Mass in

Kähler Geometry

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Joint work with
Joint work with

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Definition. A complete, non-compact Riemannian $n$-manifold $(M^n, g)$
**Definition.** A complete, non-compact Riemannian $n$-manifold $(M^n, g)$ is called asymptotically Euclidean (AE) if there is a compact set $K \subset M$ such that each component of $M - K$ is diffeomorphic to $\mathbb{R}^n$. If $n \geq 3$.
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Seems to depend on choice of coordinates!
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**Chruściel-type fall-off:**

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**Bartnik-type fall-off:**

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When $n = 3$, ADM mass in general relativity.
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also has mass \( m \).
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Local matter density $\geq 0$
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Local matter density $\geq 0 \implies$ total mass $\geq 0$. 
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Conjectured true in ALE case, too.

L 1986:
ALE counter-examples.
Scalar-flat Kähler metrics
Positive Mass Conjecture:
Any AE manifold with \( s \geq 0 \) has \( m \geq 0 \).

Schoen-Yau 1979:
Proved in dimension \( n \leq 7 \).

Witten 1981:
Proved for spin manifolds (implicitly, for any \( n \)).

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Conjectured true in ALE case, too.

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ALE counter-examples.
Scalar-flat Kähler metrics
on line bundles \( L \to \mathbb{CP}_1 \) of Chern-class \( \leq -3 \).
Mass of ALE Kähler manifolds?
Mass of ALE Kähler manifolds?

Scalar-flat Kähler case?
Mass of ALE Kähler manifolds?

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Lemma.
Mass of ALE Kähler manifolds?

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**Lemma.** Any ALE Kähler manifold
Mass of ALE Kähler manifolds?

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**Lemma.** Any ALE Kähler manifold has only one end.
Mass of ALE Kähler manifolds?

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Mass of \textbf{ALE Kähler} manifolds?

Scalar-flat Kähler case?

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\[ n = 2m \geq 4 \]
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each end is pseudo-convex at infinity.
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Mass of ALE Kähler manifolds?

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intersection form on $H^2$ of compactification.
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**Lemma.** *Any ALE Kähler manifold has only one end.*
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**Upshot:**
Mass of ALE Kähler manifolds?

Scalar-flat Kähler case?

Lemma. Any ALE Kähler manifold has only one end.

Upshot:

Mass of an ALE Kähler manifold is unambiguous.
Mass of **ALE Kähler** manifolds?

**Scalar-flat Kähler case?**

**Lemma.** *Any ALE Kähler manifold has only one end.*

**Upshot:**

Mass of an ALE Kähler manifold is unambiguous.

Does not depend on the choice of an end!
We begin with the scalar-flat Kähler case.
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Theorem A.
We begin with the scalar-flat Kähler case.

**Theorem A.** *The mass*
We begin with the scalar-flat Kähler case.

**Theorem A.** *The mass of an ALE*
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**Theorem A.** The mass of an ALE scalar-flat Kähler manifold
We begin with the scalar-flat Kähler case.

**Theorem A.** The mass of an ALE scalar-flat Kähler manifold is a topological invariant.
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Theorem A. The mass of an ALE scalar-flat Kähler manifold is a topological invariant. That is, \( m(M, g, J) \) is completely determined by

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- the smooth manifold $M$,
- the first Chern class $c_1 = c_1(M, J) \in H^2(M)$ of the complex structure, and
- the Kähler class $[\omega] \in H^2(M)$ of the metric.

