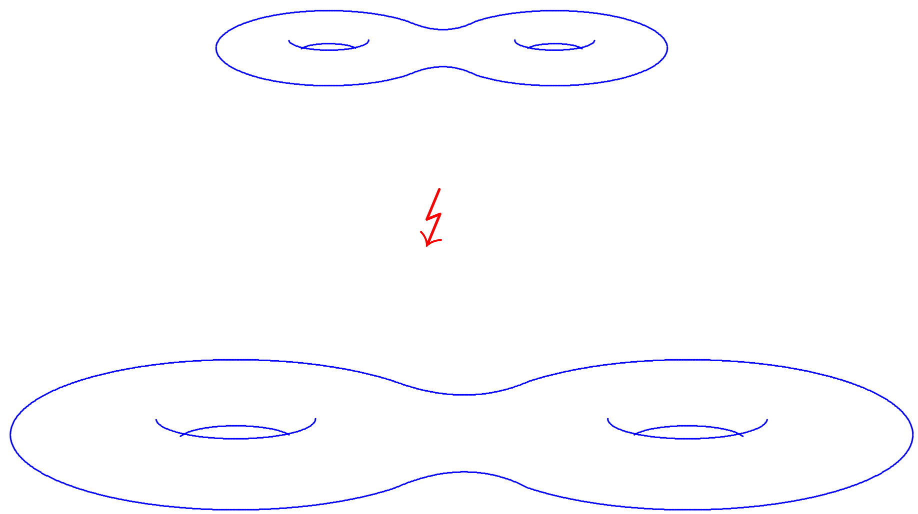
Of course! Rescale!



Least curved metric?

Given a smooth compact manifold M, is there a Riemannian metric on M which has curvature uniformly close to zero?

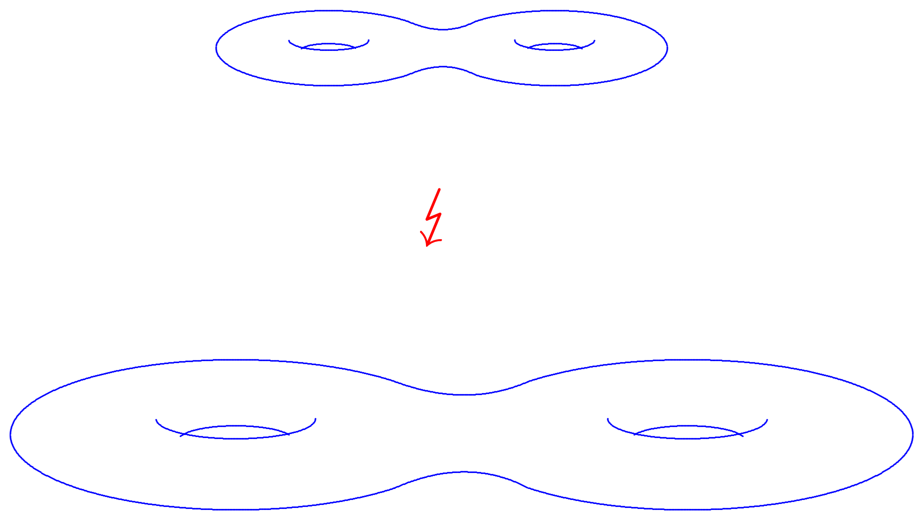
Of course! Rescale!



Least curved metric?

Given a smooth compact manifold M, is there a Riemannian metric on M which has curvature uniformly close to zero?

Of course! Rescale!



$$g \rightsquigarrow cg \implies |\mathcal{R}| \rightsquigarrow c^{-1}|\mathcal{R}|$$

Trade-off: small curvature $\longleftrightarrow$ *big volume*.

Trade-off: small curvature $\longleftrightarrow$ *big volume*.

Need scale-invariant measure of curvature!

Trade-off: small curvature $\longleftrightarrow$ *big volume*.

Need scale-invariant measure of curvature!

Natural choice: Riemannian functional

$$g \longmapsto \mathcal{K}(g) := \int_M |\mathcal{R}|_g^{n/2} d\mu_g.$$

Trade-off: small curvature $\longleftrightarrow$ *big volume*.

Need scale-invariant measure of curvature!

Natural choice: Riemannian functional

$$g \longmapsto \mathcal{K}(g) := \int_M |\mathcal{R}|_g^{n/2} d\mu_g.$$

Definition (Berger)**.** *Let M^n be a smooth compact n-manifold, $n \geq 3$. A Riemannian metric g on M will be called an* optimal metric *if it is an absolute minimizer of the functional $\mathcal{K}$.*

In dimension four,

$$g \longmapsto \mathcal{K}(g) := \int_M |\mathcal{R}|_g^2 d\mu_g,$$

cf. Yang-Mills functional; Calabi's functionals.

In dimension four,

$$g \longmapsto \mathcal{K}(g) := \int_M |\mathcal{R}|_g^2 d\mu_g,$$

cf. Yang-Mills functional; Calabi's functionals.

Berger's motivation: *Einstein metrics.*

In dimension four,

$$g \longmapsto \mathcal{K}(g) := \int_M |\mathcal{R}|_g^2 d\mu_g,$$

cf. Yang-Mills functional; Calabi's functionals.

Berger's motivation: *Einstein metrics.*

Definition. *A Riemannian metric is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$*.*

Proposition (Berger). *Let* (M^4, g) *be a compact Einstein* 4*-manifold. Then* g *is an optimal metric. Moreover, every other optimal metric* $\tilde{g}$ *on* M *is also Einstein.*

Proposition (Berger). *Let* (M^4, g) *be a compact Einstein* 4*-manifold. Then* g *is an optimal metric. Moreover, every other optimal metric* $\tilde{g}$ *on* M *is also Einstein.*

This statement is false in every other dimension!
Standard S^{2k+1}, $S^{2k+1} \times S^3$ not optimal…

What's so special about dimension 4?

What's so special about dimension 4?

The Lie group $SO(4)$ is *not simple*

What's so special about dimension 4?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

What's so special about dimension 4?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g),

What's so special about dimension 4?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g), $\Longrightarrow$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

What's so special about dimension 4?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g), $\Longrightarrow$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^\pm$ are (± 1)-eigenspaces of

$$\star : \Lambda^2 \to \Lambda^2,$$

What's so special about dimension 4?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g), $\Longrightarrow$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^\pm$ are (± 1)-eigenspaces of

$$\star : \Lambda^2 \to \Lambda^2,$$

$$\star^2 = 1.$$

Λ^+ self-dual 2-forms.

Λ^- anti-self-dual 2-forms.

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \longrightarrow \Lambda^2$$

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \longrightarrow \Lambda^2$$

splits into 4 irreducible pieces:

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \longrightarrow \Lambda^2$$

splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^+		
Λ^-		

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \longrightarrow \Lambda^2$$

splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^+	$W_+ + \frac{s}{12}$	$\mathring{r}$
Λ^-	$\mathring{r}$	$W_- + \frac{s}{12}$

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \longrightarrow \Lambda^2$$

splits into 4 irreducible pieces:

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W_- + \frac{s}{12} \end{array} \right).$$

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \longrightarrow \Lambda^2$$

splits into 4 irreducible pieces:

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W_- + \frac{s}{12} \end{array} \right).$$

where

$$\begin{aligned} s &= \text{scalar curvature} \\ \mathring{r} &= \text{trace-free Ricci curvature} \\ W_+ &= \text{self-dual Weyl curvature} \\ W_- &= \text{anti-self-dual Weyl curvature} \end{aligned}$$

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W_- + \frac{s}{12} \end{array} \right).$$

where

$$\begin{aligned} s &= \text{scalar curvature} \\ \mathring{r} &= \text{trace-free Ricci curvature} \\ W_+ &= \text{self-dual Weyl curvature } \textit{(conformally invariant)} \\ W_- &= \text{anti-self-dual Weyl curvature} \quad \text{''} \end{aligned}$$

(M, g) compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(\frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu$$

(M, g) compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(\frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu$$

4-dimensional Hirzebruch signature formula

$$\tau(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|^2 - |W_-|^2 \right) d\mu$$

for signature $\tau(M) = b_+(M) - b_-(M)$.

$$\mathcal{K}(g) = \int_M |\mathcal{R}|_g^2 d\mu_g$$

$$
\begin{aligned}
\mathcal{K}(g) &= \int_M |\mathcal{R}|_g^2 \, d\mu_g \\
&= 8\pi^2 \chi(M) + \int_M |\mathring{r}|^2 \, d\mu_g
\end{aligned}
$$

$$\mathcal{K}(g) = \int_M |\mathcal{R}|_g^2 d\mu_g$$
$$= 8\pi^2\chi(M) + \int_M |\mathring{r}|^2 d\mu_g$$

Berger: Einstein $\Longrightarrow$ optimal.

$$
\begin{aligned}
\mathcal{K}(g) &= \int_M |\mathcal{R}|_g^2 d\mu_g \\
&= 8\pi^2\chi(M) + \int_M |\mathring{r}|^2 d\mu_g \\
&= -8\pi^2(\chi + 3\tau)(M) + 2\int_M \left(\frac{s^2}{24} + 2|W_+|^2 \right) d\mu_g .
\end{aligned}
$$

$$\begin{aligned}\mathcal{K}(g) &= \int_M |\mathcal{R}|_g^2 d\mu_g \\ &= 8\pi^2\chi(M) + \int_M |\mathring{r}|^2 d\mu_g \\ &= -8\pi^2(\chi + 3\tau)(M) + 2\int_M \left(\frac{s^2}{24} + 2|W_+|^2\right) d\mu_g.\end{aligned}$$

Definition. *A Riemannian metric g on a smooth oriented 4-manifold M is called* anti-self-dual *(ASD) if it satisfies*

$$W_+ \equiv 0.$$

$$
\begin{aligned}
\mathcal{K}(g) &= \int_M |\mathcal{R}|_g^2 d\mu_g \\
&= 8\pi^2\chi(M) + \int_M |\mathring{r}|^2 d\mu_g \\
&= -8\pi^2(\chi + 3\tau)(M) + 2\int_M \left(\frac{s^2}{24} + 2|W_+|^2\right) d\mu_g.
\end{aligned}
$$

Definition. *A Riemannian metric g on a smooth oriented 4-manifold M is called* anti-self-dual *(ASD) if it satisfies*

$$W_+ \equiv 0.$$

If also

$$s \equiv 0$$

then called scalar-flat anti-self-dual *(SFASD).*

Proposition (Lafontaine). *If smooth compact oriented* M^4 *carries SFASD metric* g*, then* g *is optimal*

Proposition (Lafontaine). *If smooth compact oriented M^4 carries SFASD metric g, then g is optimal; and every other optimal metric $\tilde{g}$ on M is SFASD, too.*

Proposition (Lafontaine). *If smooth compact oriented M^4 carries SFASD metric g, then g is optimal; and every other optimal metric $\tilde{g}$ on M is SFASD, too.*

$$\mathcal{K}(g) = -8\pi^2(\chi+3\tau)(M)+2\int_M \left(\frac{s^2}{24} + 2|W_+|^2\right) d\mu_g.$$

Proposition (Lafontaine). *If smooth compact oriented M^4 carries SFASD metric g, then g is optimal; and every other optimal metric $\tilde{g}$ on M is SFASD, too.*

Also get topological obstruction:

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu_g$$

Proposition (Lafontaine). *If smooth compact oriented M^4 carries SFASD metric g, then g is optimal; and every other optimal metric $\tilde{g}$ on M is SFASD, too.*

Also get topological obstruction:

$$
\begin{aligned}
(2\chi + 3\tau)(M) &= \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu_g \\
&\leq 0.
\end{aligned}
$$

Proposition (Lafontaine). *If smooth compact oriented M^4 carries SFASD metric g, then g is optimal; and every other optimal metric $\tilde{g}$ on M is SFASD, too.*

Also get topological obstruction:

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu_g$$
$$\leq 0.$$

Reverse Hitchin-Thorpe!

