Einstein Metrics,

Harmonic Forms, &

Conformally Kähler Geometry

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Stony Brook University

Seminario di Geometria
Terza Università di Roma
23 maggio, 2019

Dedicato

Dedicato al mio
allievo, collega, ed amico

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in occasione del convegno
MAX LX.

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"…the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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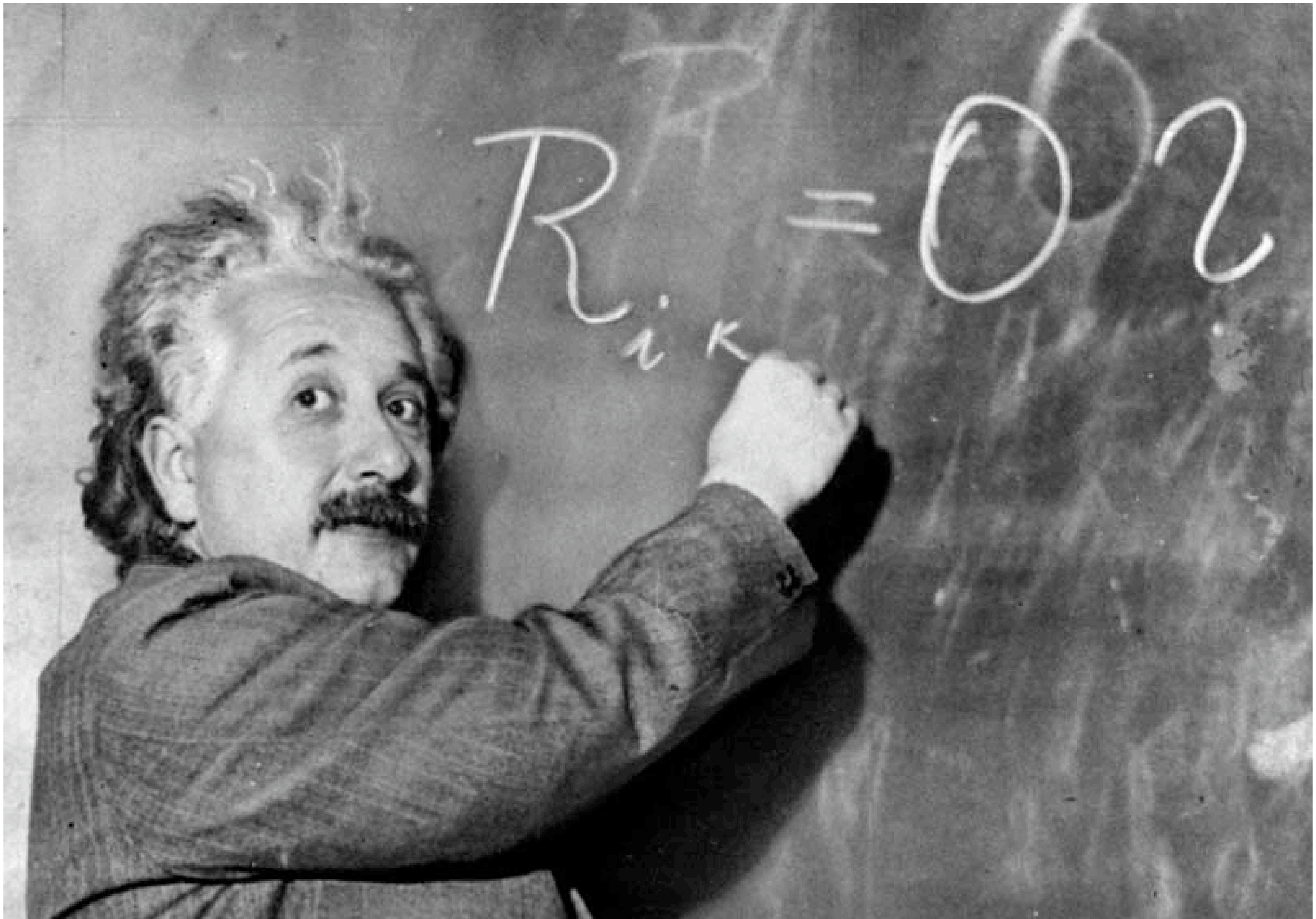

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$$s = r_j^{\ j} = \mathcal{R}^{ij}{}_{ij}.$$

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When $n = 4$, situation is more encouraging...

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One key question:

Does enough rigidity really hold in dimension four to make this a genuine geometrization?

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Some Suggestive Questions. *If (M^4, ω) is a symplectic 4-manifold, when does M^4 admit an Einstein metric h (unrelated to ω)? What if we also require $\lambda \geq 0$?*

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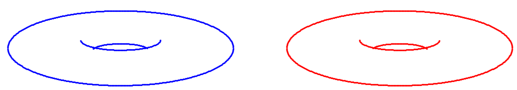

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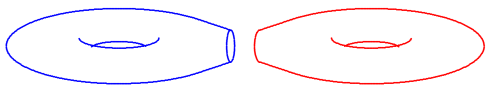

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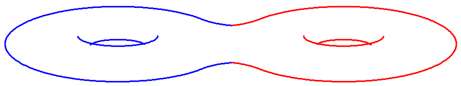

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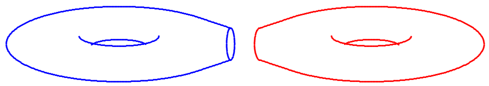

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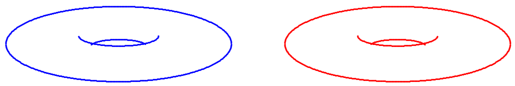

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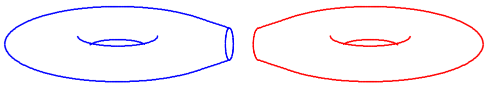

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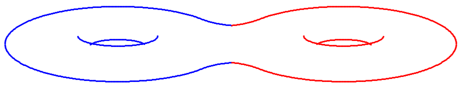

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Calabi/Yau: Admits Ricci-flat Kähler metrics.

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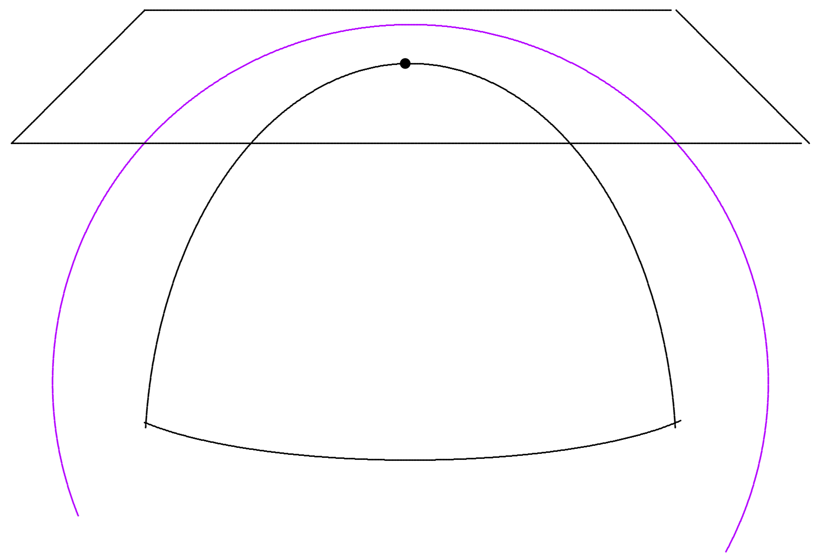

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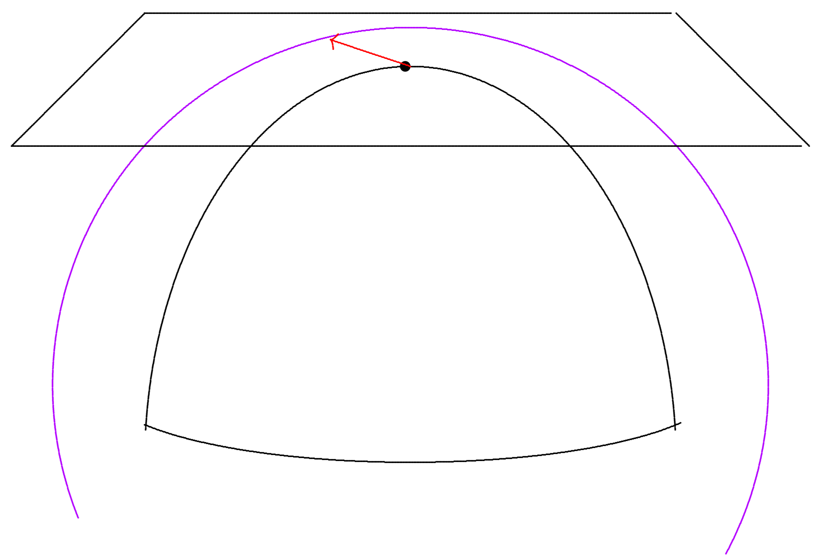

