Curvature Functionals,

Kähler Metrics, &

the Geometry of 4-Manifolds III

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Stony Brook University

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where $F_A{}^+ =$ self-dual part curvature of A, and $\sigma : \mathbb{V}_+ \to \Lambda^+$ is a natural real-quadratic map,

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$$d^*(A - A_0) = 0$$

imposed to eliminate automorphisms of $L \to M$.

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Bootstrapping with gauge-fixed equations, one gets L^p_k bounds for (Φ, A) for all k, p.

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SW invariant $\in \mathbb{Z}_2$ means mod-2 mapping degree.

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Implies non-existence of metrics g for which $s > 0$.

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Hitchin-Thorpe Inequality:

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu_g$$

$$\text{Einstein} \Rightarrow \quad = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2\right) d\mu_g$$

Theorem (Hitchin-Thorpe Inequality). *If smooth compact oriented M^4 admits Einstein g, then*

$$(2\chi + 3\tau)(M) \geq 0,$$

with equality only if (M, g) is locally hyper-Kähler. The latter case happens only if M finitely covered by flat T^4 or $K3$.

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Key point: SW $\Rightarrow$ $s > 0$ impossible when Kod $= 2$.

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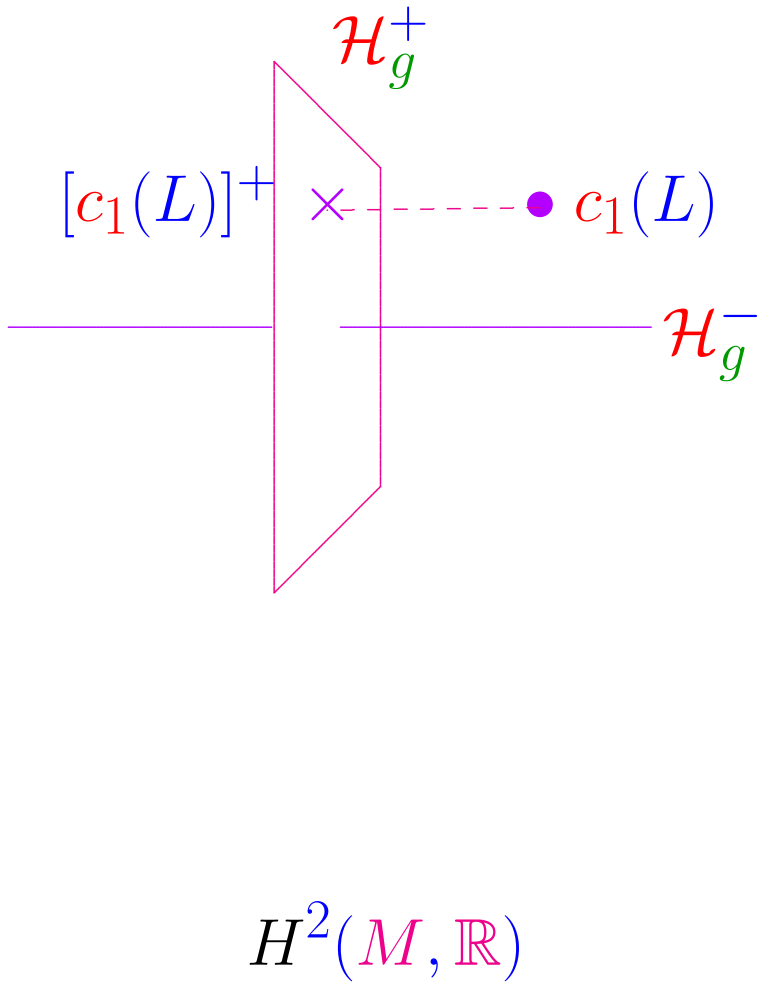
where $c_1(L)^+ \in \mathcal{H}^+_g$ is self-dual part of

$$c_1(L) \in H^2(M, \mathbb{R}) = \mathcal{H}^+_g \oplus \mathcal{H}^-_g$$



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Curvature Estimates:

If *SW equations* have solution $\forall \tilde{g} \in [g]$,
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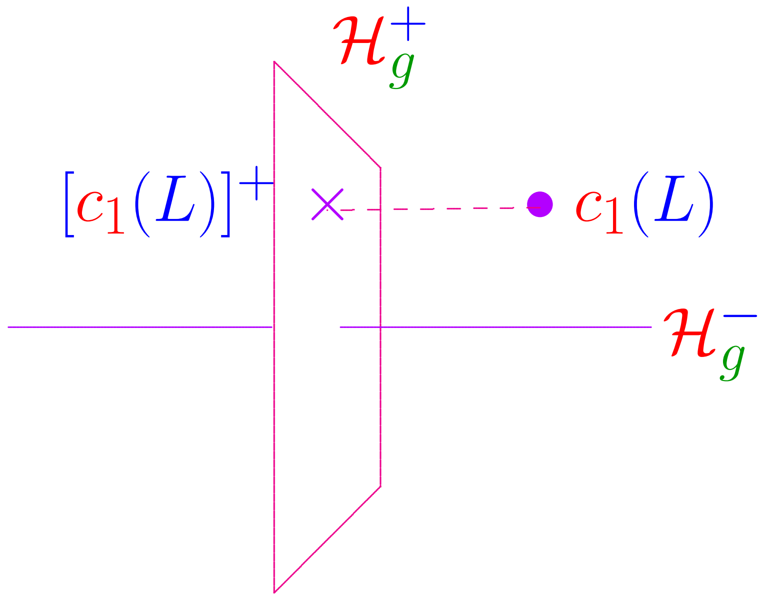
We need metric-independent improvement!



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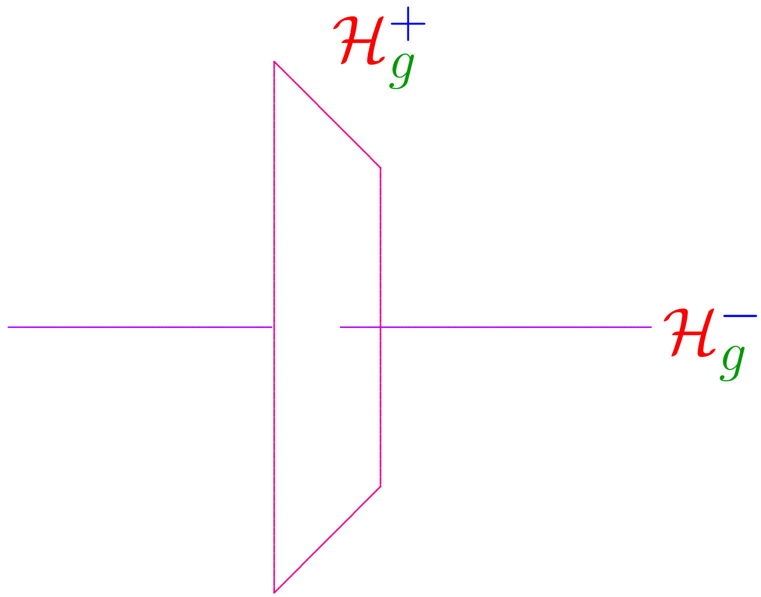

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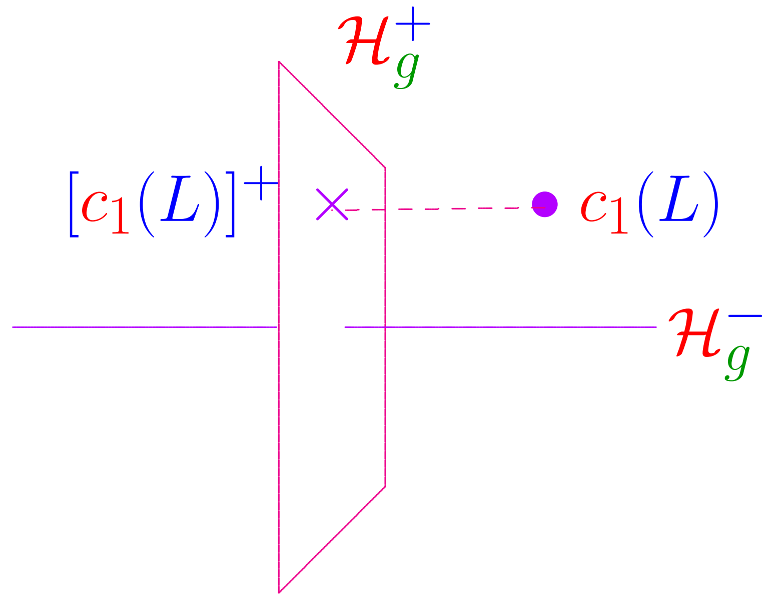