Proposition. *Let M^4 be simply connected smooth compact. If M admits a scalar-flat anti-self-dual metric, then*

- M *is homeomorphic to* $k\overline{\mathbb{CP}}_2$, $k \geq 5$; *or*
- M *is diffeomorphic to* $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $k \geq 10$; *or*
- M *is diffeomorphic to* $K3$.

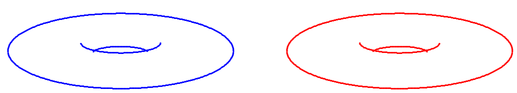
Recall:

$\overline{\mathbb{CP}}_2$ = reverse oriented $\mathbb{CP}_2$.

Recall:

$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

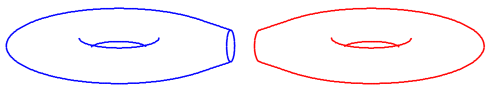
Connected sum:



Recall:

$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

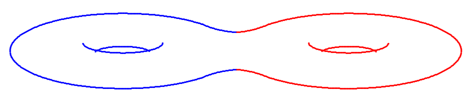
Connected sum:



Recall:

$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

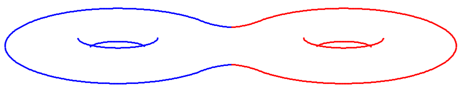
Connected sum:



Recall:

$\overline{\mathbb{CP}}_2$ = reverse oriented $\mathbb{CP}_2$.

Connected sum:

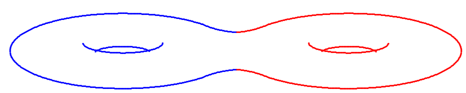


$K3$ = Kummer-Kähler-Kodaira manifold.

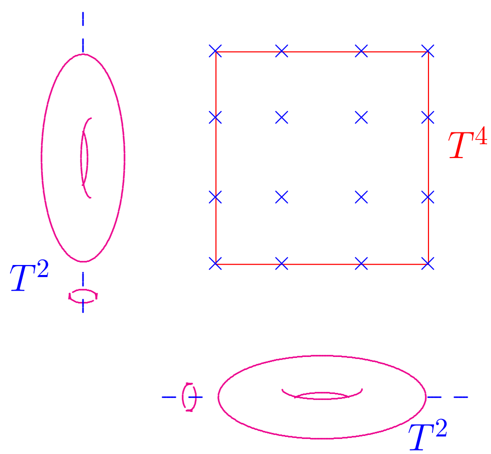
Recall:

$\overline{\mathbb{CP}}_2$ = reverse oriented $\mathbb{CP}_2$.

Connected sum:



$K3$ = Kummer-Kähler-Kodaira manifold.



Proposition. *Let M^4 be simply connected smooth compact. If M admits a scalar-flat anti-self-dual metric, then*

- M *is homeomorphic to* $k\overline{\mathbb{CP}_2}$, $k \geq 5$; *or*
- M *is diffeomorphic to* $\mathbb{CP}_2 \# k\overline{\mathbb{CP}_2}$, $k \geq 10$; *or*
- M *is diffeomorphic to* $K3$.

Proposition. *Let M^4 be simply connected smooth compact. If M admits a scalar-flat anti-self-dual metric, then*

- *M is homeomorphic to $k\overline{\mathbb{CP}}_2$, $k \geq 5$; or*
- *M is diffeomorphic to $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $k \geq 10$; or*
- *M is diffeomorphic to $K3$.*

Weitzenböck formula for $\varphi \in \Gamma(\Lambda^+)$:

$$(d + d^*)^2\varphi = \nabla^*\nabla\varphi - 2W_+(\varphi, \cdot) + \frac{s}{3}\varphi$$

Proposition. *Let M^4 be simply connected smooth compact. If M admits a scalar-flat anti-self-dual metric, then*

- *M is homeomorphic to $k\overline{\mathbb{CP}}_2$, $k \geq 5$; or*
- *M is diffeomorphic to $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $k \geq 10$; or*
- *M is diffeomorphic to $K3$.*

Weitzenböck formula $\Rightarrow$

$$b_+(M) = \dim\{\varphi \in \Gamma(\Lambda^+) \mid \nabla\varphi = 0\}.$$

Proposition. *Let M^4 be simply connected smooth compact. If M admits a scalar-flat anti-self-dual metric, then*

- *M is homeomorphic to $k\overline{\mathbb{CP}}_2$, $k \geq 5$; or*
- *M is diffeomorphic to $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $k \geq 10$; or*
- *M is diffeomorphic to $K3$.*

Weitzenböck formula $\Rightarrow$

$$b_+(M) = \dim\{\varphi \in \Gamma(\Lambda^+) \mid \nabla\varphi = 0\}.$$

Kähler case: $\varphi \in \Gamma(K^\ell) \Longrightarrow$

$$2(\bar{\partial} + \bar{\partial}^*)^2\varphi = \nabla^*\nabla\varphi + \frac{\ell s}{2}\varphi$$

Proposition. *Let M^4 be simply connected smooth compact. If M admits a scalar-flat anti-self-dual metric, then*

- *M is homeomorphic to $k\overline{\mathbb{CP}}_2$, $k \geq 5$; or*
- *M is diffeomorphic to $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $k \geq 10$; or*
- *M is diffeomorphic to $K3$.*

Weitzenböck formula $\Rightarrow$

$$b_+(M) = \dim\{\varphi \in \Gamma(\Lambda^+) \mid \nabla\varphi = 0\}.$$

Kähler case: $h^0(\mathcal{O}(K^\ell)) = 0$ or K^ℓ trivial.

Proposition. *Let M^4 be simply connected smooth compact. If M admits a scalar-flat anti-self-dual metric, then*

- *M is homeomorphic to $k\overline{\mathbb{CP}}_2$, $k \geq 5$; or*
- *M is diffeomorphic to $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $k \geq 10$; or*
- *M is diffeomorphic to $K3$.*

Weitzenböck formula $\Rightarrow$

$$b_+(M) = \dim\{\varphi \in \Gamma(\Lambda^+) \mid \nabla\varphi = 0\}.$$

Lafontaine:

$$(2\chi + 3\tau)(M) \leq 0.$$

Proposition. *Let M^4 be simply connected smooth compact. If M admits a scalar-flat anti-self-dual metric, then*

- *M is homeomorphic to $k\overline{\mathbb{CP}}_2$, $k \geq 5$; or*
- *M is diffeomorphic to $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $k \geq 10$; or*
- *M is diffeomorphic to $K3$.*

Weitzenböck formula $\Rightarrow$

$$b_+(M) = \dim\{\varphi \in \Gamma(\Lambda^+) \mid \nabla\varphi = 0\}.$$

Lafontaine:

$$(2\chi + 3\tau)(M) \leq 0.$$

Enriques, Kodaira

Proposition. *Let M^4 be simply connected smooth compact. If M admits a scalar-flat anti-self-dual metric, then*

- *M is homeomorphic to $k\overline{\mathbb{CP}}_2$, $k \geq 5$; or*
- *M is diffeomorphic to $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $k \geq 10$; or*
- *M is diffeomorphic to $K3$.*

Weitzenböck formula $\Rightarrow$

$$b_+(M) = \dim\{\varphi \in \Gamma(\Lambda^+) \mid \nabla\varphi = 0\}.$$

Lafontaine:

$$(2\chi + 3\tau)(M) \leq 0.$$

Enriques, Kodaira, Donaldson, Freedman.

Proposition. *Let* M^4 *be simply connected smooth compact. If* M *admits a scalar-flat anti-self-dual metric, then*

- M *is homeomorphic to* $k\overline{\mathbb{CP}}_2$, $k \geq 5$*; or*
- M *is diffeomorphic to* $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $k \geq 10$*; or*
- M *is diffeomorphic to* $K3$.

Theorem A. *Simply connected smooth compact M^4 actually admits a scalar-flat anti-self-dual metric if*

- M *is diffeomorphic to* $k\overline{\mathbb{CP}}_2$, $k \geq 5$*; or*
- M *is diffeomorphic to* $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $k \geq 10$*; or*
- M *is diffeomorphic to* $K3$.

Proposition. *Let M^4 be simply connected smooth compact. If M admits a scalar-flat anti-self-dual metric, then*

- *M is homeomorphic to $k\overline{\mathbb{CP}}_2$, $k \geq 5$; or*
- *M is diffeomorphic to $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $k \geq 10$; or*
- *M is diffeomorphic to $K3$.*

Theorem A. *Simply connected smooth compact M^4 actually admits a scalar-flat anti-self-dual metric if*

- M *is diffeomorphic to* $k\overline{\mathbb{CP}}_2$, $k \geq 5$; *or*
- M *is diffeomorphic to* $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $k \geq 10$; *or*
- M *is diffeomorphic to* $K3$.

Theorem A. *Simply connected smooth compact M^4 actually admits a scalar-flat anti-self-dual metric if*

- M *is diffeomorphic to* $k\overline{\mathbb{CP}}_2$, $k \geq 5$*; or*
- M *is diffeomorphic to* $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $k \geq 10$*; or*
- M *is diffeomorphic to* $K3$.