In fact, we will see that there is an explicit formula for the mass in terms of these data!
The explicit formula reproduces the mass in cases where it previously had been laboriously computed from the definition.
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(Discovered independently by Rollin, Singer, & Şuvaina, using different methods.)
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\[
H^2_c(M) \to H^2_{dR}(M)
\]
Explicit formula depends on a topological fact:

**Lemma.** Let $(M, g)$ be any ALE manifold of real dimension $n \geq 4$. Then the natural map

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H^1(\partial \overline{M}) \to H^2(\overline{M}, \partial \overline{M}) \to H^2(\overline{M}) \to H^2(\partial \overline{M})
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induced by the inclusion of compactly supported smooth forms into all forms.
We can now state our mass formula:

**Theorem C.**
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**Theorem C. Any ALE Kähler manifold** \((M, g, J)\)
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$$m(M, g) = -\langle c_1, [\omega]_m \rangle (2^m - 1) \pi^{m-1} + (m-1)! 4(2^m - 1) \pi^m \int_M s_g d\mu_g$$
We can now state our mass formula:

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where

- \(s = \text{scalar curvature};\)
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- \(s = \text{scalar curvature};\)
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- \([\omega] \in H^2(M)\) is Kähler class of \((g, J)\); and
- \(\langle , \rangle\) is pairing between \(H_c^2(M)\) and \(H^{2m-2}(M)\).
\[ m(M, g) = -\langle \clubsuit (c_1), [\omega]^{m-1} \rangle \frac{(m - 1)!}{(2m - 1) \pi^{m-1}} + \frac{(m - 1)!}{4(2m - 1) \pi^m} \int_M s_g d\mu_g \]
\[
\frac{4\pi^m (2m-1)}{(m-1)!} m(M, g) = -\frac{4\pi}{(m-1)!} \langle \clubsuit (c_1), [\omega]^{m-1} \rangle + \int_M sg d\mu_g
\]
For a compact Kähler manifold \((M^{2m}, g, J)\),

\[
\int_M s_g d\mu_g = \frac{4\pi}{(m - 1)!} \langle c_1, [\omega]^{m-1} \rangle
\]
For a compact Kähler manifold \((M^{2m}, g, J)\),

\[
0 = -\frac{4\pi}{(m-1)!}\langle c_1, [\omega]^{m-1}\rangle + \int_M s_g d\mu_g
\]
For an ALE Kähler manifold \((M^{2m}, g, J)\),

\[
m(M, g) = -\frac{\langle \heart(c_1), [\omega]^{m-1} \rangle}{(2m - 1)\pi^{m-1}} + \frac{(m - 1)!}{4(2m - 1)\pi^m} \int_M s_g d\mu_g
\]
For an ALE Kähler manifold \((M^{2m}, g, J)\),

\[
\frac{4\pi^m(2m-1)}{(m-1)!} m(M, g) = -\frac{4\pi}{(m-1)!}\langle \clubsuit (c_1), [\omega]^{m-1}\rangle + \int_M s_g d\mu_g
\]

So the mass is a “boundary correction” to the topological formula for the total scalar curvature.
Theorem C. Any ALE Kähler manifold \((M, g, J)\) of complex dimension \(m\) has mass given by

\[
m(M, g) = -\frac{\langle \clubsuit (c_1), [\omega]^{m-1} \rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m} \int_M s_g d\mu_g
\]
Corollary. Any ALE scalar-flat Kähler manifold $(M, g, J)$ of complex dimension $m$ has mass given by

$$m(M, g) = -\frac{\langle ♣(c_1), [\omega]^{m-1}\rangle}{(2m - 1)\pi^{m-1}}.$$
Corollary. Any ALE scalar-flat Kähler manifold \((M, g, J)\) of complex dimension \(m\) has mass given by

\[
m(M, g) = -\frac{\langle \diamondsuit (c_1), [\omega]^{m-1} \rangle}{(2m - 1)\pi^{m-1}}.
\]

So Theorem A is an immediate consequence!
Rough Idea of Proof:
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Special Case: Suppose
Rough Idea of Proof:

Special Case: Suppose

- $m = 2$, $n = 4$;
Rough Idea of Proof:

**Special Case:** Suppose

- $m = 2$, $n = 4$;
- Scalar flat: $s \equiv 0$; and
Rough Idea of Proof:

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g^{jk} (g_{j\ell,k} - g_{jk,\ell}) \nu^\ell \alpha_E = -\star d \log \left( \sqrt{\det g} \right) + O(\varrho^{-3-\varepsilon}).
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Thus

