Mass in

Kähler Geometry

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New Perspectives in Differential Geometry
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Joint work with
Joint work with

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Definition. A complete, non-compact Riemannian $n$-manifold $(M^n, g)$
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Definition. A complete, non-compact Riemannian $n$-manifold $(M^n, g)$ is called asymptotically Euclidean if there is a compact set $K \subset M$ such that each component of $M - K$ is diffeomorphic to $\mathbb{R}^n - D^n$ in such a manner that $n \geq 3$.
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With weak fall-off conditions, the mass is well-defined & coordinate independent.
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also has mass \( m \).
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Local matter density $\geq 0$
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Local matter density \( \geq 0 \) \( \implies \) total mass \( \geq 0 \).
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Lemma.
Mass of \textbf{ALE} Kähler manifolds?

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\textbf{Lemma.} \textit{Any ALE Kähler manifold}
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\[ n = 2m \geq 4 \]
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\underline{Upshot:}
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**Lemma.** *Any ALE Kähler manifold has only one end.*

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Mass of an ALE Kähler manifold is unambiguous.
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**Lemma.** Any **ALE Kähler manifold has only one end.**

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**Upshot:**

Mass of an **ALE Kähler manifold is unambiguous.**

Does not depend on the choice of an end!
We begin with the scalar-flat Kähler case.
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**Theorem A.**
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- the Kähler class \( [\omega] \in H^2(M) \) of the metric.

In fact, we will see that there is an explicit formula for the mass in terms of these data!
The explicit formula reproduces the mass in cases where it previously had been laboriously computed from the definition.
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(Discovered independently by Rollin, Singer, & Şuvaina, using different methods.)
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**Lemma.** Let \((M, g)\) be any ALE manifold of real dimension \(n \geq 4\). Then the natural map
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Here

\[ H^p_c(M) := \frac{\ker d : \mathcal{E}^p_c(M) \to \mathcal{E}^{p+1}_c(M)}{d\mathcal{E}^{p-1}_c(M)} \]
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**Lemma.** Let $(M, g)$ be any ALE manifold of real dimension $n \geq 4$. Then the natural map

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to denote the inverse of the natural map
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induced by the inclusion of compactly supported smooth forms into all forms.
We can now state our mass formula:

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m(M, g) = -\langle c_1, \omega \rangle_m - \frac{(m-1)!}{4(2m-1)\pi^{m-1}} + \int_M s_g d\mu_g
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where

- \(s = \text{scalar curvature};\)
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- \([\omega] \in H^2(M)\) is Kähler class of \((g, J)\); and
- \(\langle \ , \ \rangle\) is pairing between \(H^2_c(M)\) and \(H^{2m-2}(M)\).
\[ m(M, g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1}\rangle}{(2m - 1)\pi^{m-1}} + \frac{(m - 1)!}{4(2m - 1)\pi^m} \int_M s_g d\mu_g \]
\[
\frac{(2m - 1) \pi^m}{(m - 1)!} m(M,g) = -\frac{4\pi}{(m - 1)!} \langle \bigstar(c_1), [\omega]^{m-1} \rangle + \int_M s g d\mu_g
\]
For a compact Kähler manifold \((M^{2m}, g, J)\),

\[
\int_M s_g d\mu_g = \frac{4\pi}{(m-1)!}\langle c_1, [\omega]^{m-1}\rangle
\]
For a compact Kähler manifold \((M^{2m}, g, J)\),

\[
0 = -\frac{4\pi}{(m - 1)!} \langle c_1, [\omega]^{m-1} \rangle + \int_M s_g \, d\mu_g
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For an ALE Kähler manifold \((M^{2m}, g, J)\),

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\]

So the mass is a “boundary correction” to the topological formula for the total scalar curvature.
Theorem C. Any ALE Kähler manifold \((M, g, J)\) of complex dimension \(m\) has mass given by

\[
m(M, g) = -\frac{\langle ♣(c_1), [\omega]^{m-1} \rangle}{(2m - 1)\pi^{m-1}} + \frac{(m - 1)!}{4(2m - 1)\pi^m} \int_M s_g d\mu_g
\]
Corollary. Any ALE scalar-flat Kähler manifold \((M, g, J)\) of complex dimension \(m\) has mass given by

\[
m(M, g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1} \rangle}{(2m - 1)\pi^{m-1}}.
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Corollary. Any ALE scalar-flat Kähler manifold \((M, g, J)\) of complex dimension \(m\) has mass given by