Corollary. *These* M^4 *admit optimal metrics.*

Theorem A. *Simply connected smooth compact* M^4 *actually admits a scalar-flat anti-self-dual metric if*

- M *is diffeomorphic to* $k\overline{\mathbb{CP}}_2$, $k \geq 5$*; or*
- M *is diffeomorphic to* $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $k \geq 10$*; or*
- M *is diffeomorphic to* $K3$*.*

Corollary. *These* M^4 *admit optimal metrics.*

For $M^4 \neq K3$, optimal, but not Einstein.

Corollary. *A compact simply connected topological 4-manifold M carries a smooth structure for which there is a compatible SFASD* metric g *iff M is homeomorphic to*

- $k\overline{\mathbb{CP}}_2$, $k \geq 5$;
- $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $k \geq 10$; *or*
- $K3$.

Theorem A. *Simply connected smooth compact M^4 actually admits a scalar-flat anti-self-dual metric if*

- M *is diffeomorphic to* $k\overline{\mathbb{CP}}_2$, $k \geq 5$; *or*
- M *is diffeomorphic to* $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $k \geq 10$; *or*
- M *is diffeomorphic to* $K3$.

Theorem A. *Simply connected smooth compact* M^4 *actually admits a scalar-flat anti-self-dual metric if*

- M *is diffeomorphic to* $k\overline{\mathbb{CP}}_2$, $k \geq 5$*; or*
- M *is diffeomorphic to* $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $k \geq 10$*; or*
- M *is diffeomorphic to* $K3$.

Pieces of proof:

Theorem A. *Simply connected smooth compact M^4 actually admits a scalar-flat anti-self-dual metric if*

- M *is diffeomorphic to* $k\overline{\mathbb{CP}}_2$, $k \geq 5$*; or*
- M *is diffeomorphic to* $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $k \geq 10$*; or*
- M *is diffeomorphic to* $K3$.

Pieces of proof:

Yau (1978)

Theorem A. *Simply connected smooth compact* M^4 *actually admits a scalar-flat anti-self-dual metric if*

- M *is diffeomorphic to* $k\overline{\mathbb{CP}}_2$, $k \geq 5$*; or*
- M *is diffeomorphic to* $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $k \geq 10$*; or*
- M *is diffeomorphic to* $K3$*.*

Pieces of proof:

Yau (1978)

Kim-LeBrun-Pontecorvo (1993)

Theorem A. *Simply connected smooth compact* M^4 *actually admits a scalar-flat anti-self-dual metric if*

- M *is diffeomorphic to* $k\overline{\mathbb{CP}}_2$, $k \geq 5$*; or*
- M *is diffeomorphic to* $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $k \geq 10$*; or*
- M *is diffeomorphic to* $K3$*.*

Pieces of proof:

Yau (1978)

Kim-LeBrun-Pontecorvo (1993)

Singer-Rollin (2004)

Theorem A. *Simply connected smooth compact M^4 actually admits a scalar-flat anti-self-dual metric if*

- M *is diffeomorphic to* $k\overline{\mathbb{CP}}_2$, $k \geq 5$*; or*
- M *is diffeomorphic to* $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $k \geq 10$*; or*
- M *is diffeomorphic to* $K3$.

Pieces of proof:

Yau (1978)

Kim-LeBrun-Pontecorvo (1993)

Singer-Rollin (2004)

LeBrun (2004)

Theorem A. *Simply connected smooth compact M^4 actually admits a scalar-flat anti-self-dual metric if*

- *M is diffeomorphic to $k\overline{\mathbb{CP}}_2$, $k \geq 5$; or*
- *M is diffeomorphic to $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $k \geq 10$; or*
- *M is diffeomorphic to $K3$.*

Pieces of proof:

Yau (1978)

Kim-LeBrun-Pontecorvo (1993)

Singer-Rollin (2004)

LeBrun (2004)

LeBrun-Maskit (2008)

When $b_+(M) \neq 0$, Weitzenböck formula

$$(d + d^*)^2 \varphi = \nabla^* \nabla \varphi - 2W_+(\varphi, \cdot) + \frac{s}{3}\varphi$$

shows $\nexists$ ASD metrics with $s > 0$.

When $b_+(M) \neq 0$, Weitzenböck formula

$$(d + d^*)^2\varphi = \nabla^*\nabla\varphi - 2W_+(\varphi, \cdot) + \frac{s}{3}\varphi$$

shows $\nexists$ ASD metrics with $s > 0$.

But when $b_+(M) = 0$, key is to find

family g_t of ASD metrics s.t. s changes sign.

Proposition. *For any integer* $k \geq 5$, *the connected sum*

$$k\overline{\mathbb{CP}}_2 = \underbrace{\overline{\mathbb{CP}}_2 \# \cdots \# \overline{\mathbb{CP}}_2}_{k}$$

admits 1*-parameter family of* ASD *conformal metrics* $[g_t]$, $t \in [-1, 1]$, *such that*

- $\exists g_{-1} \in [g_{-1}]$ *with* $s < 0$; *and*
- $\exists g_1 \in [g_1]$ *with* $s > 0$.

Twistor picture of anti-self-duality condition:

Twistor picture of anti-self-duality condition:

Oriented $(M^4, g) \leftrightsquigarrow (Z, J)$.

Twistor picture of anti-self-duality condition:

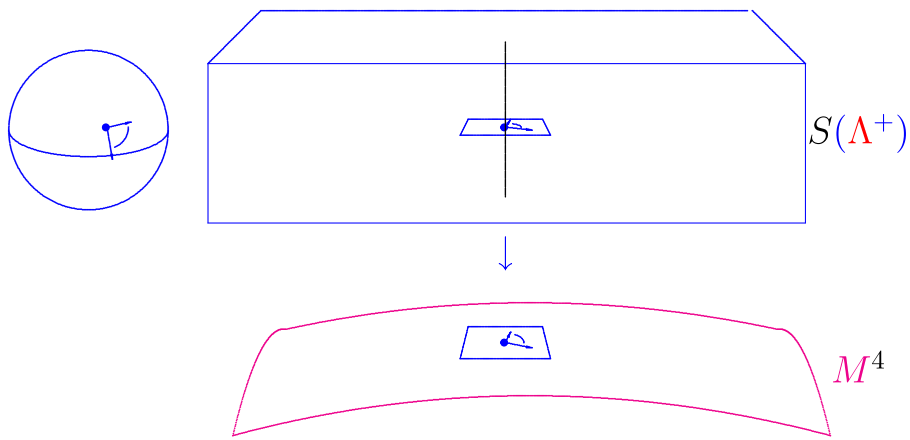
Oriented $(M^4, g) \leftrightsquigarrow (Z, J)$.

$Z = S(\Lambda^+)$, $J : TZ \to TZ$, $J^2 = -1$:

Twistor picture of anti-self-duality condition:

Oriented $(M^4, g) \leftrightsquigarrow (Z, J)$.

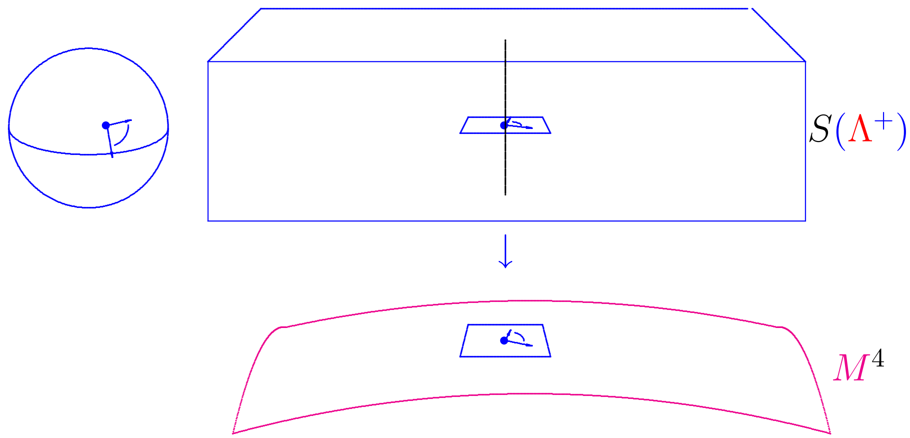
$Z = S(\Lambda^+)$, $J : TZ \to TZ$, $J^2 = -1$:



Twistor picture of anti-self-duality condition:

Oriented $(M^4, g) \leftrightsquigarrow (Z, J)$.

$Z = S(\Lambda^+)$, $J : TZ \to TZ$, $J^2 = -1$:



Theorem (Atiyah-Hitchin-Singer). (Z, J) *is a complex 3-manifold iff* $W_+ = 0$.

ASD condition is conformally invariant.

ASD condition is conformally invariant.

But most ASD conformal classes on compact M will *not* contain metric with $s \equiv 0$.

ASD condition is conformally invariant.

But most ASD conformal classes on compact M will *not* contain metric with $s \equiv 0$.

Problem $\Longleftrightarrow$ sign of Yamabe constant.

ASD condition is conformally invariant.

But most ASD conformal classes on compact M will *not* contain metric with $s \equiv 0$.

Problem $\Longleftrightarrow$ sign of Yamabe constant.

$\Longleftrightarrow$ sign λ_0 for Yamabe Laplacian

$$(\Delta + s/6)$$

Proposition. *For any integer* $k \geq 5$, *the connected sum*

$$k\overline{\mathbb{CP}}_2 = \underbrace{\overline{\mathbb{CP}}_2 \# \cdots \# \overline{\mathbb{CP}}_2}_{k}$$

admits 1*-parameter family of* ASD *conformal metrics* $[g_t]$, $t \in [-1, 1]$, *such that*

- $\exists g_{-1} \in [g_{-1}]$ *with* $s < 0$; *and*
- $\exists g_1 \in [g_1]$ *with* $s > 0$.

Proposition. *For any integer* $k \geq 5$, *the connected sum*

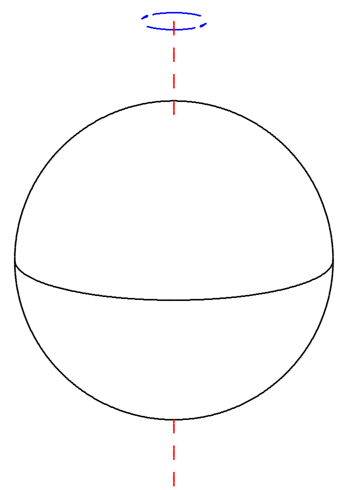
$$k\overline{\mathbb{CP}}_2 = \underbrace{\overline{\mathbb{CP}}_2 \# \cdots \# \overline{\mathbb{CP}}_2}_{k}$$

admits 1*-parameter family of* ASD *conformal metrics* $[g_t]$, $t \in [-1, 1]$, *such that*

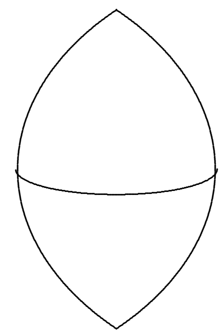
- $\exists g_{-1} \in [g_{-1}]$ *with* $s < 0$; *and*
- $\exists g_1 \in [g_1]$ *with* $s > 0$.