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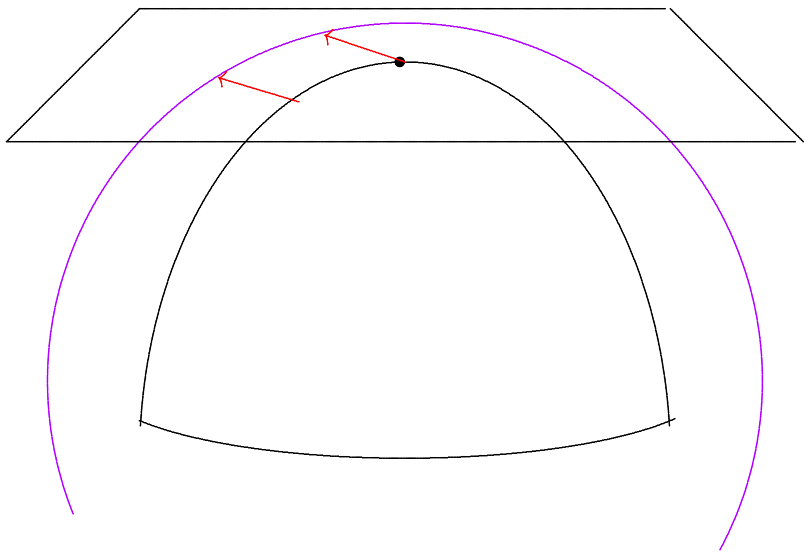

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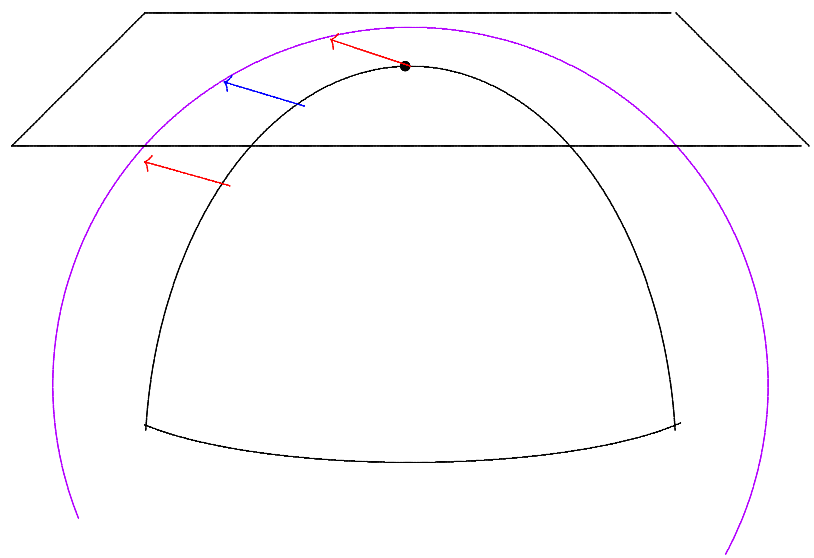

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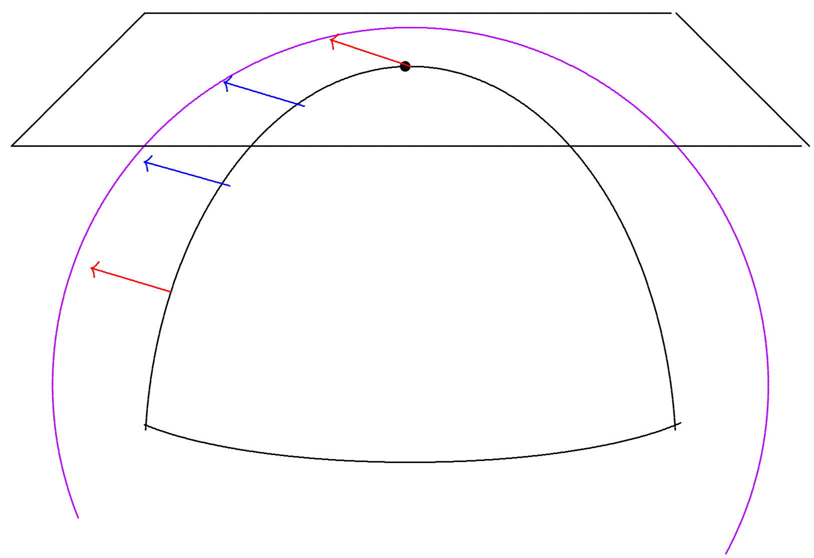

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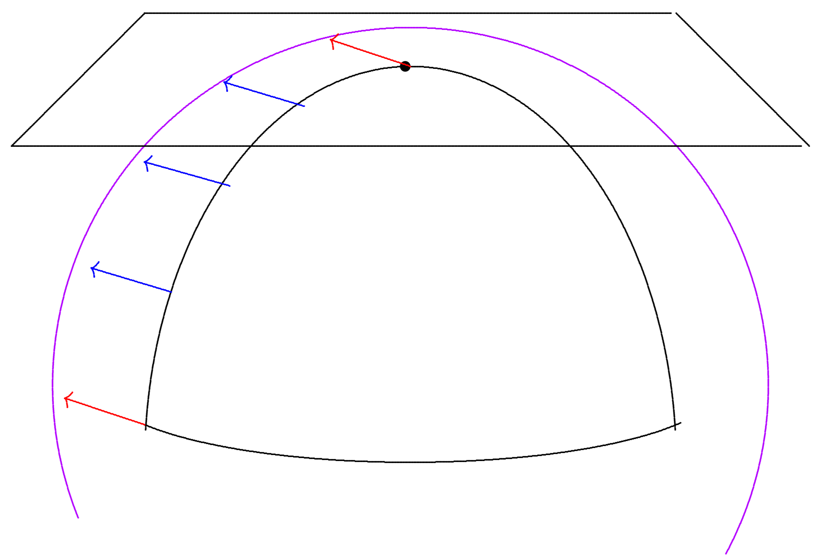

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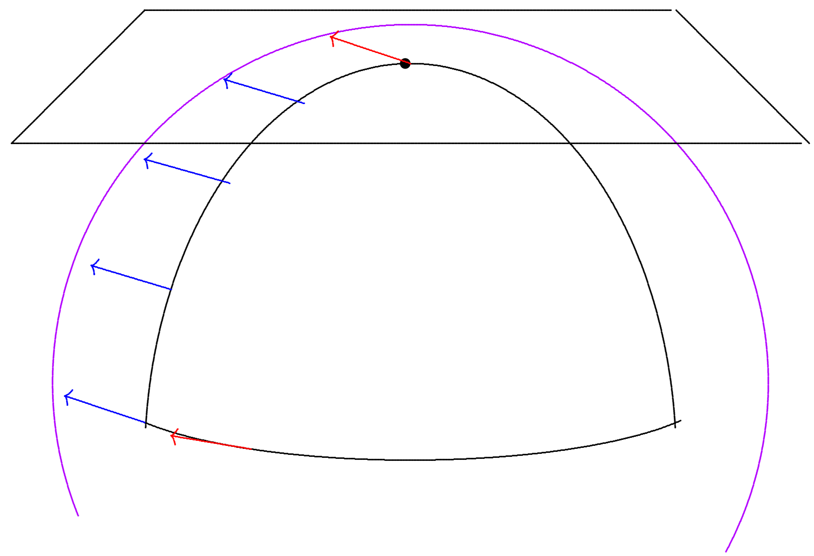

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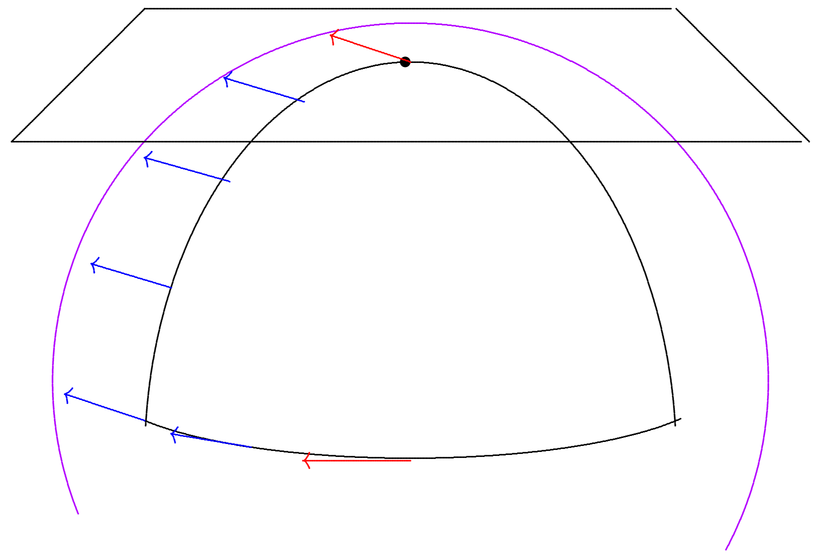