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m(M, g) = -\frac{\langle ♣(c_1), [\omega]^{m-1} \rangle}{(2m - 1)\pi^{m-1}}.
\]

So Theorem A is an immediate consequence!
Rough Idea of Proof:
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Special Case: Suppose
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- \( m = 2, \ n = 4 \);
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Special Case: Suppose

- \( m = 2, \ n = 4; \)
- Scalar flat: \( s \equiv 0; \) and
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\[
(M - K, J) \approx_{\text{bih}} (\mathbb{C}^2 - \overline{B^4})/\Gamma.
\]
Rough Idea of Proof:

**Special Case:** Suppose

- $m = 2$, $n = 4$;
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- Complex structure $J$ standard at infinity:
  
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Since $g$ is Kähler, the complex coordinates
Rough Idea of Proof:

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Since \( g \) is Kähler, the complex coordinates

\[ (z^1, z^2) = (x^1 + ix^2, x^3 + ix^4) \]

are harmonic.
Rough Idea of Proof:

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Rough Idea of Proof:

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$$(M - K, J) \approx_{\text{bih}} (\mathbb{C}^2 - B^4)/\Gamma.$$ 

Since $g$ is Kähler, the complex coordinates

$$(z^1, z^2) = (x^1 + ix^2, x^3 + ix^4)$$

are harmonic. So $x^j$ are harmonic, too, and

$$g^{jk} (g_{j\ell,k} - g_{jk,\ell}) \nu^\ell \alpha_E = -\ast d \log \left( \sqrt{\det g} \right) + O(\varrho^{-3-\varepsilon}).$$
\[ m(M, g) = - \lim_{\rho \to \infty} \frac{1}{12\pi^2} \int_{S_\rho/\Gamma} \star d \left( \log \sqrt{\text{det} g} \right) \]
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Now set \( \theta = \frac{i}{2}(\partial - \bar{\partial}) \left( \log \sqrt{\det g} \right) \), so that
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Thus
\[ m(M, g) = - \lim_{\varrho \to \infty} \frac{1}{6\pi^2} \int_{S_\varrho/\Gamma} \theta \wedge \omega \]

However, since \( s = 0 \),
\[ d(\theta \wedge \omega) = \rho \wedge \omega = \frac{s}{4} \omega^2 = 0. \]
\[ m(M, g) = - \lim_{\rho \to \infty} \frac{1}{12\pi^2} \int_{S_\rho/\Gamma} \star d \left( \log \sqrt{\text{det} g} \right) \]

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Thus

\[ m(M, g) = - \frac{1}{6\pi^2} \int_{S_\rho/\Gamma} \theta \wedge \omega \]
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![Graph of a smooth cut-off function with a transition from 0 to 1 near infinity, labeled "end" on the x-axis. The y-axis ranges from 0 to 1.]
Let $f : M \to \mathbb{R}$ be smooth cut-off function:

\begin{itemize}
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\end{itemize}
Let $f : M \to \mathbb{R}$ be smooth cut-off function:

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Compactly supported, because $d\theta = \rho$ near infinity.
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by Stokes’ theorem.
Let \( f : M \to \mathbb{R} \) be smooth cut-off function:
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So

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as claimed.
We assumed:
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- \( m = 2 \);
- \( s \equiv 0 \); and
- Complex structure \( J \) standard at infinity.
General case:
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- General $m \geq 2$: 
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Argument proceeds by osculation:
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Argument proceeds by osculation:

\[
J = J_0 + O(\varrho^{-3}), \quad \nabla J = O(\varrho^{-4})
\]

in suitable asymptotic coordinates adapted to $g$. 
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Moduli space carries \( \mathcal{O} \) projective structure

with many totally geodesic hypersurfaces.
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Compactify $M$ itself by adding $\mathbb{CP}^{m-1}$ at infinity.
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$$\overline{M} \rightarrow \mathbb{CP}^{m}$$

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Thus obtain holomorphic map

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This has some interesting consequences...
Theorem D (Positive Mass Theorem).
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**Theorem D** (Positive Mass Theorem). *Any AE Kähler manifold with non-negative scalar curvature has non-negative mass:*