Strategy:

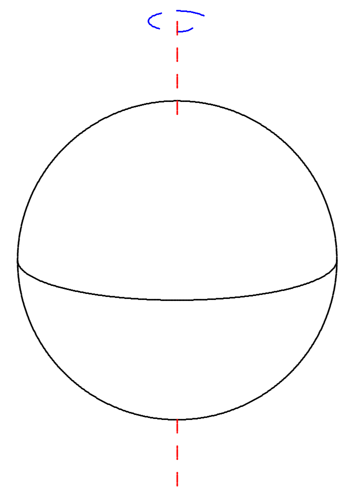
- Find such metrics on related orbifold.
- Then smooth singularities.



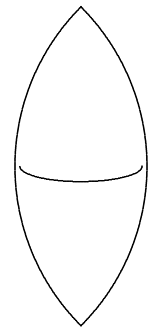
$$\mathbb{Z}_2 \curvearrowright S^4$$



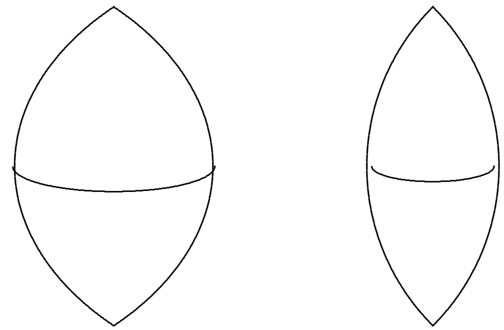
$S^4/\mathbb{Z}_2$



$$\mathbb{Z}_\ell \curvearrowright S^4, \quad \ell \geq 3$$

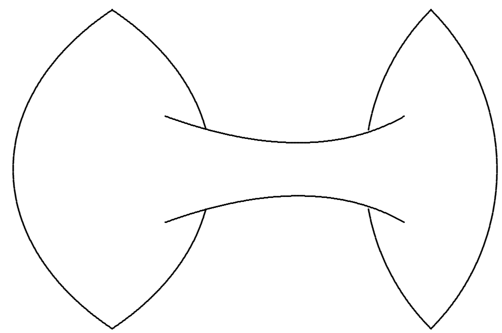


$S^4/\mathbb{Z}_\ell, \quad \ell \geq 3$

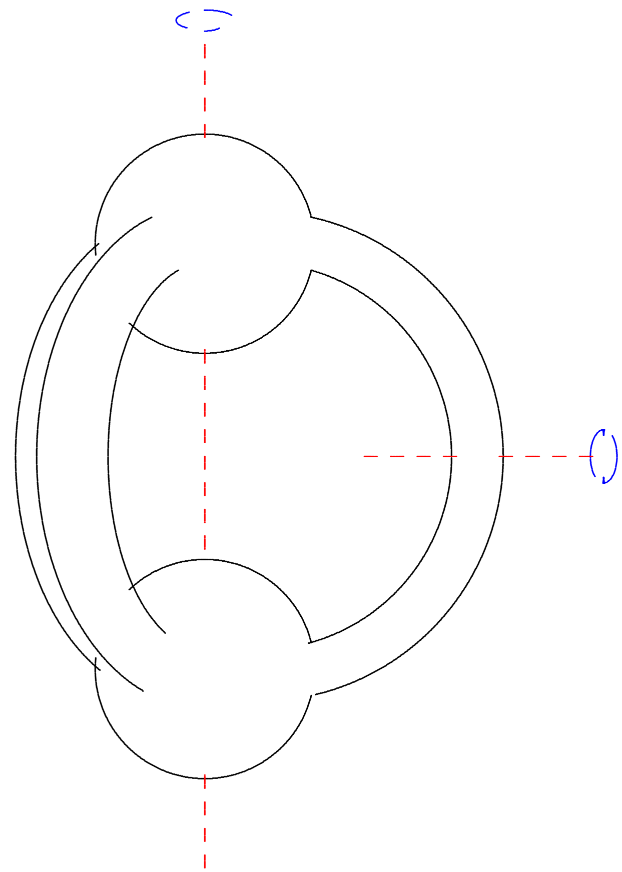


$S^4/\mathbb{Z}_2$ $\quad$ $S^4/\mathbb{Z}_\ell, \quad \ell \geq 3$

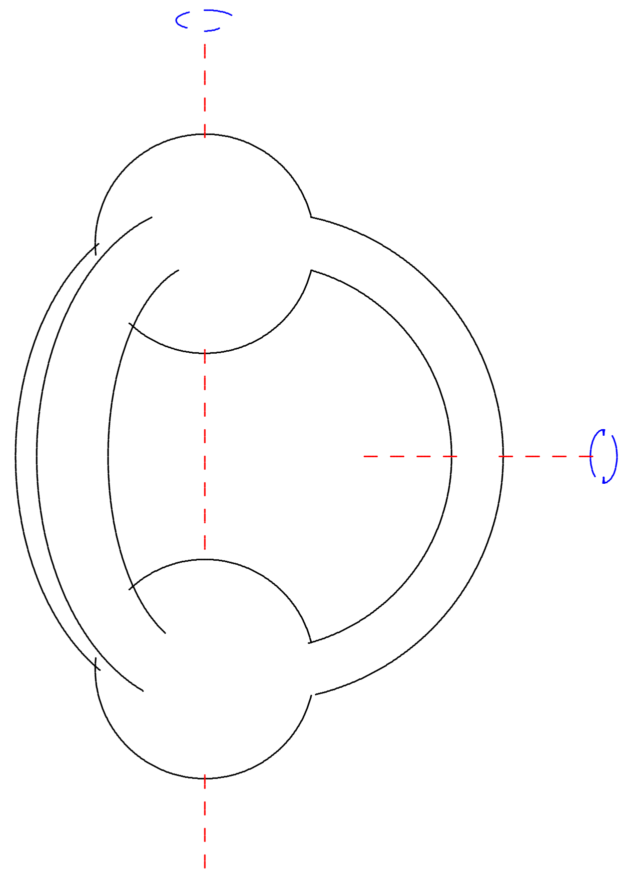
Relevant orbifold:



$$V = (S^4/\mathbb{Z}_2)\#(S^4/\mathbb{Z}_\ell), \quad \ell \geq 3.$$



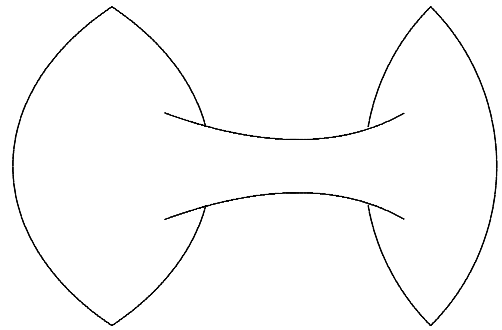
$$V = X/D_\ell$$



$$V = X / D_\ell$$

where

$$X = \underbrace{(S^3 \times S^1) \# \cdots \# (S^3 \times S^1)}_{\ell - 1}$$



Lemma. $\exists$ *smooth family* g_t, $t \in [-1, 1]$, *on* V *s.t.*

- $\forall t$, g_t *conformally flat orbifold metric; and*
- $\forall t$, s_{g_t} *has same sign as* t, *everywhere.*

Lemma (Schoen-Yau, Nayatani). *Let* $(M, [g])$ *be compact, conformally flat* n*-manifold,* $n \geq 3$*, which can be uniformized as*

$$M = \Omega/\Gamma,$$

where $\Gamma \subset SO(n+1, 1)$*,* $\Omega \subset S^n$*.*

Lemma (Schoen-Yau, Nayatani). *Let* $(M, [g])$ *be compact, conformally flat* n*-manifold,* $n \geq 3$*, which can be uniformized as*

$$M = \Omega/\Gamma,$$

where $\Gamma \subset SO(n+1, 1)$, $\Omega \subset S^n$*. Let* $g \in [g]$ *be metric on* M *s.t.* s *has fixed sign.*

Lemma (Schoen-Yau, Nayatani). *Let* $(M, [g])$ *be compact, conformally flat* n*-manifold,* $n \geq 3$*, which can be uniformized as*

$$M = \Omega/\Gamma,$$

where $\Gamma \subset SO(n+1,1)$*,* $\Omega \subset S^n$*. Let* $g \in [g]$ *be metric on* M *s.t.* s *has fixed sign. Assume limit set* Λ *of* Γ *infinite, and let* $\dim(\Lambda)$ *denotes its Hausdorff dimension.*

Lemma (Schoen-Yau, Nayatani). *Let* $(M,[g])$ *be compact, conformally flat* n*-manifold,* $n \geq 3$*, which can be uniformized as*

$$M = \Omega/\Gamma,$$

where $\Gamma \subset SO(n+1,1)$, $\Omega \subset S^n$. *Let* $g \in [g]$ *be metric on* M *s.t.* s *has fixed sign. Assume limit set* Λ *of* Γ *infinite, and let* $\dim(\Lambda)$ *denotes its Hausdorff dimension. Then*

$$s > 0 \iff \dim(\Lambda) < \frac{n}{2} - 1$$

Lemma (Schoen-Yau, Nayatani). *Let* $(M, [g])$ *be compact, conformally flat* n*-manifold,* $n \geq 3$*, which can be uniformized as*

$$M = \Omega/\Gamma,$$

where $\Gamma \subset SO(n+1, 1)$*,* $\Omega \subset S^n$*. Let* $g \in [g]$ *be metric on* M *s.t.* s *has fixed sign. Assume limit set* Λ *of* Γ *infinite, and let* $\dim(\Lambda)$ *denotes its Hausdorff dimension. Then*

$$s > 0 \iff \dim(\Lambda) < \frac{n}{2} - 1$$

$$s = 0 \iff \dim(\Lambda) = \frac{n}{2} - 1$$

Lemma (Schoen-Yau, Nayatani). *Let* $(M, [g])$ *be compact, conformally flat* n*-manifold,* $n \geq 3$*, which can be uniformized as*

$$M = \Omega / \Gamma,$$

where $\Gamma \subset SO(n+1, 1)$*,* $\Omega \subset S^n$*. Let* $g \in [g]$ *be metric on* M *s.t.* s *has fixed sign. Assume limit set* Λ *of* Γ *infinite, and let* $\dim(\Lambda)$ *denotes its Hausdorff dimension. Then*

$$s > 0 \iff \dim(\Lambda) < \frac{n}{2} - 1$$
$$s = 0 \iff \dim(\Lambda) = \frac{n}{2} - 1$$
$$s < 0 \iff \dim(\Lambda) > \frac{n}{2} - 1.$$

Proposition. *For any* $\ell \geq 3$, *there exists* 1-*parameter family of Kleinian groups*

$$\mathbb{Z}_2 * \mathbb{Z}_\ell \hookrightarrow SL(2,\mathbb{C}) \subset GL(2,\mathbb{H})$$

such that $\dim(\Lambda)$ *ranges over* $(0,2)$.