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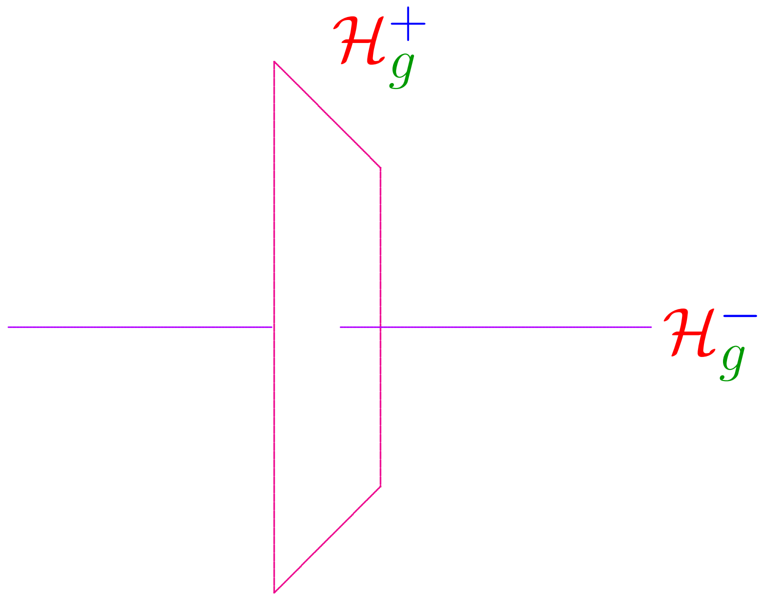

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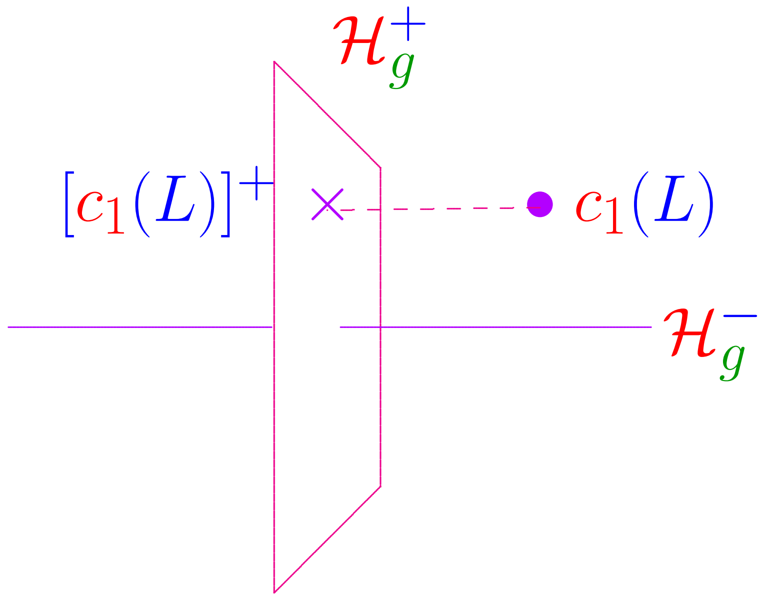