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Theorem D (Positive Mass Theorem). Any AE Kähler manifold with non-negative scalar curvature has non-negative mass:

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Moreover, \( m = 0 \iff \)
Theorem D (Positive Mass Theorem). Any AE Kähler manifold with non-negative scalar curvature has non-negative mass:

\[ AE \ & \ Kähler \ & \ s \geq 0 \implies m(M, g) \geq 0. \]

Moreover, \( m = 0 \iff (M, g) \) is Euclidean space.
**Theorem D** (Positive Mass Theorem). Any AE Kähler manifold with non-negative scalar curvature has non-negative mass:

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Proof actually shows something stronger!
Theorem E (Penrose Inequality).
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Theorem E (Penrose Inequality). Let $(M^{2m}, g, J)$ be an AE Kähler manifold with scalar curvature $s \geq 0$. Then $(M, J)$ carries a canonical divisor $D$ that is expressed as a sum $\sum j n_j D_j$ of compact complex hypersurfaces with positive integer coefficients, with the property that $\bigcup j D_j \neq \emptyset$ whenever $(M, J) \neq \mathbb{C}^m$. 
Theorem E (Penrose Inequality). Let $(\mathcal{M}^{2m}, g, J)$ be an AE Kähler manifold with scalar curvature $s \geq 0$. Then $(\mathcal{M}, J)$ carries a canonical divisor $D$ that is expressed as a sum $\sum j n_j D_j$ of compact complex hypersurfaces with positive integer coefficients, with the property that $\bigcup j D_j \neq \emptyset$ whenever $(\mathcal{M}, J) \neq \mathbb{C}^m$. In terms of this divisor,
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$$m(M, g) \geq \sum \text{Vol}(D_j)$$
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Theorem E (Penrose Inequality). Let $(M^{2m}, g, J)$ be an AE Kähler manifold with scalar curvature $s \geq 0$. Then $(M, J)$ carries a canonical divisor $D$ that is expressed as a sum $\sum_j n_j D_j$ of compact complex hypersurfaces with positive integer coefficients, with the property that $\bigcup_j D_j \neq \emptyset$ whenever $(M, J) \neq \mathbb{C}^m$. In terms of this divisor, we then have

$$m(M, g) \geq \frac{(m - 1)!}{(2m - 1)\pi^{m-1}} \sum n_j \text{Vol}(D_j)$$

with $= \iff (M, g, J)$ is scalar-flat Kähler.
This follows from existence of a holomorphic map
\[ \Phi : M \to \mathbb{C}^m \]
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Indeed, we then have a holomorphic section

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and
\[-\langle \clubsuit(c_1), \frac{\omega^{m-1}}{(m-1)!} \rangle = \sum n_j \text{Vol}(D_j) \]
This follows from existence of a holomorphic map
\[ \Phi : M \rightarrow \mathbb{C}^m \]
which is a biholomorphism near infinity.

Indeed, we then have a holomorphic section
\[ \varphi = \Phi^* d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^m \]
of the canonical line bundle which vanishes exactly at the critical points of \( \Phi \).

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\[ D = \sum n_j D_j \]
and
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so the mass formula implies the claim.
After all this heavy discussion of the mass,
After all this heavy discussion of the mass,

let me end on a lighter note,
After all this heavy discussion of the mass,

let me end on a lighter note,

by expressing my appreciation of an old friend,
After all this heavy discussion of the mass,
let me end on a lighter note,
by expressing my appreciation of an old friend,
whose talents
whose talents sometimes seem to defy gravity,
whose talents sometimes seem to defy gravity,
and whose questions
and whose questions never fail to elicit deep thought.
and whose questions never fail to elicit deep thought.
When we were graduate students at Oxford,
When we were graduate students at Oxford, Simon introduced me to holonomy,
When we were graduate students at Oxford, Simon introduced me to holonomy, and quaternion-Kähler geometry,
When we were graduate students at Oxford, Simon introduced me to holonomy, and quaternion-Kähler geometry, eventually leading to a very successful collaboration many years later.
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Ed era lui che mi ha convinto a imparare l’italiano!
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But as this audience will attest, I am no isolated case.
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Tanti auguri! And Happy Birthday!