Proposition. *For any* $\ell \geq 3$, *there exists* 1-*parameter family of Kleinian groups*

$$\mathbb{Z}_2 * \mathbb{Z}_\ell \hookrightarrow SL(2,\mathbb{C}) \subset GL(2,\mathbb{H})$$

such that $\dim(\Lambda)$ *ranges over* $(0,2)$.

Fixed points of $\mathbb{Z}_2$ and $\mathbb{Z}_\ell$:

Proposition. *For any* $\ell \geq 3$, *there exists* 1-*parameter family of Kleinian groups*

$$\mathbb{Z}_2 * \mathbb{Z}_\ell \hookrightarrow SL(2, \mathbb{C}) \subset GL(2, \mathbb{H})$$

such that $\dim(\Lambda)$ *ranges over* $(0, 2)$.

Fixed points of $\mathbb{Z}_2$ and $\mathbb{Z}_\ell$:

4-points in $\mathbb{CP}_1$.

Proposition. *For any* $\ell \geq 3$*, there exists* 1-*parameter family of Kleinian groups*

$$\mathbb{Z}_2 * \mathbb{Z}_\ell \hookrightarrow SL(2, \mathbb{C}) \subset GL(2, \mathbb{H})$$

such that $\dim(\Lambda)$ *ranges over* $(0, 2)$*.*

Fixed points of $\mathbb{Z}_2$ and $\mathbb{Z}_\ell$:

4-points in $\mathbb{CP}_1$.

Change cross-ratio!

Proposition. *For any* $\ell \geq 3$, *there exists* 1-*parameter family of Kleinian groups*

$$\mathbb{Z}_2 * \mathbb{Z}_\ell \hookrightarrow SL(2, \mathbb{C}) \subset GL(2, \mathbb{H})$$

such that $\dim(\Lambda)$ *ranges over* $(0, 2)$.

Fixed points of $\mathbb{Z}_2$ and $\mathbb{Z}_\ell$:

4-points in $\mathbb{CP}_1$.

Change cross-ratio!

Construction related to non-classical Schottky groups.

Proposition. *For any* $\ell \geq 3$, *there exists* 1-*parameter family of Kleinian groups*

$$\mathbb{Z}_2 * \mathbb{Z}_\ell \hookrightarrow SL(2, \mathbb{C}) \subset GL(2, \mathbb{H})$$

such that $\dim(\Lambda)$ *ranges over* $(0, 2)$.

Fixed points of $\mathbb{Z}_2$ and $\mathbb{Z}_\ell$:

4-points in $\mathbb{CP}_1$.

Change cross-ratio!

Construction related to non-classical Schottky groups.

When $\dim(\Lambda) < 1$, $s > 0$ on V.

Proposition. *For any* $\ell \geq 3$, *there exists* 1-*parameter family of Kleinian groups*

$$\mathbb{Z}_2 * \mathbb{Z}_\ell \hookrightarrow SL(2, \mathbb{C}) \subset GL(2, \mathbb{H})$$

such that $\dim(\Lambda)$ *ranges over* $(0, 2)$.

Fixed points of $\mathbb{Z}_2$ and $\mathbb{Z}_\ell$:

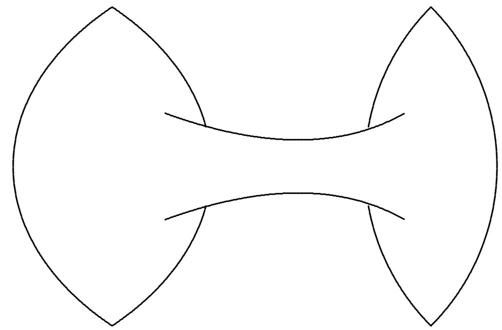
4-points in $\mathbb{CP}_1$.

Change cross-ratio!

Construction related to non-classical Schottky groups.

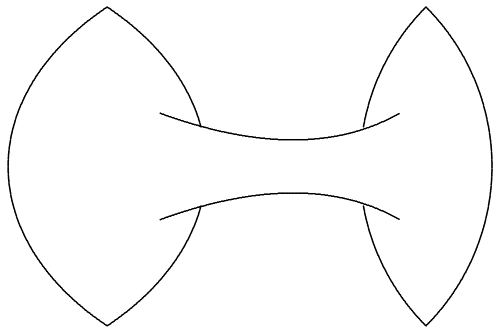
When $\dim(\Lambda) < 1$, $s > 0$ on V.

When $\dim(\Lambda) > 1$, $s < 0$ on V.



Lemma. $\exists$ *smooth family* g_t, $t \in [-1, 1]$, *on* V *s.t.*

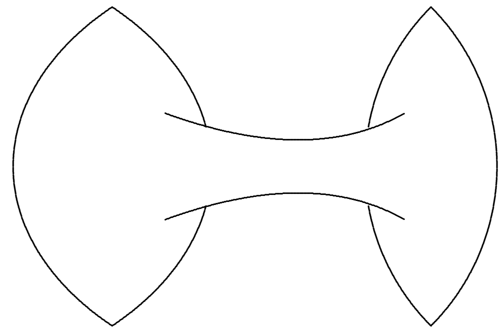
- $\forall t$, g_t *conformally flat orbifold metric; and*
- $\forall t$, s_{g_t} *has same sign as* t, *everywhere.*



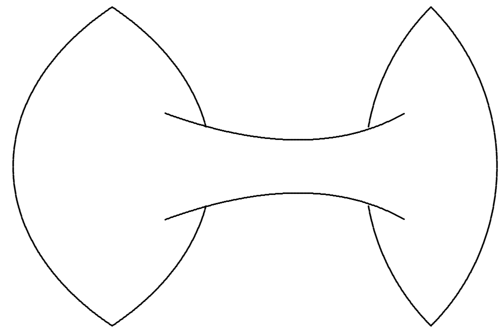
Lemma. $\exists$ *smooth family* g_t, $t \in [-1, 1]$, *on* V *s.t.*

- $\forall t$, g_t *conformally flat orbifold metric; and*
- $\forall t$, s_{g_t} *has same sign as* t, *everywhere.*

Now we take these lemons, and make lemonade!

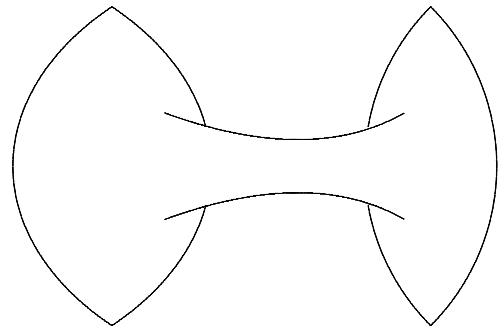


$$V = (S^4/\mathbb{Z}_2)\#(S^4/\mathbb{Z}_\ell), \quad \ell \geq 3.$$



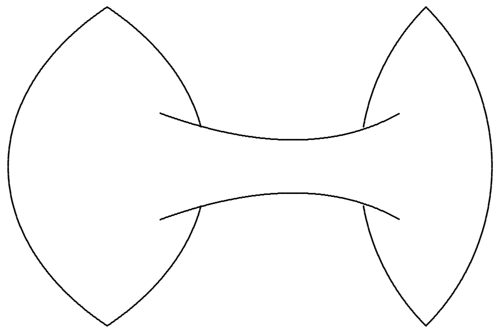
$$V = (S^4/\mathbb{Z}_2)\#(S^4/\mathbb{Z}_\ell), \quad \ell \geq 3.$$

Four orbifold singularities.



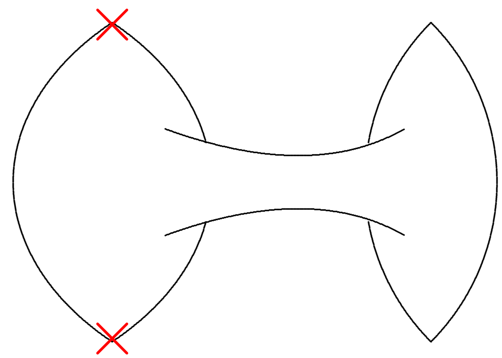
$$V = (S^4/\mathbb{Z}_2)\#(S^4/\mathbb{Z}_\ell), \quad \ell \geq 3.$$