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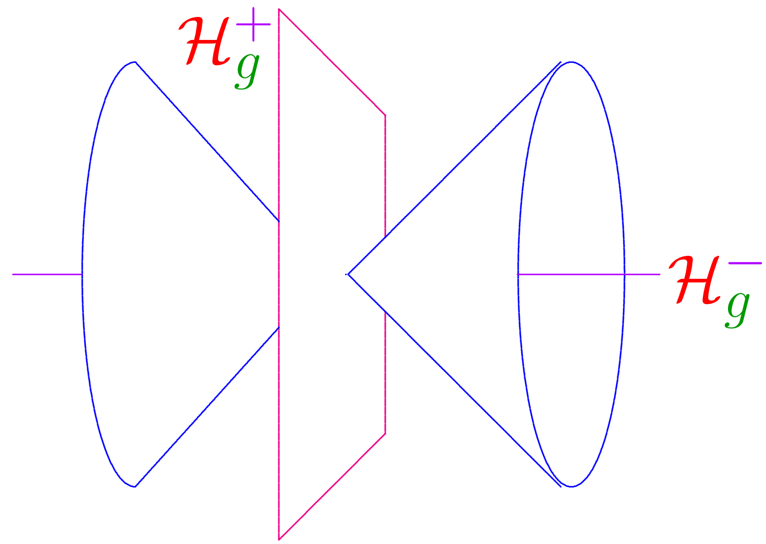

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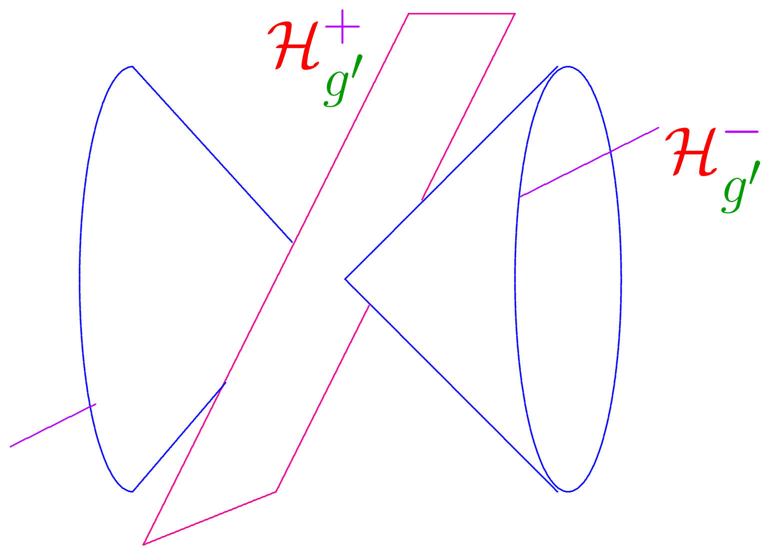

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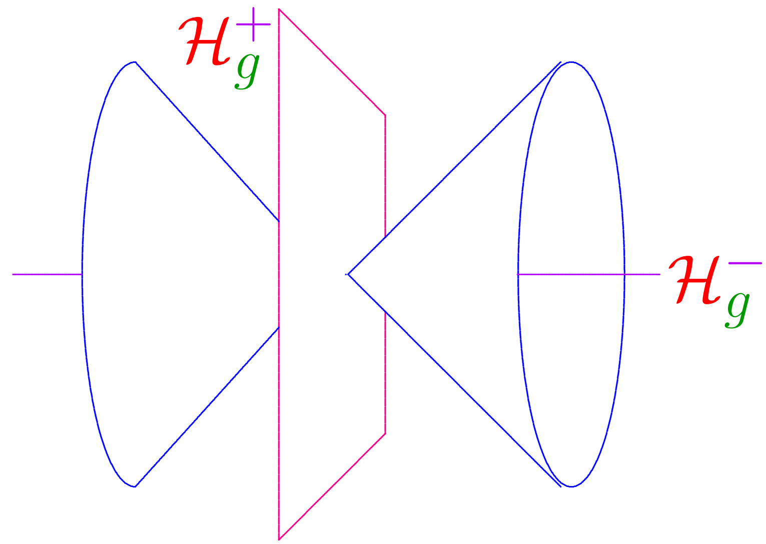