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However, since \( s = 0 \),

\[ d(\theta \wedge \omega) = \rho \wedge \omega = \frac{s}{4} \omega^2 = 0. \]
\[ m(M, g) = - \lim_{\varrho \to \infty} \frac{1}{12\pi^2} \int_{S_\varrho/\Gamma} \star d \left( \log \sqrt{\det g} \right) \]

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![Diagram of a smooth cut-off function with a transition from 0 to 1 as the radius approaches the "end".](image-url)
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Compactly supported, because \( d\theta = \rho \) near infinity.
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where \( M_\rho \) defined by radius \( \leq \rho \).
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because scalar-flat $\implies \rho \wedge \omega = 0.$
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<table>
<thead>
<tr>
<th>Equivalent near</th>
</tr>
</thead>
<tbody>
<tr>
<td>away from end,</td>
</tr>
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as claimed.
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One argument proceeds by osculation:

\[
J = J_0 + O(\varrho^{-3}), \quad \nabla J = O(\varrho^{-4})
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in suitable asymptotic coordinates adapted to \( g \).
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Cap off $\tilde{M}_\infty$ by adding $\mathbb{CP}^{m-1}$ at infinity.
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To understand $J$ at infinity:

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Added hypersurface $\mathbb{CP}^{m-1}$ has normal bundle $O(1)$.

Belongs to $m$-dimensional family of hypersurfaces.

Moduli space carries $O$ projective structure

with many totally geodesic hypersurfaces.

So flat if $m \geq 3$.

When $m = 2$, Cotton tensor may be non-zero, but “flatter” than might naively expect.
To understand $J$ at infinity:
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AE case:

Compactify $M$ itself by adding $\mathbb{CP}^{m-1}$ at infinity.
To understand $J$ at infinity:

**AE case:**

Compactify $M$ itself by adding $\mathbb{CP}_{m-1}$ at infinity.

Linear system of $\mathbb{CP}_{m-1}$ gives holomorphic map

\[
\overline{M} \rightarrow \mathbb{CP}_m
\]

which is biholomorphism near $\mathbb{CP}_{m-1}$. 
To understand $J$ at infinity:

**AE case:**

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Linear system of $\mathbb{CP}^{m-1}$ gives holomorphic map $\overline{M} \rightarrow \mathbb{CP}^m$ which is biholomorphism near $\mathbb{CP}^{m-1}$.

Thus obtain holomorphic map $\Phi : M \rightarrow \mathbb{C}^m$ which is biholomorphism near infinity.
To understand \( J \) at infinity:

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This has some interesting consequences...
Theorem D (Positive Mass Theorem).
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*Moreover, \( m = 0 \iff \)*)
Theorem D (Positive Mass Theorem). Any AE Kähler manifold with non-negative scalar curvature has non-negative mass:

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Moreover, $$m = 0 \iff (M, g) \text{ is Euclidean space.}$$
**Theorem D** (Positive Mass Theorem). *Any* AE Kähler manifold *with non-negative scalar curvature has non-negative mass:*

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*Moreover,* \( m = 0 \iff (M, g) \text{ is Euclidean space.} \)

Proof actually shows something stronger!
Theorem E (Penrose Inequality).

Let \((M,g,J)\) be an AE Kähler manifold with scalar curvature \(s \geq 0\). Then \((M,J)\) carries a canonical divisor \(D\) that is expressed as a sum \(\sum n_j D_j\) of compact complex hypersurfaces with positive integer coefficients, with the property that \(\bigcup n_j D_j \neq \emptyset\) whenever \((M,J) \neq \mathbb{C}^m\). In terms of this divisor, we then have...
Theorem E (Penrose Inequality). \( \text{Let } (M^{2m}, g, J) \)
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so the mass formula implies the claim.
\[ m(M, g) = -\frac{\langle \spadesuit(c_1), [\omega]^{m-1} \rangle}{(2m - 1)\pi^{m-1}} + \frac{(m - 1)!}{4(2m - 1)\pi^m} \int_M s_g d\mu_g \]
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