Four orbifold singularities. Models:



$$V = (S^4/\mathbb{Z}_2)\#(S^4/\mathbb{Z}_\ell), \quad \ell \geq 3.$$

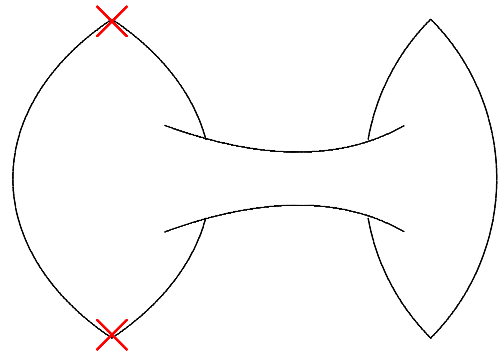
Four orbifold singularities. Models: $\mathbb{C}^2/\Gamma$



$$V = (S^4/\mathbb{Z}_2)\#(S^4/\mathbb{Z}_\ell), \quad \ell \geq 3.$$

Four orbifold singularities. Models: $\mathbb{C}^2/\Gamma$, for

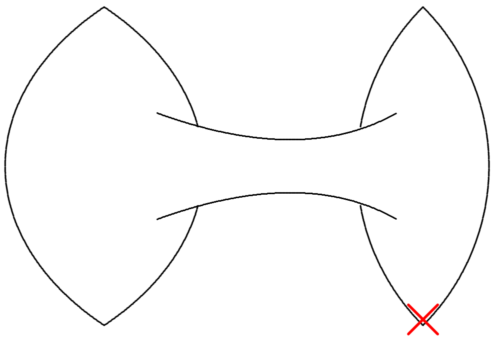
- $\Gamma = \mathbb{Z}_2$: generator $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$



$$V = (S^4/\mathbb{Z}_2)\#(S^4/\mathbb{Z}_\ell), \quad \ell \geq 3.$$

Four orbifold singularities. Models: $\mathbb{C}^2/\Gamma$, for

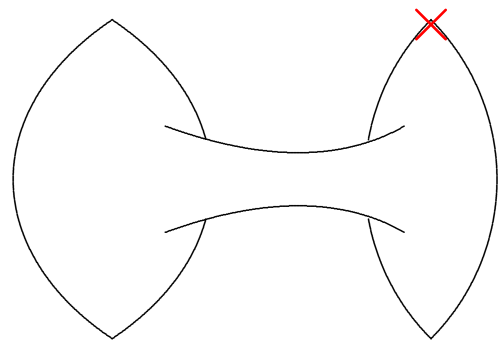
- $\Gamma = \mathbb{Z}_2$: generator $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (two of these)



$$V = (S^4/\mathbb{Z}_2)\#(S^4/\mathbb{Z}_\ell), \quad \ell \geq 3.$$

Four orbifold singularities. Models: $\mathbb{C}^2/\Gamma$, for

- $\Gamma = \mathbb{Z}_2$: generator $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (two of these)
- $\Gamma = \mathbb{Z}_\ell$: generator $\begin{pmatrix} e^{2\pi i/\ell} & 0 \\ 0 & e^{2\pi i/\ell} \end{pmatrix}$



$$V = (S^4/\mathbb{Z}_2)\#(S^4/\mathbb{Z}_\ell), \quad \ell \geq 3.$$

Four orbifold singularities. Models: $\mathbb{C}^2/\Gamma$, for

- $\Gamma = \mathbb{Z}_2$: generator $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (two of these)
- $\Gamma = \mathbb{Z}_\ell$: generator $\begin{pmatrix} e^{2\pi i/\ell} & 0 \\ 0 & e^{2\pi i/\ell} \end{pmatrix}$
- $\Gamma = \mathbb{Z}_\ell$: generator $\begin{pmatrix} e^{2\pi i/\ell} & 0 \\ 0 & e^{-2\pi i/\ell} \end{pmatrix}$

Key objective:

Key objective:

replace orbifold singularities $\rightsquigarrow$ ASD ALE spaces.

Key objective:

replace orbifold singularities $\rightsquigarrow$ ASD ALE spaces.

$$Y - K \approx (\mathbb{C}^2 - B)/\Gamma$$

Key objective:

replace orbifold singularities $\rightsquigarrow$ ASD ALE spaces.

$$Y - K \approx (\mathbb{C}^2 - B)/\Gamma$$

$$g_{jk} = \delta_{jk} + O(\varrho^{-2})$$

Key objective:

replace orbifold singularities $\rightsquigarrow$ ASD ALE spaces.

$$Y - K \approx (\mathbb{C}^2 - B)/\Gamma$$

$$g_{jk} = \delta_{jk} + O(\varrho^{-2})$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \rightsquigarrow \text{ Eguchi-Hanson } [\mathcal{O}(-2) \longrightarrow \mathbb{CP}_1]$$

Key objective:

replace orbifold singularities $\rightsquigarrow$ ASD ALE spaces.

$$Y - K \approx (\mathbb{C}^2 - B)/\Gamma$$

$$g_{jk} = \delta_{jk} + O(\varrho^{-2})$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \rightsquigarrow \text{ Eguchi-Hanson } [\mathcal{O}(-2) \longrightarrow \mathbb{CP}_1]$$

$$\begin{pmatrix} e^{2\pi i/\ell} & 0 \\ 0 & e^{2\pi i/\ell} \end{pmatrix} \rightsquigarrow \text{ LeBrun } [\mathcal{O}(-\ell) \longrightarrow \mathbb{CP}_1]$$

Key objective:

replace orbifold singularities $\rightsquigarrow$ ASD ALE spaces.

$$Y - K \approx (\mathbb{C}^2 - B)/\Gamma$$

$$g_{jk} = \delta_{jk} + O(\varrho^{-2})$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \rightsquigarrow \text{ Eguchi-Hanson } [\mathcal{O}(-2) \to \mathbb{CP}_1]$$

$$\begin{pmatrix} e^{2\pi i/\ell} & 0 \\ 0 & e^{2\pi i/\ell} \end{pmatrix} \rightsquigarrow \text{ LeBrun } [\mathcal{O}(-\ell) \to \mathbb{CP}_1]$$

$$\begin{pmatrix} e^{2\pi i/\ell} & 0 \\ 0 & e^{-2\pi i/\ell} \end{pmatrix} \rightsquigarrow \text{ Gibbons-Hawking } A_{\ell-1}$$

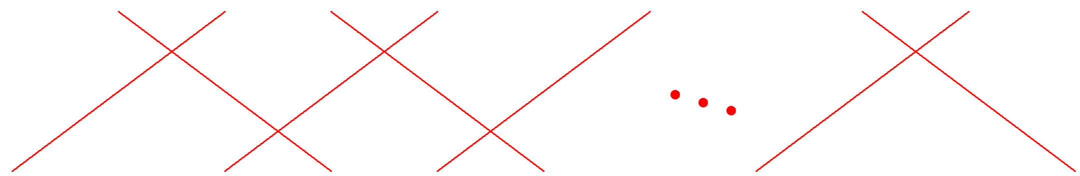
Gibbons-Hawking $A_{\ell-1}$ spaces:

Gibbons-Hawking $A_{\ell-1}$ spaces:

Construct manifold by plumbing together 2-spheres

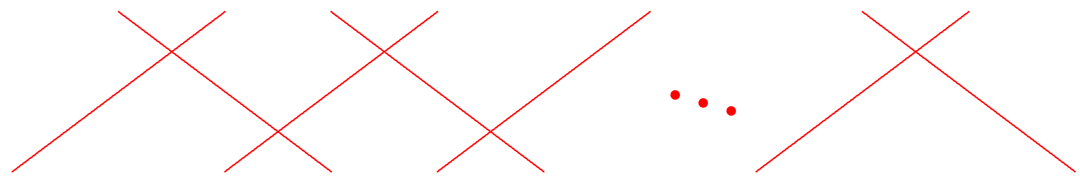
Gibbons-Hawking $A_{\ell-1}$ spaces:

Construct manifold by plumbing together 2-spheres



Gibbons-Hawking $A_{\ell-1}$ spaces:

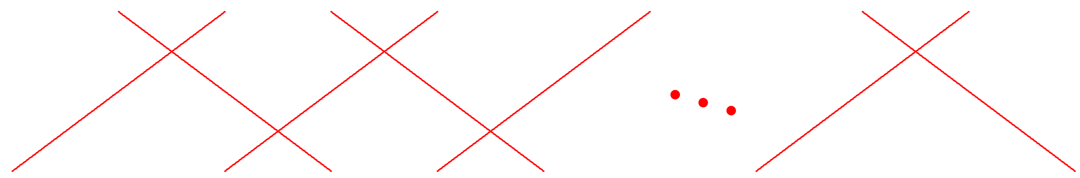
Construct manifold by plumbing together 2-spheres



each with tubular neighborhood diffeo to T^*S^2

Gibbons-Hawking $A_{\ell-1}$ spaces:

Construct manifold by plumbing together 2-spheres

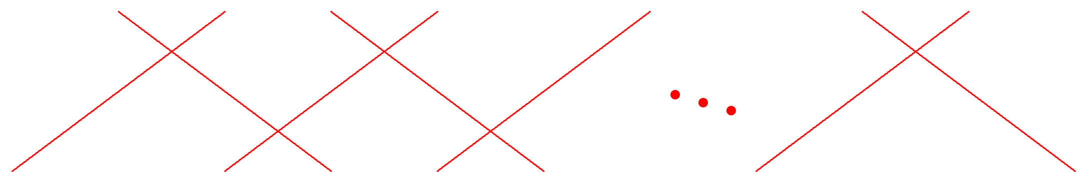


each with tubular neighborhood diffeo to T^*S^2

intersection pattern dual to Dynkin diagram $A_{\ell-1}$

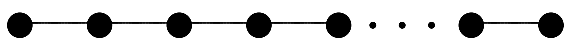
Gibbons-Hawking $A_{\ell-1}$ spaces:

Construct manifold by plumbing together 2-spheres



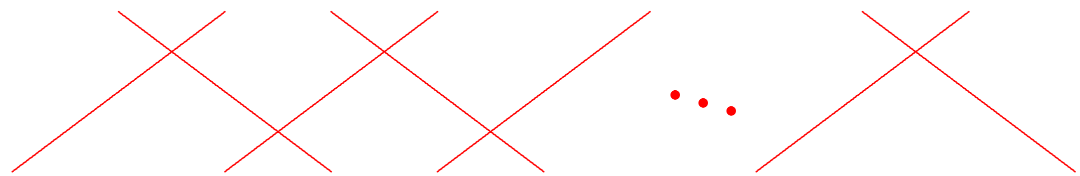
each with tubular neighborhood diffeo to T^*S^2

intersection pattern dual to Dynkin diagram $A_{\ell-1}$



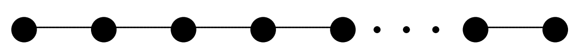
Gibbons-Hawking $A_{\ell-1}$ spaces:

Construct manifold by plumbing together 2-spheres



each with tubular neighborhood diffeo to T^*S^2

intersection pattern dual to Dynkin diagram $A_{\ell-1}$



Minimal resolution of rational double-point

$$xy = z^{\ell}$$

Gibbons-Hawking $A_{\ell-1}$ metrics:

Gibbons-Hawking $A_{\ell-1}$ metrics:

Metric explicit, hyper-Kähler ALE:

Gibbons-Hawking $A_{\ell-1}$ metrics:

Metric explicit, hyper-Kähler ALE:

$$g_{GH} = U(dx^2 + dy^2 + dz^2) + U^{-1}(dt + \theta)^2$$
$$\Delta U = 0, \quad d \star^{(3)} \theta = dU$$

Gibbons-Hawking $A_{\ell-1}$ metrics:

Metric explicit, hyper-Kähler ALE:

$$g_{GH} = U(dx^2 + dy^2 + dz^2) + U^{-1}(dt + \theta)^2$$
$$\Delta U = 0, \quad d \star^{(3)} \theta = dU$$

Data: configuration of ℓ points $p_1, \ldots, p_\ell \in \mathbb{R}^3$.