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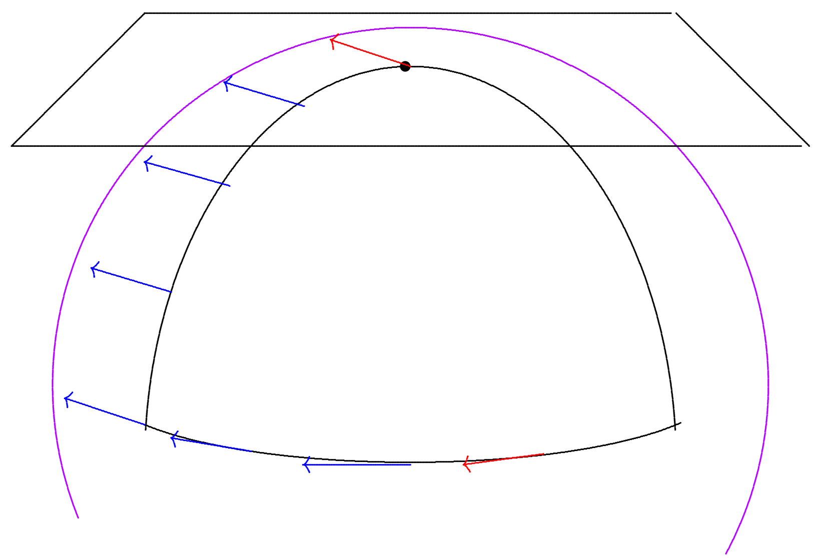

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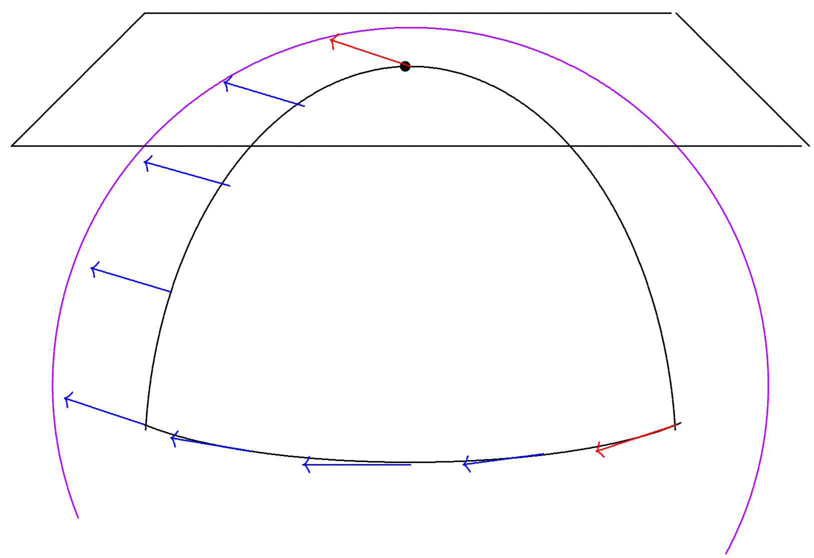

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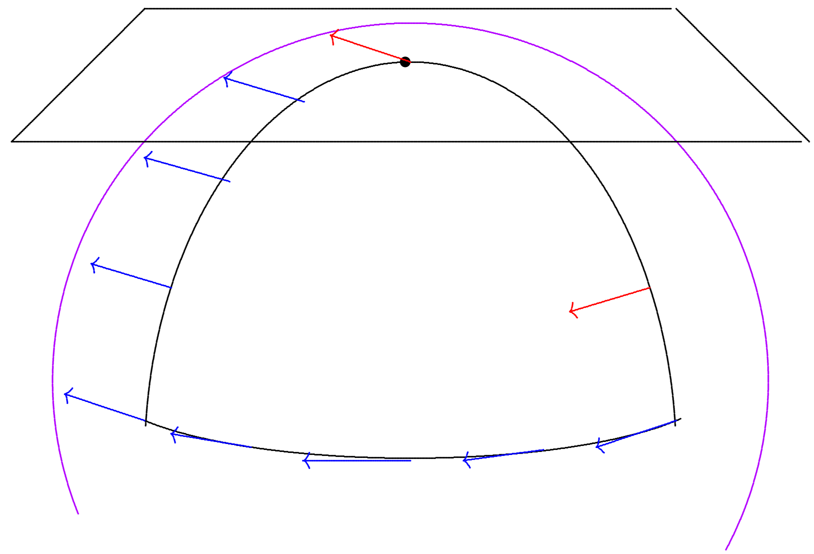

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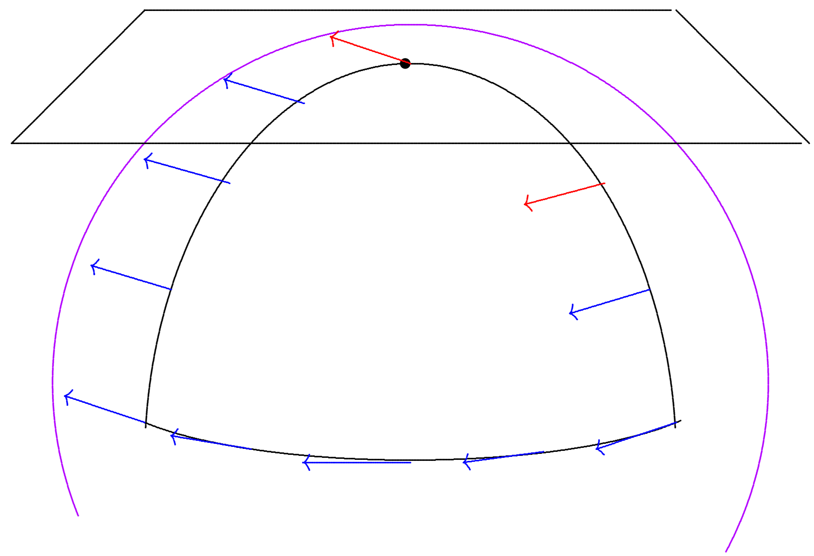

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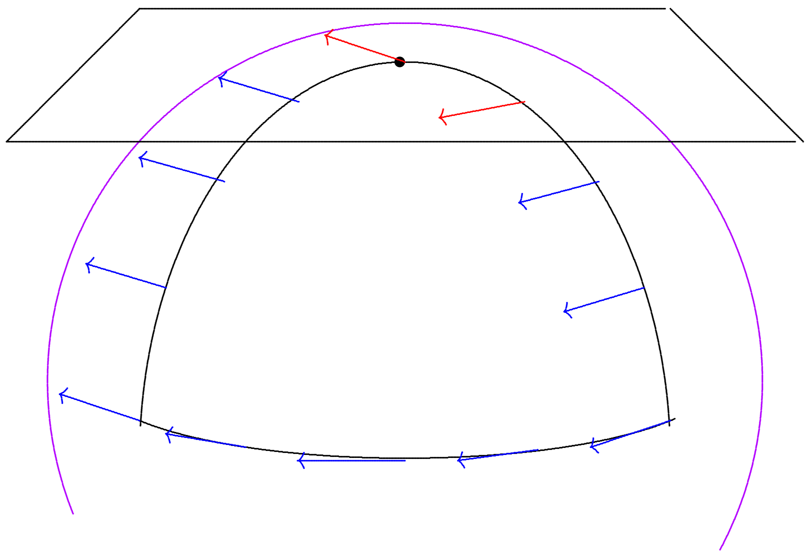