Gibbons-Hawking $A_{\ell-1}$ metrics:

Metric explicit, hyper-Kähler ALE:

$$g_{GH} = U(dx^2 + dy^2 + dz^2) + U^{-1}(dt + \theta)^2$$
$$\Delta U = 0, \quad d \star^{(3)} \theta = dU$$

Data: configuration of ℓ points $p_1, \ldots, p_\ell \in \mathbb{R}^3$.

$$U = \frac{1}{2} \sum_{j=1}^{\ell} \frac{1}{\rho_j}$$

Gibbons-Hawking $A_{\ell-1}$ metrics:

Metric explicit, hyper-Kähler ALE:

$$g_{GH} = U(dx^2 + dy^2 + dz^2) + U^{-1}(dt + \theta)^2$$
$$\Delta U = 0, \quad d \star^{(3)} \theta = dU$$

Data: configuration of ℓ points $p_1, \ldots, p_\ell \in \mathbb{R}^3$.

$$U = \frac{1}{2} \sum_{j=1}^{\ell} \frac{1}{\rho_j}$$

where ρ_j = Euclidean distance to p_j.

Line Bundle Metrics:

Line Bundle Metrics:

Defined on total space of $\mathcal{O}(-\ell) \longrightarrow \mathbb{CP}_1$

Line Bundle Metrics:

Defined on total space of $\mathcal{O}(-\ell) \to \mathbb{CP}_1$

Metric scalar-flat

Line Bundle Metrics:

Defined on total space of $\mathcal{O}(-\ell) \to \mathbb{CP}_1$

Metric scalar-flat Kähler

Line Bundle Metrics:

Defined on total space of $\mathcal{O}(-\ell) \to \mathbb{CP}_1$

Metric scalar-flat Kähler ALE

Line Bundle Metrics:

Defined on total space of $\mathcal{O}(-\ell) \to \mathbb{CP}_1$

Metric scalar-flat Kähler ALE, cohomogeneity one.

Line Bundle Metrics:

Defined on total space of $\mathcal{O}(-\ell) \to \mathbb{CP}_1$

Metric scalar-flat Kähler ALE, cohomogeneity one.

On complement of zero section, explicitly given

Line Bundle Metrics:

Defined on total space of $\mathcal{O}(-\ell) \to \mathbb{CP}_1$

Metric scalar-flat Kähler ALE, cohomogeneity one.

On complement of zero section, explicitly given by

$$g_{LB} = \frac{d\varrho^2}{F_\ell(\varrho)} + \varrho^2 \left(\theta_1^2 + \theta_2^2 + F_\ell(\varrho)\theta_3^2 \right)$$

Line Bundle Metrics:

Defined on total space of $\mathcal{O}(-\ell) \to \mathbb{CP}_1$

Metric scalar-flat Kähler ALE, cohomogeneity one.

On complement of zero section, explicitly given by

$$g_{LB} = \frac{d\varrho^2}{F_\ell(\varrho)} + \varrho^2 \left(\theta_1^2 + \theta_2^2 + F_\ell(\varrho)\theta_3^2 \right)$$

for $\varrho > 1$

Line Bundle Metrics:

Defined on total space of $\mathcal{O}(-\ell) \to \mathbb{CP}_1$

Metric scalar-flat Kähler ALE, cohomogeneity one.

On complement of zero section, explicitly given by

$$g_{LB} = \frac{d\varrho^2}{F_\ell(\varrho)} + \varrho^2 \left(\theta_1^2 + \theta_2^2 + F_\ell(\varrho)\theta_3^2\right)$$

for $\varrho > 1$, where

$$F_\ell(\varrho) = 1 + \frac{\ell - 2}{\varrho^2} - \frac{\ell - 1}{\varrho^4}$$

Line Bundle Metrics:

Defined on total space of $\mathcal{O}(-\ell) \to \mathbb{CP}_1$

Metric scalar-flat Kähler ALE, cohomogeneity one.

On complement of zero section, explicitly given by

$$g_{LB} = \frac{d\varrho^2}{F_\ell(\varrho)} + \varrho^2 \left(\theta_1^2 + \theta_2^2 + F_\ell(\varrho)\theta_3^2 \right)$$

for $\varrho > 1$, where

$$F_\ell(\varrho) = 1 + \frac{\ell - 2}{\varrho^2} - \frac{\ell - 1}{\varrho^4}$$

$\{\theta_j \mid j = 1, 2, 3\}$ left-invariant co-frame on S^3.

Eguchi-Hanson Metric:

Eguchi-Hanson Metric:

Intersection of these two constructions:

Eguchi-Hanson Metric:

Intersection of these two constructions:

Exactly the $\ell = 2$ case of either GH or LB.

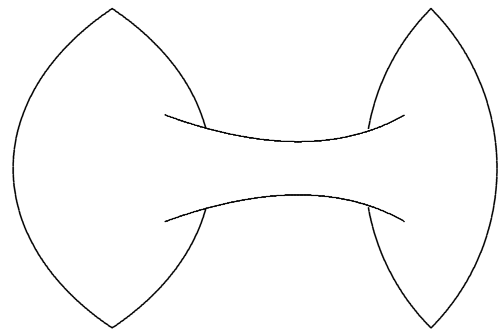
Eguchi-Hanson Metric:

Intersection of these two constructions:

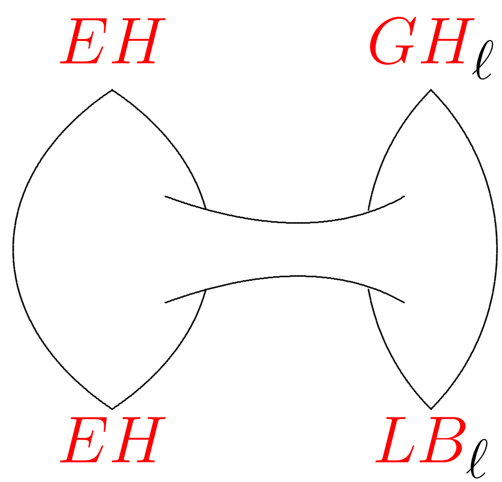
Exactly the $\ell = 2$ case of either GH or LB.

Lives on T^*S^2.

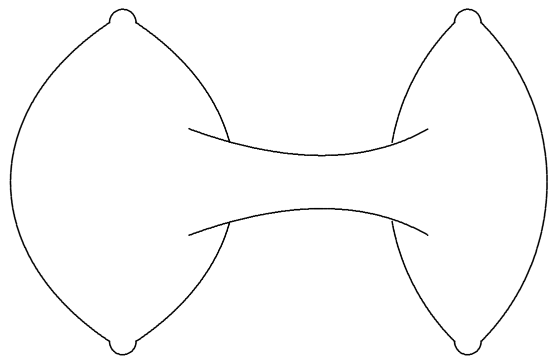
ASD gluing:



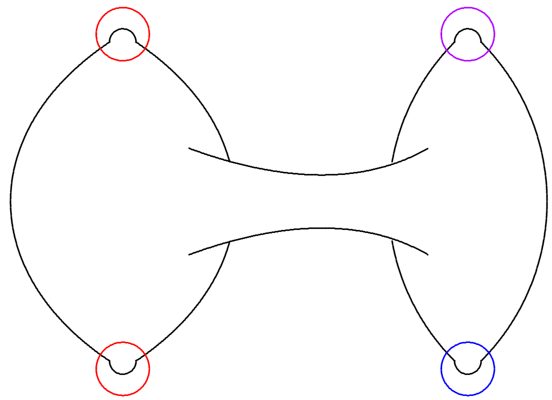
ASD gluing:



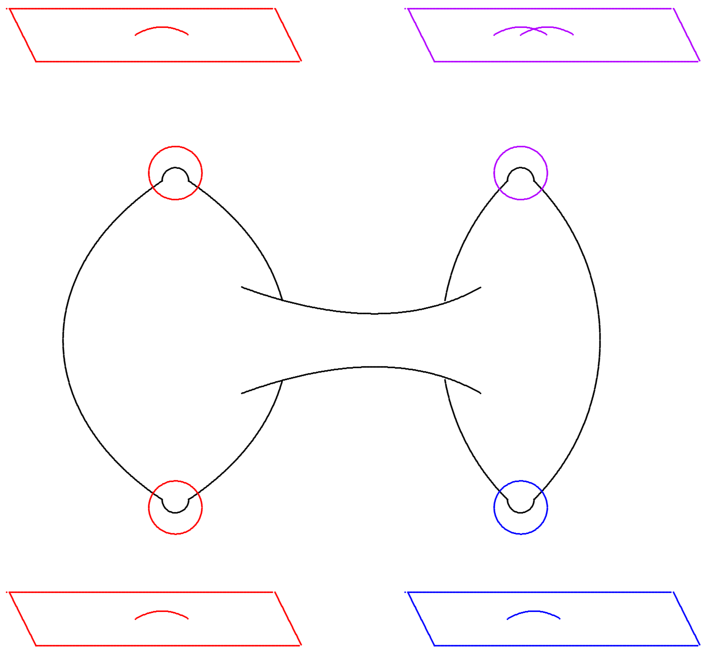
ASD gluing:



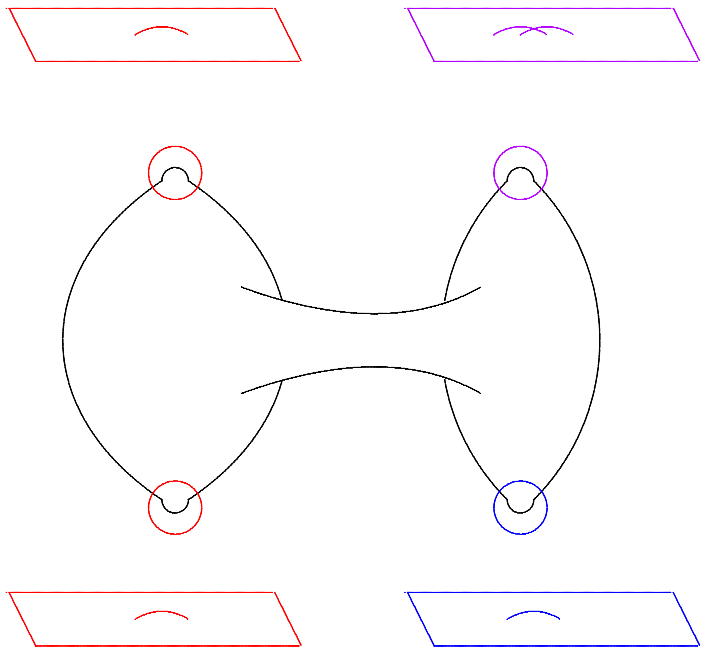
ASD gluing:



ASD gluing:



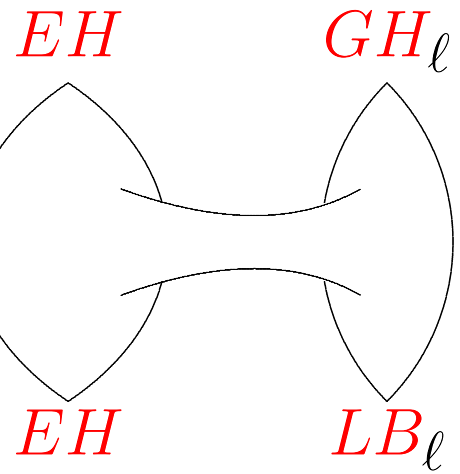
ASD gluing:



Diffeomorphic to $(\ell + 2)\overline{\mathbb{CP}}_2, \ \ \ell \geq 3$

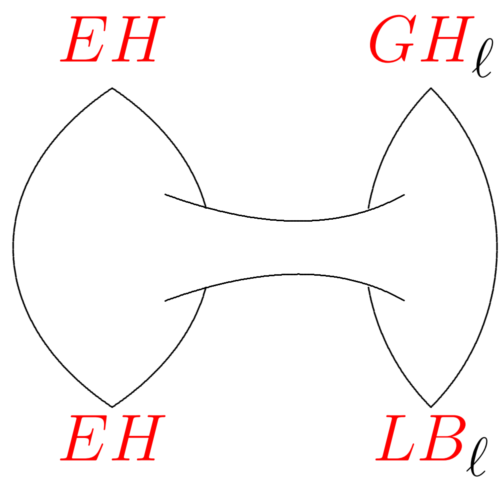
ASD gluing:

Gluing theory: Floer, Taubes, *et. al.*



ASD gluing:

Gluing theory: Floer, Taubes, *et. al.*

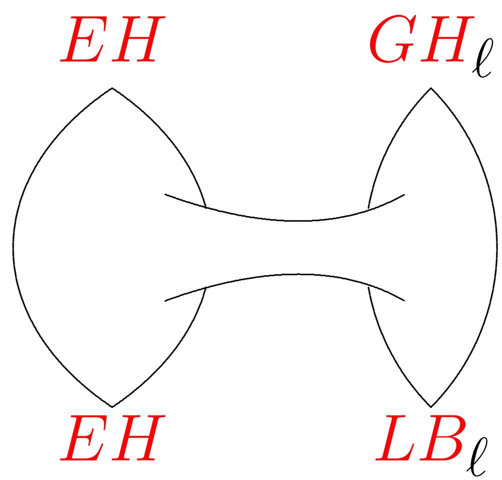


Obstruction: Surjectivity of

$$DW_+ : C^\infty(\odot^2_0\Lambda^1) \to C^\infty(\odot^2_0\Lambda^+)$$

ASD gluing:

Gluing theory: Floer, Taubes, *et. al.*



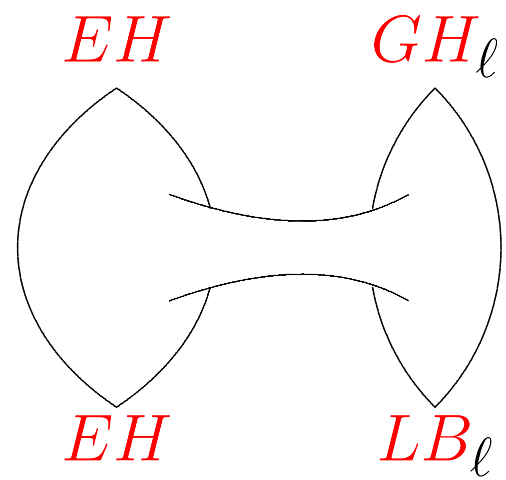
Obstruction: Surjectivity of

$$DW_+ : C^\infty(\odot_0^2\Lambda^1) \to C^\infty(\odot_0^2\Lambda^+)$$

on V & orbifold compactifications of ALE spaces.

ASD gluing:

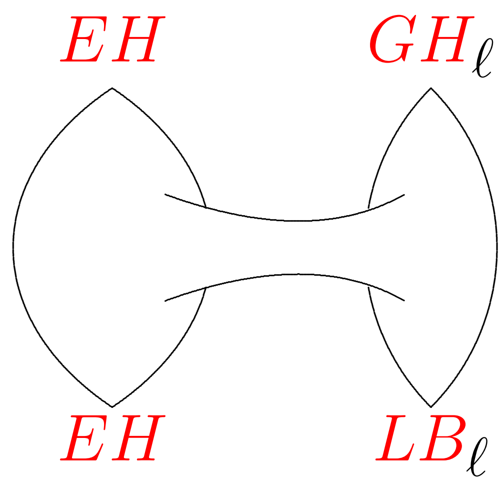
Gluing theory: Floer, Taubes, *et. al.*



Lemma. $\ker DW_+{}^* = 0$ *for* V *& for orbifold compactifications of relevant ALE spaces.*

ASD gluing:

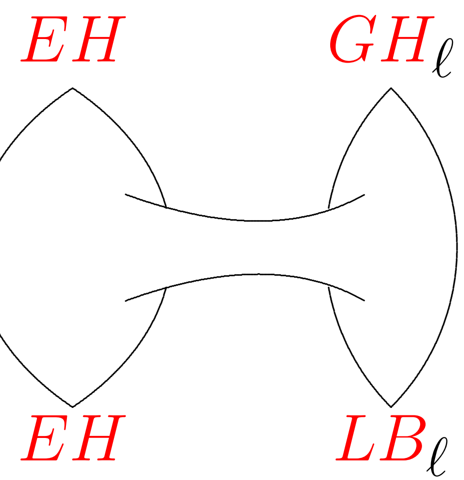
Gluing theory: Floer, Taubes, *et. al.*



Then find metrics with s of desired sign in constructed conformal classes.

ASD gluing:

Gluing theory: Floer, Taubes, *et. al.*

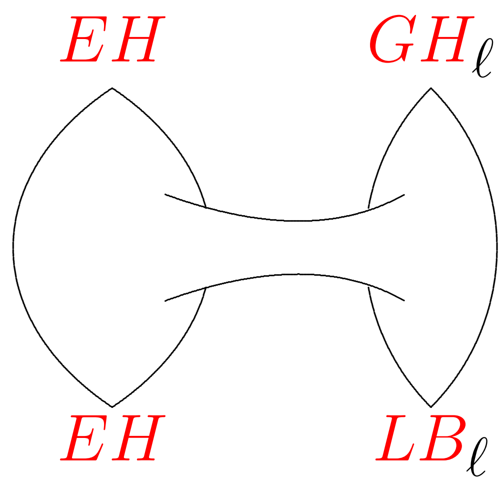


Then find metrics with s of desired sign in constructed conformal classes.

$s < 0$ very easy.

ASD gluing:

Gluing theory: Floer, Taubes, *et. al.*



Then find metrics with s of desired sign in constructed conformal classes.

$s < 0$ very easy.

$s > 0$ much more delicate.

Theorem A. *Simply connected smooth compact M^4 actually admits a scalar-flat anti-self-dual metric if*

- M *is diffeomorphic to* $k\overline{\mathbb{CP}}_2$, $k \geq 5$; *or*
- M *is diffeomorphic to* $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $k \geq 10$; *or*
- M *is diffeomorphic to* $K3$.

Theorem A also tells us that

$$\mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2$$

admits optimal metrics if $k \geq 10$.

Theorem A also tells us that

$$\mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2$$

admits optimal metrics if $k \geq 10$.

What about $k \leq 9$?

Theorem A also tells us that

$$\mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2$$

admits optimal metrics if $k \geq 10$.

What about $k \leq 9$?

Theorem. *For all* $k \leq 8$, $\mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2$ *admits an Einstein metric. Such a metric is optimal.*

Theorem A also tells us that

$$\mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2$$

admits optimal metrics if $k \geq 10$.

What about $k \leq 9$?

Theorem. *For all* $k \leq 8$, $\mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2$ *admits an Einstein metric. Such a metric is optimal.*

Pieces of proof:

Page (1979)

Tian-Yau (1987)

Chen-LeBrun-Weber (2007)

Theorem A also tells us that

$$\mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2$$

admits optimal metrics if $k \geq 10$.

What about $k \leq 9$?

Theorem. *For all* $k \leq 8$, $\mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2$ *admits an Einstein metric. Such a metric is optimal.*

Pieces of proof:

Page (1979)

Tian-Yau (1987)

Chen-LeBrun-Weber (2007)

Conformal geometry crucial to last case!

Theorem A also tells us that

$$\mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2$$

admits optimal metrics if $k \geq 10$.

However. . .

Theorem A also tells us that

$$\mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2$$

admits optimal metrics if $k \geq 10$.

However. . .

Proposition. *There is no optimal metric on* $\mathbb{CP}_2 \# 9 \overline{\mathbb{CP}}_2$.

Similarly:

Theorem. *If* $j \geq 2$ *and* $k \geq 9j$, *the smooth simply connected* 4-*manifold* $j\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$ *does not admit optimal metrics.*

Similarly:

Theorem. *If $j \geq 2$ and $k \geq 9j$, the smooth simply connected 4-manifold $j\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$ does not admit optimal metrics.*

Key is to produce a sequence of metrics g_j on smooth compact oriented M^4 for which

$$\int s^2 d\mu \to 0 \quad \text{and} \quad \int |W_+|^2 d\mu \to 0.$$

$$\mathcal{K}(g) = -8\pi^2(\chi + 3\tau)(M) + 2\int_M \left(\frac{s^2}{24} + 2|W_+|^2 \right) d\mu_g$$

Moral:

4-manifolds need not carry optimal metrics.

Moral:

4-manifolds need not carry optimal metrics.

Geometrization of 3-manifolds:

Wrong question!

Moral:
4-manifolds need not carry optimal metrics.

Geometrization of 3-manifolds:
Wrong question!

Can 4-manifolds be canonically decomposed into, say,

- optimal and
- collapsed pieces?