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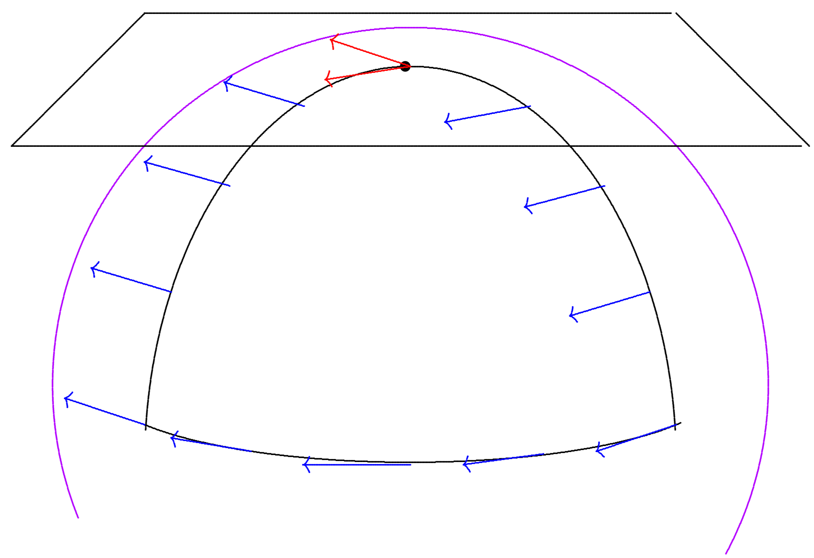

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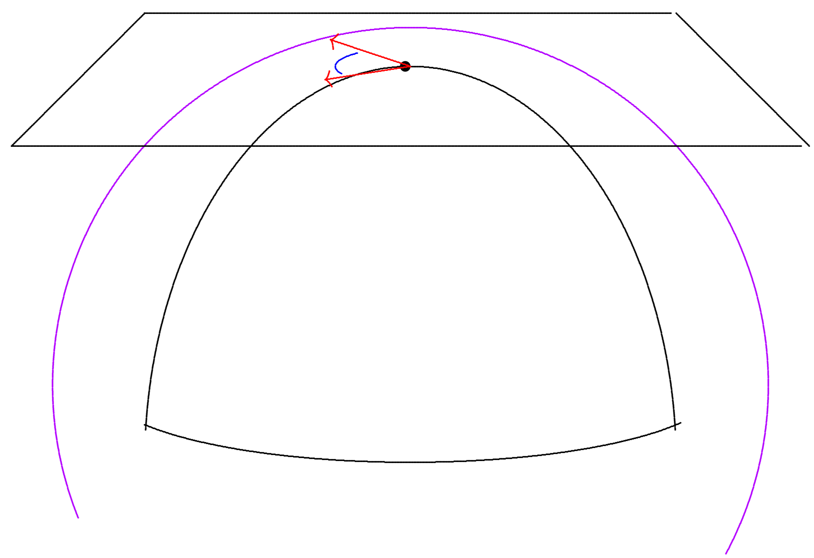
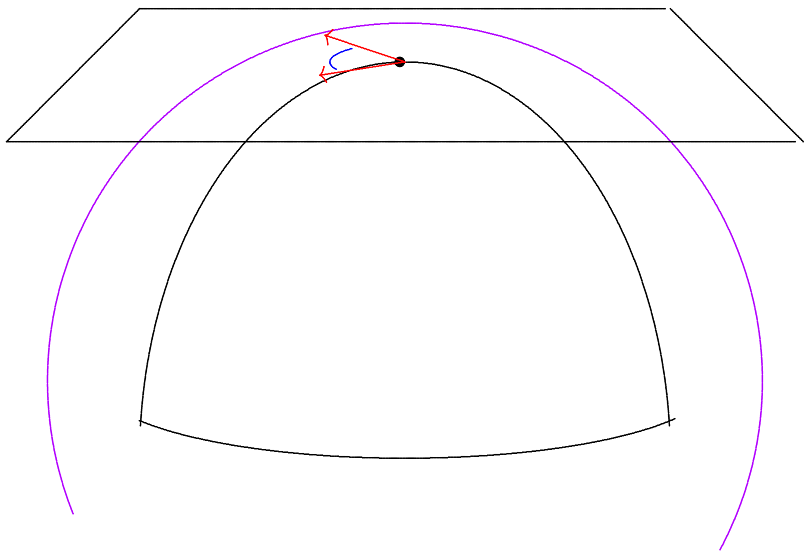
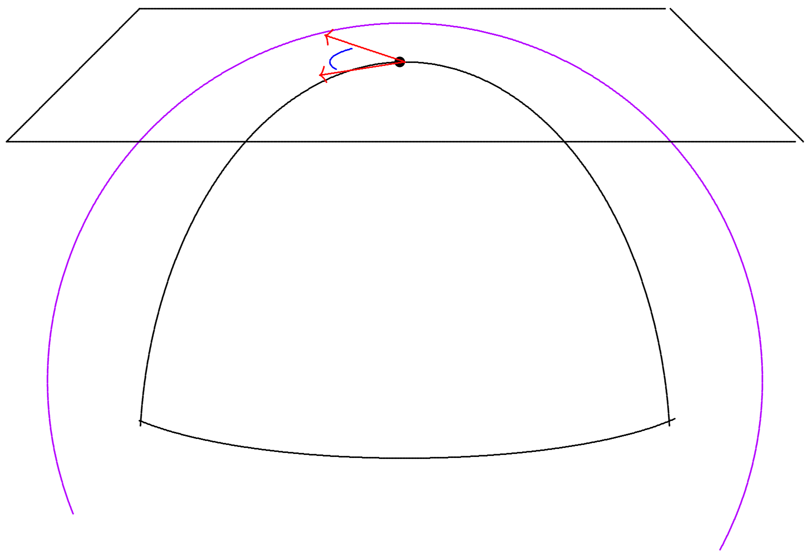


(M^n, g): holonomy $\subset \mathbf{O}(n)$



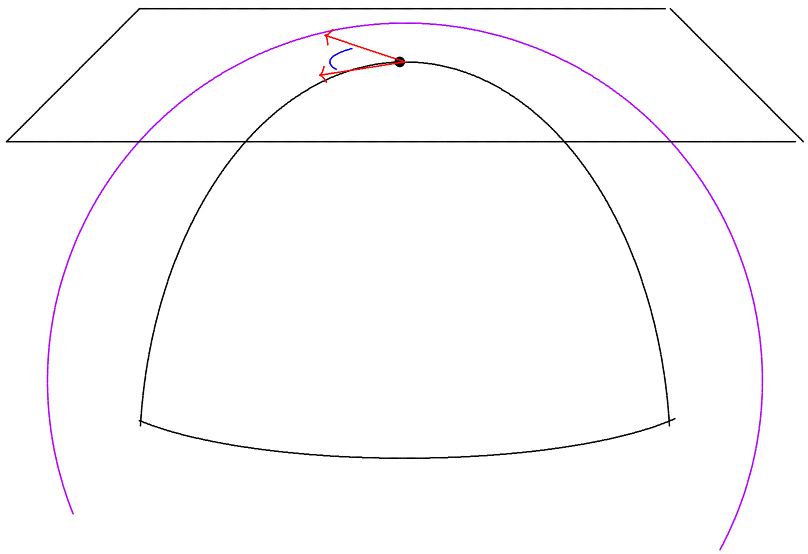
Kähler metrics:

(M^{2m}, g): holonomy



Kähler metrics:

(M^{2m}, g) Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{U}(m)$



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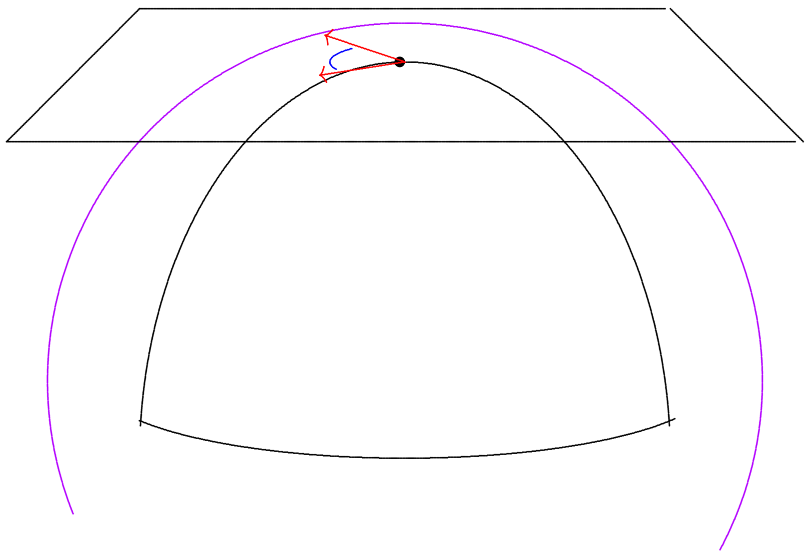

$$\mathbf{U}(m) := \mathbf{O}(2m) \cap \mathbf{GL}(m, \mathbb{C})$$

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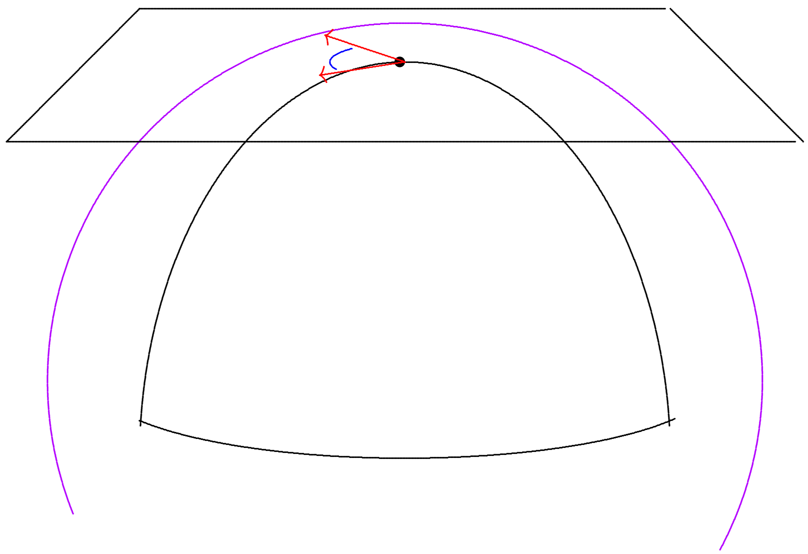

Makes tangent space a complex vector space!

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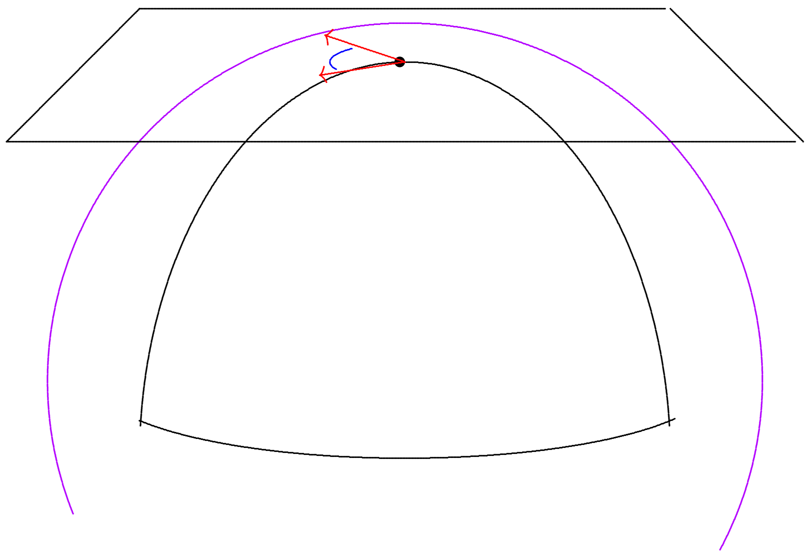

Makes tangent space a complex vector space!

$$J : TM \to TM, \qquad J^2 = -\text{identity}$$

"almost-complex structure"

Kähler metrics:

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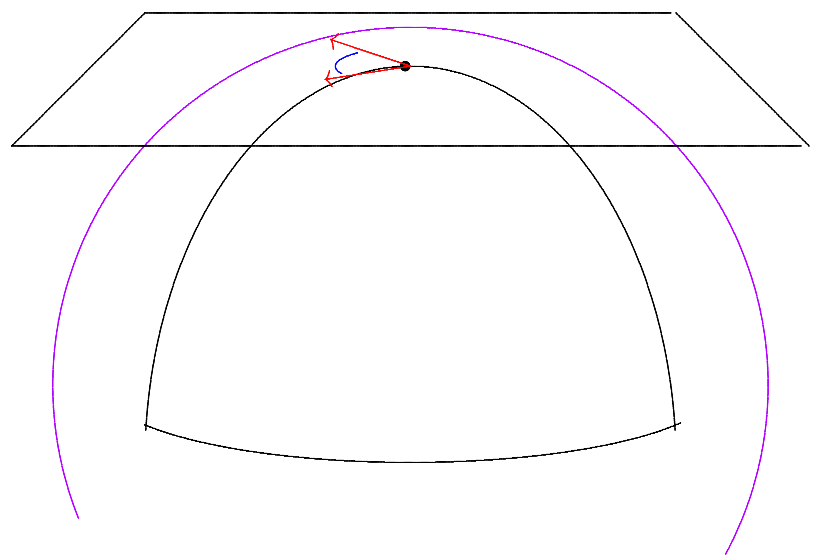

Makes tangent space a complex vector space!

Invariant under parallel transport!

Kähler metrics:

(M^{2m}, g) Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{U}(m)$

$\Longleftrightarrow$ $\exists$ almost complex-structure J with $\nabla J = 0$
and $g(J\cdot, J\cdot) = g$.

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$\Longleftrightarrow$ (M^{2m}, g) is a complex manifold & $\exists$ J-invariant closed 2-form ω such that $g = \omega(\cdot, J\cdot)$.

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ω called "Kähler form."

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$$d\omega = 0$$

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Kähler magic:

If we define the Ricci form by

$$\rho = r(J\cdot, \cdot)$$

then $i\rho$ is curvature of canonical line bundle $\Lambda^{m,0}$.

Kähler metrics:

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Kähler metrics:

(M^{2m}, g) Kähler $\iff$ holonomy $\subset \mathbf{U}(m)$

$\iff \exists$ almost complex-structure J with $\nabla J = 0$ and $g(J\cdot, J\cdot) = g$.

$\iff (M^{2m}, g)$ is a complex manifold & $\exists$ J-invariant closed 2-form ω such that $g = \omega(\cdot, J\cdot)$.

$$[\omega] \in H^2(M)$$

"Kähler class"

Kähler metrics:

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$$d\omega = 0$$

ω non-degenerate closed 2-form:

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$$d\omega = 0$$

ω non-degenerate closed 2-form: symplectic form

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω. Then M also admits an Einstein metric h with $\lambda \geq 0$ if and only if*

$$M \overset{diff}{\approx} \begin{cases} \mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{cases}$$

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Del Pezzo surfaces,
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Del Pezzo surfaces,
K3 surface, Enriques surface,
Abelian surface, Hyper-elliptic surfaces.

$$
\begin{aligned}
&\mathbb{CP}_2 \# k\overline{\mathbb{CP}_2}, \quad 0 \le k \le 8,\\
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&K3,\\
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\end{aligned}
$$

Definitive list . . .

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But we understand some cases better than others!

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Below the line:

Every Einstein metric is Ricci-flat Kähler.

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Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathscr{E}(M)$

But we understand some cases better than others!

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Below the line:

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Moduli space $\mathscr{E}(M) = \{\text{Einstein } h\}/(\text{Diffeos} \times \mathbb{R}^+)$

But we understand some cases better than others!

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Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathscr{E}(M)$ completely understood.

But we understand some cases better than others!

$$
\begin{array}{l}
\mathbb{CP}_2 \# k \overline{\mathbb{CP}_2}, \quad 0 \leq k \leq 8, \\
S^2 \times S^2, \\
\hline
K3, \\
K3/\mathbb{Z}_2, \\
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Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathscr{E}(M)$ connected!

Above the line:

Know an Einstein metric on each manifold.

$$\begin{array}{l} \mathbb{CP}_2 \# k \overline{\mathbb{CP}_2}, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \\ \hline K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{array}$$

Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathscr{E}(M)$ connected!

Above the line:

Moduli space $\mathscr{E}(M) \neq \varnothing$.

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Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathscr{E}(M)$ connected!

Above the line:

Moduli space $\mathscr{E}(M) \neq \varnothing$. But is it connected?

$$\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8,$$
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Moduli space $\mathscr{E}(M)$ connected!

Del Pezzo surfaces:

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(M^4, J) for which c_1 is a Kähler class $[\omega]$.

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Blow-up of $\mathbb{CP}_2$ at k distinct points,
in general position,

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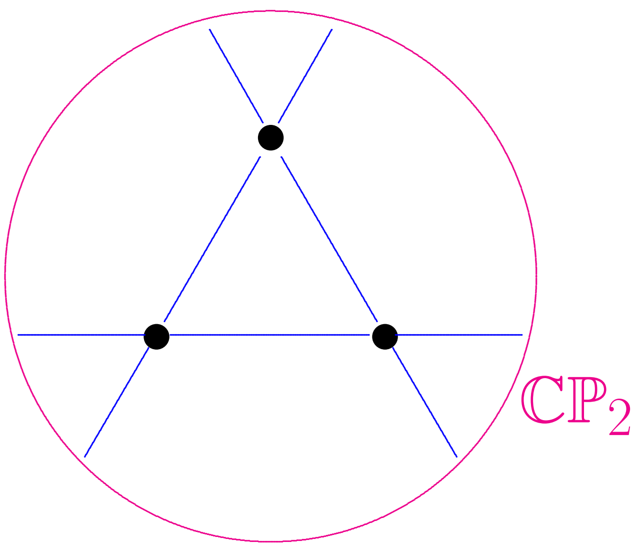

Blowing up:

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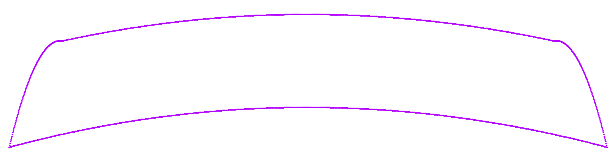
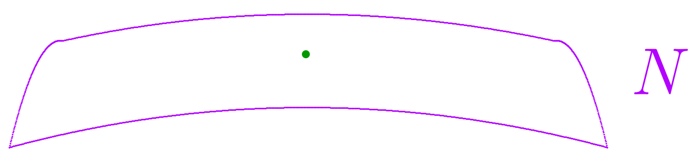
If N is a complex surface,



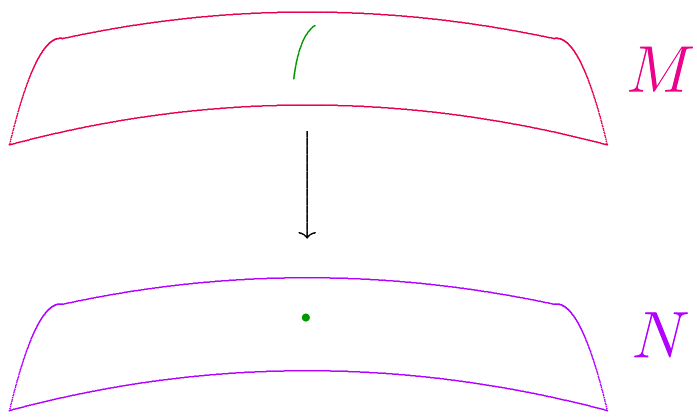
Blowing up:

If N is a complex surface, may replace $p \in N$



Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$



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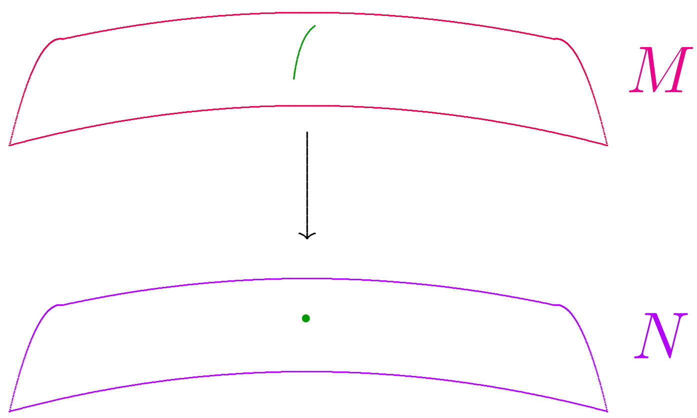

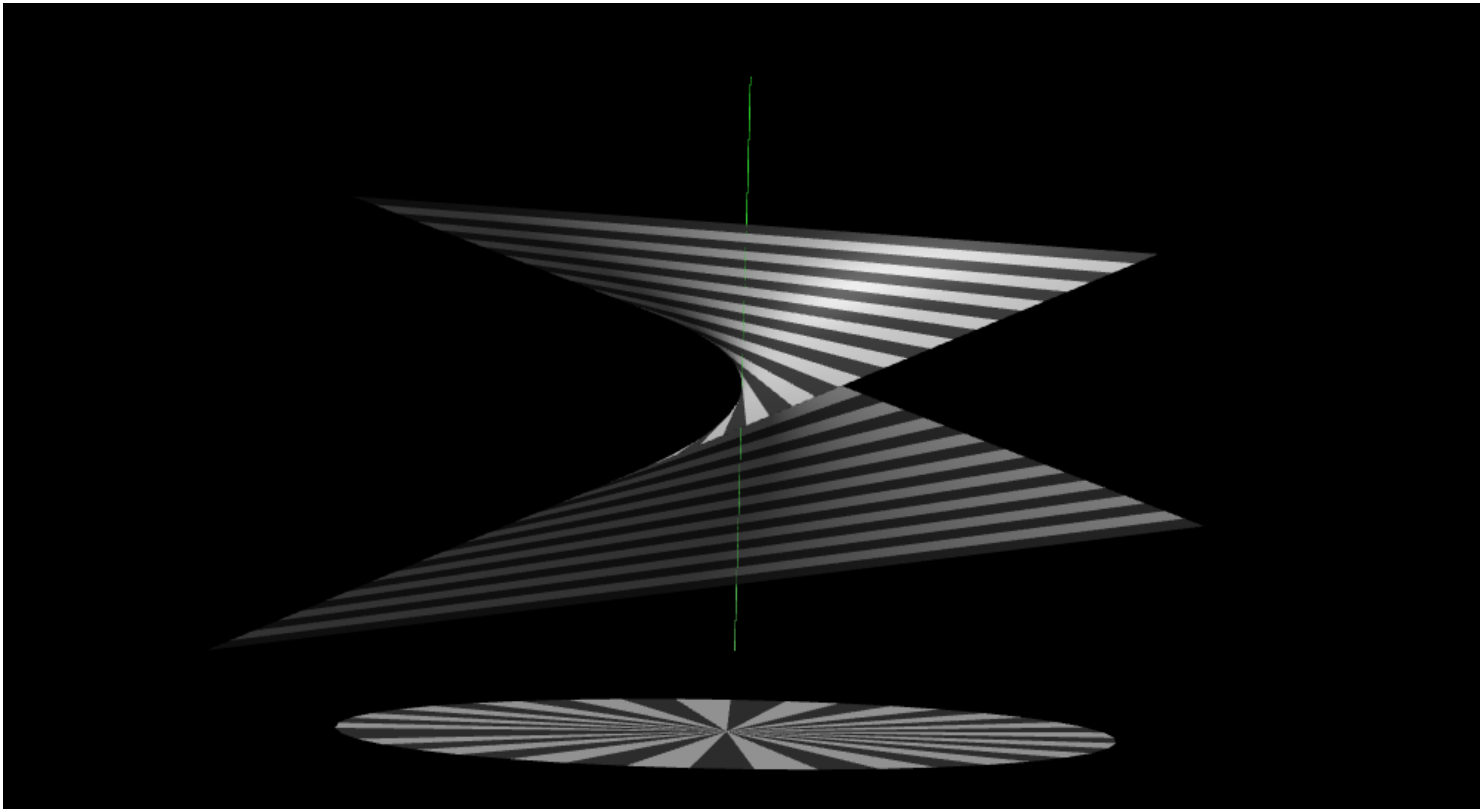

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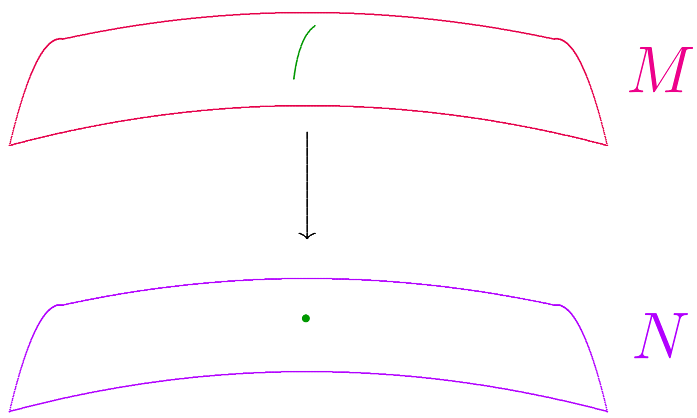

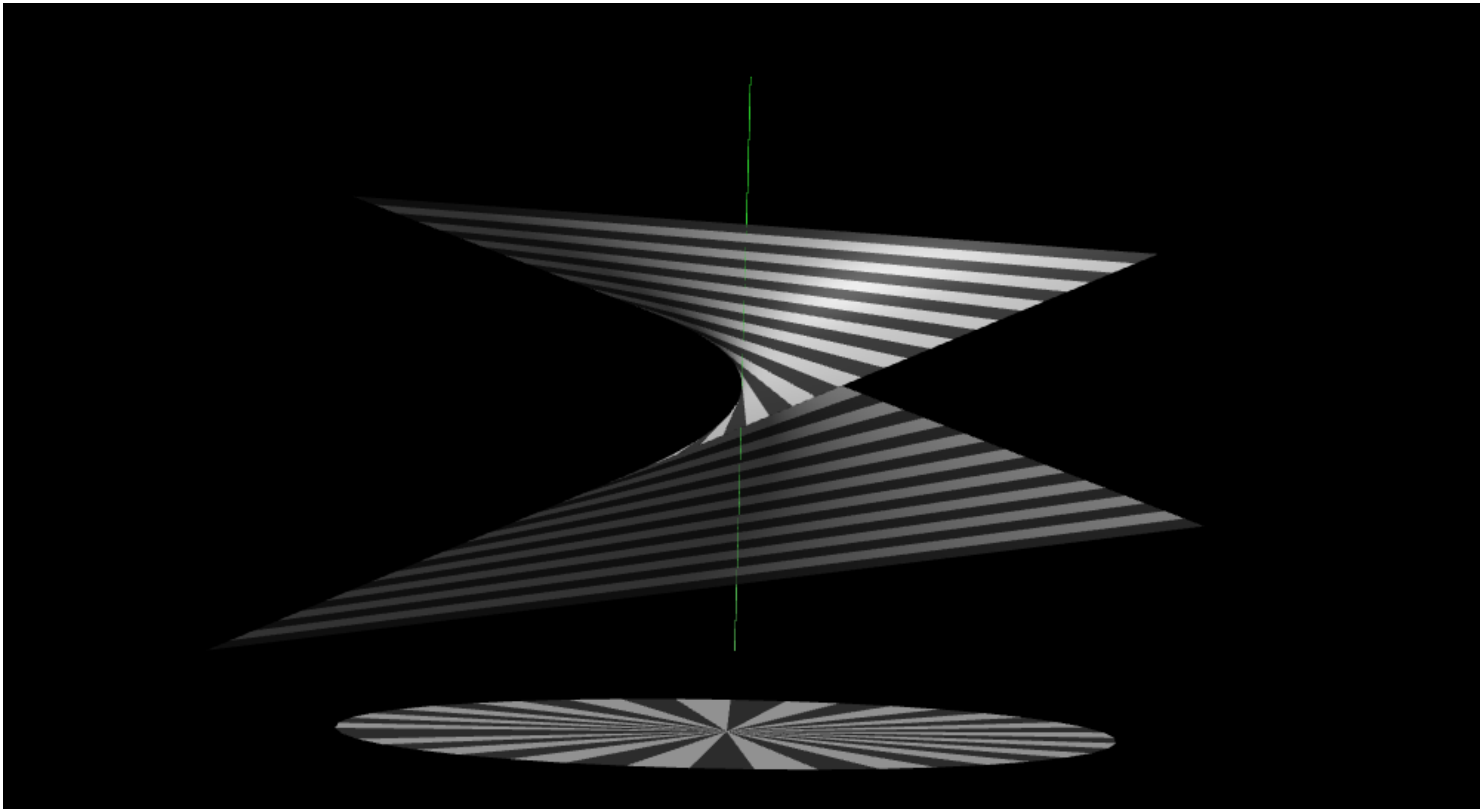

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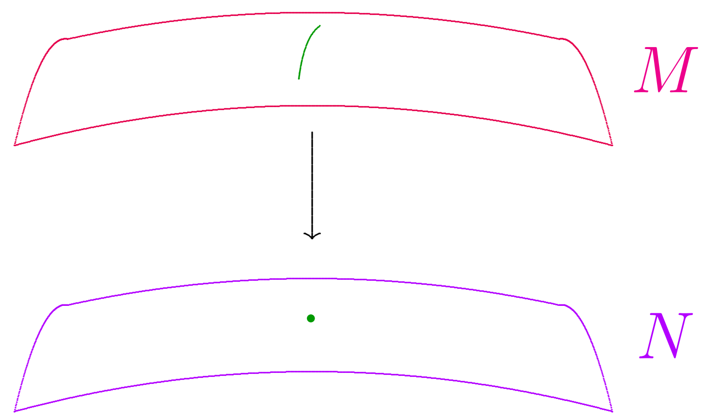

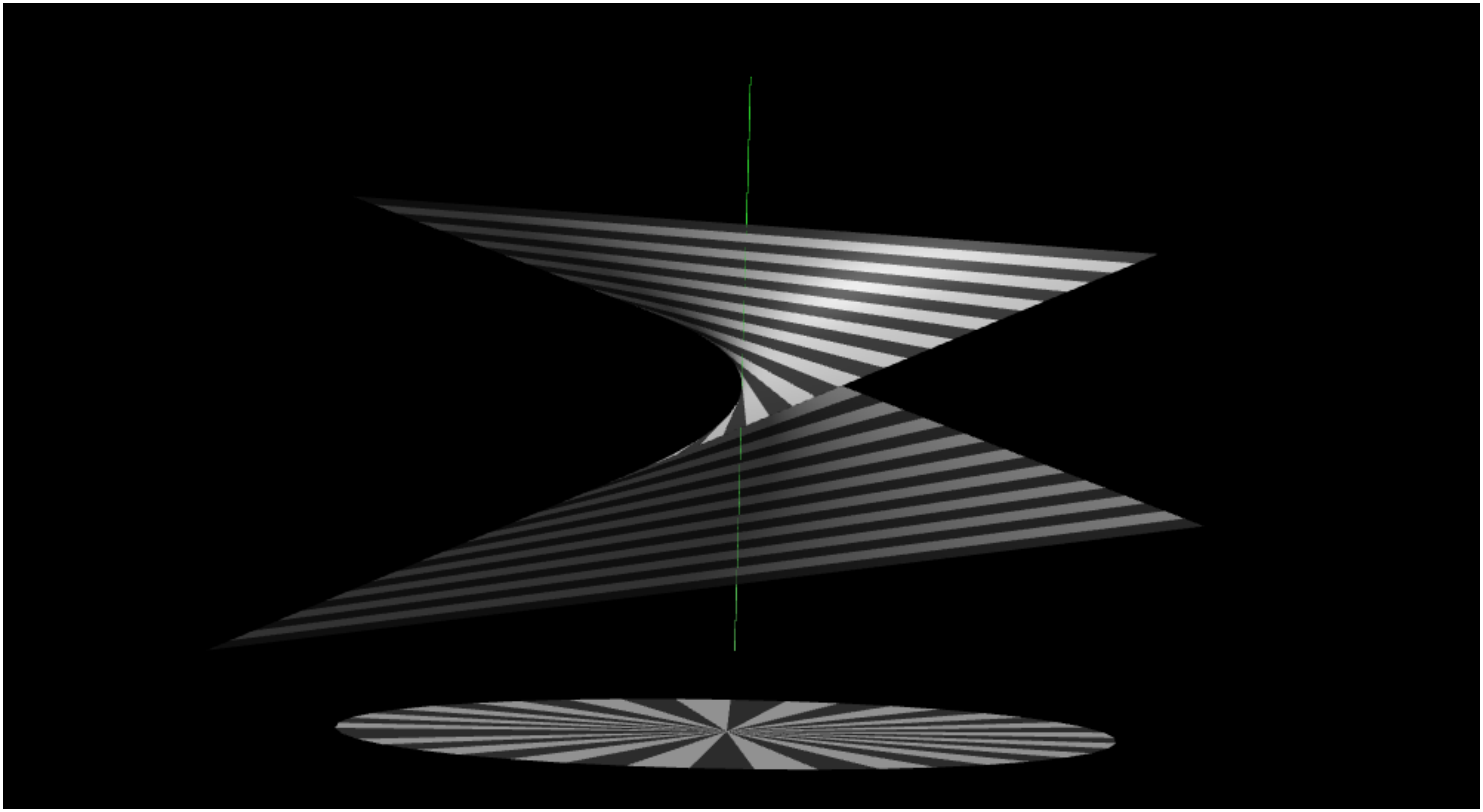

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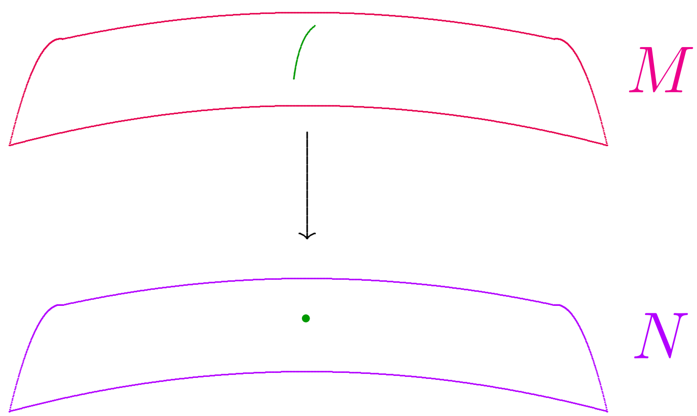

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

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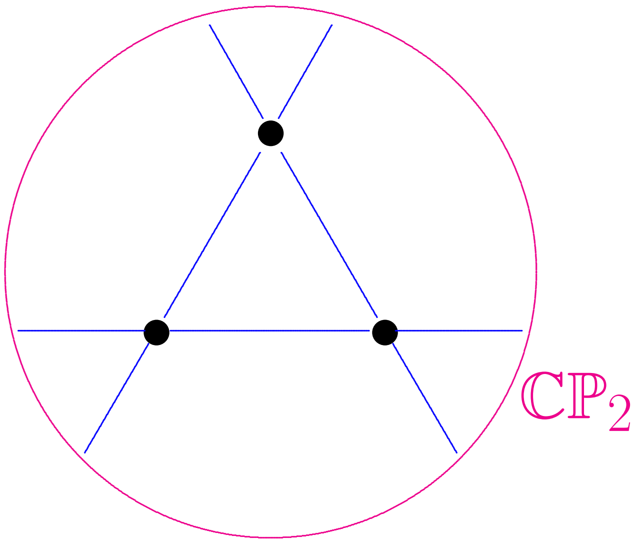

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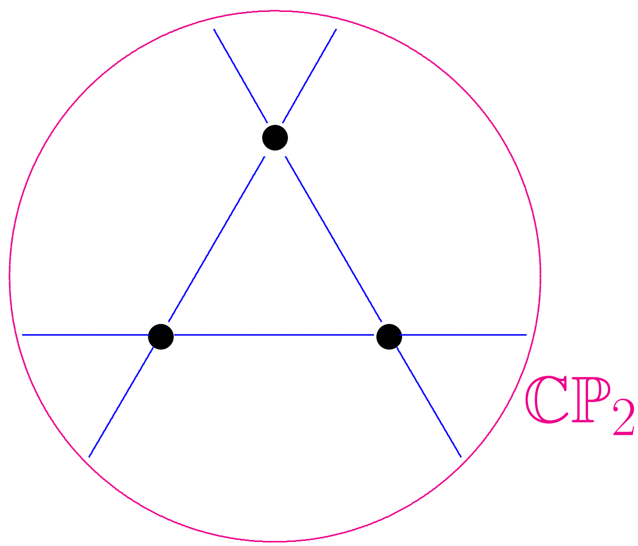

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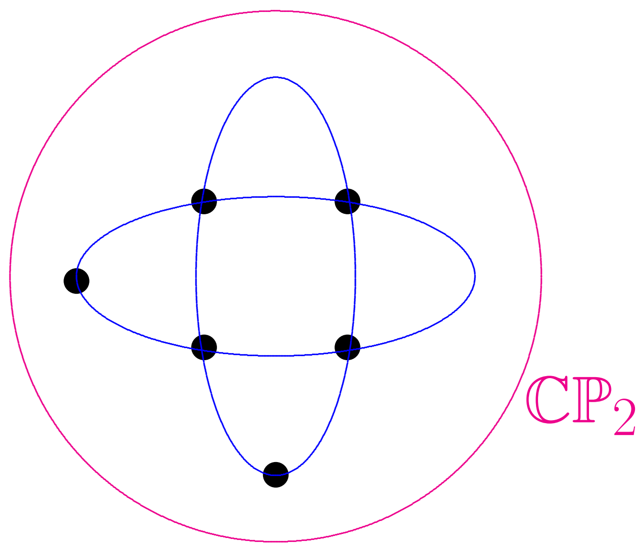

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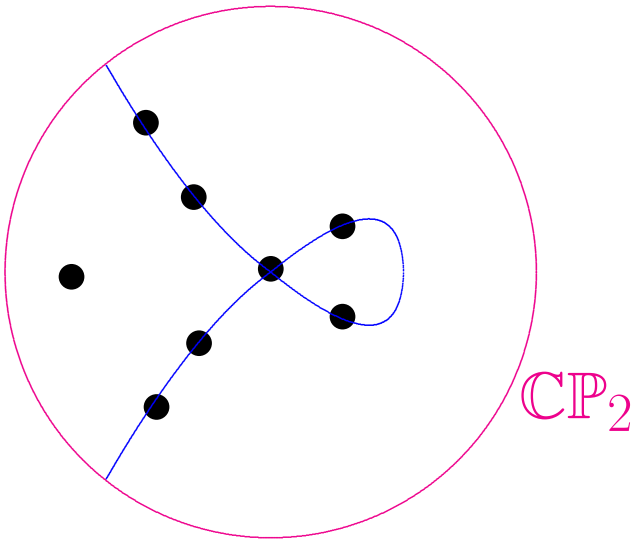

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Uniqueness: Bando-Mabuchi '87, L '12.

Above the line:

Moduli space $\mathscr{E}(M) \neq \varnothing$. But is it connected?

$$
\begin{array}{l}
\mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8, \\
S^2 \times S^2, \\
\hline
K3, \\
K3/\mathbb{Z}_2, \\
T^4, \\
T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\
T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4).
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Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathscr{E}(M)$ connected!

Basic problem:

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Understand all Einstein metrics on Del Pezzos.

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Exactly one connected component of moduli space.

Formulation depends on . . .

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More generally, their dimensions

$$b_\pm(M) = \dim \mathcal{H}^\pm_h$$

are completely metric-independent, and are oriented homotopy invariants of M.

Key background result:

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Yes — with a reasonable extra hypothesis on ω...

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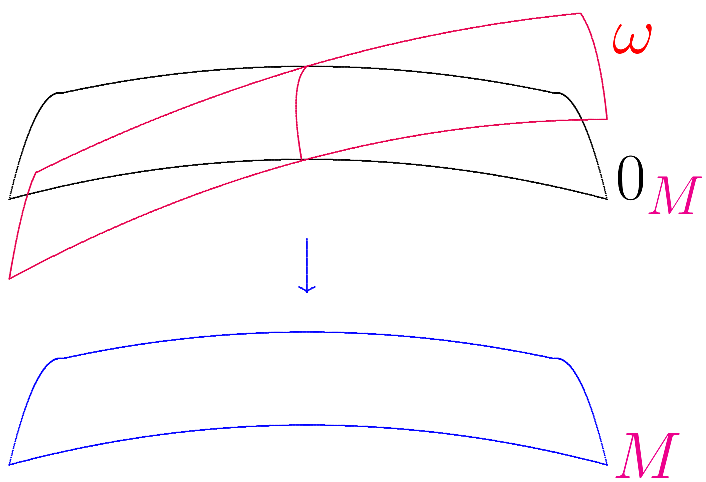

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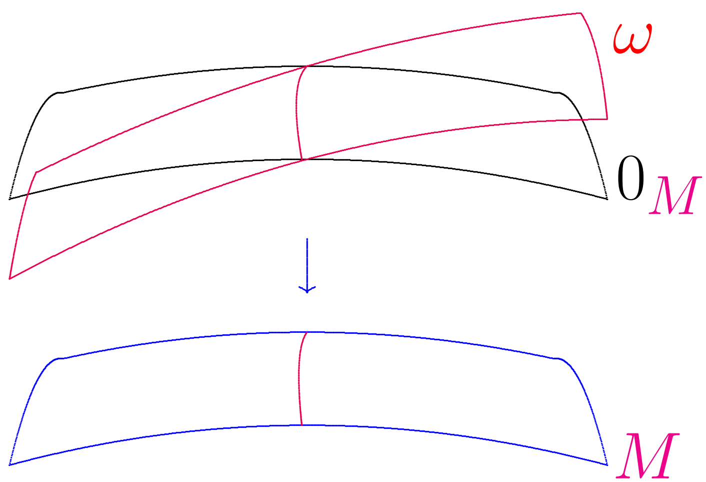

$\Longrightarrow$ Zero set Z of ω has codimension 3:

$$Z \approx \sqcup_{j=1}^{n} S^1.$$

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Theorem (Taubes, et al)**.** *If $b_+(M) \neq 0$, such forms exist for an open dense set of metrics h on M.*

Theorem A.

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Indeed, all known Einstein metrics on Del Pezzo surfaces arise this way!

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Conversely, these complex surfaces all admit $\lambda \geq 0$ *Einstein metrics* h *of the above type.*

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Moral: Taubes' genericity result does not guarantee genericity among metrics solving an equation!

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$$W^+(\omega, \omega) \geq 0, \qquad W^+(\omega, \omega) \not\equiv 0$$

Before discussing **Theorems A & B**,

consider simpler case when $W^{+}(\omega, \omega) > 0$.

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for some g-preserving almost-complex structure J.

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as proxy for Einstein equation.

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for $fW^+ \in \mathrm{End}(\Lambda^+)$.

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(for any almost-Kähler 4-manifold)

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$\therefore h \propto s^{-2} g$ globally on M.

Tanti auguri, e buon compleanno, Max!