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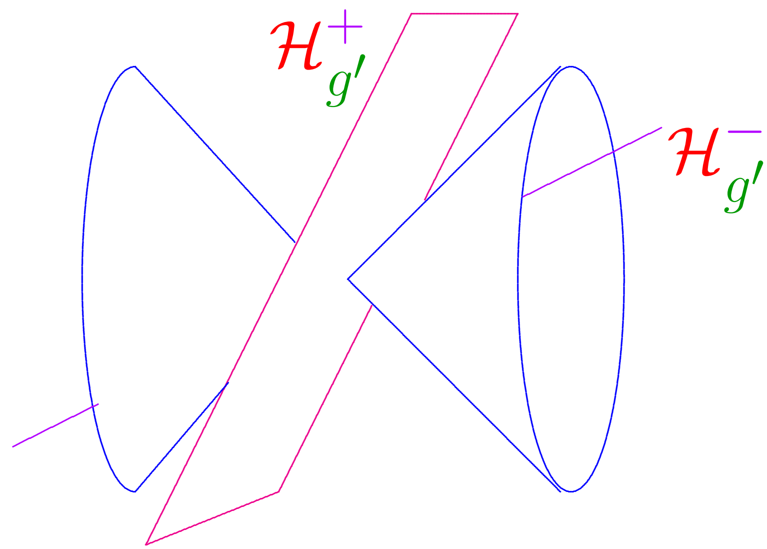

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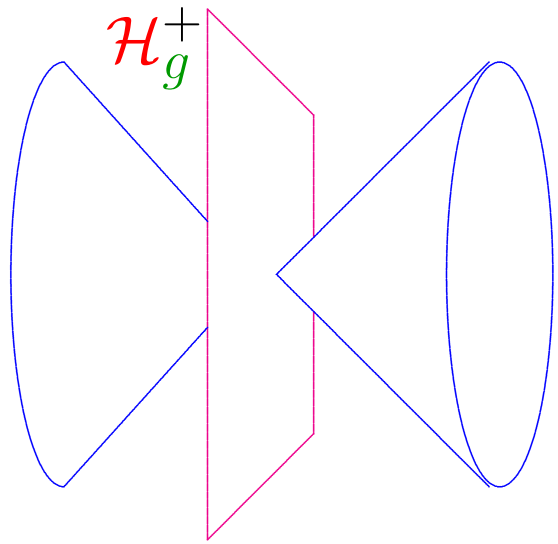

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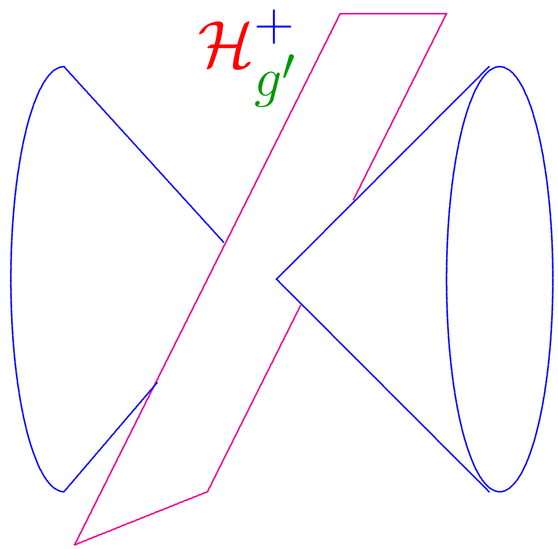

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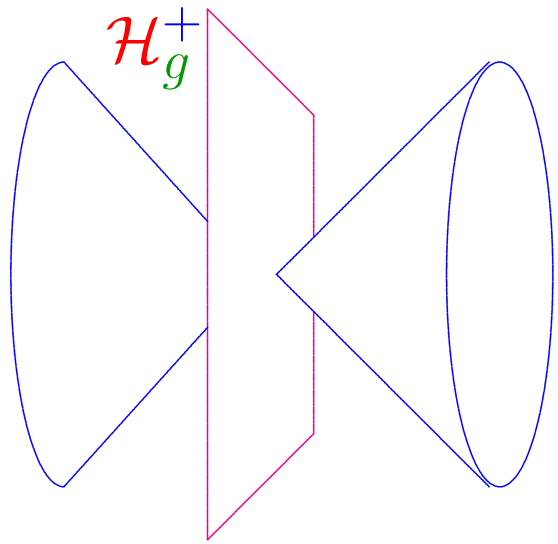

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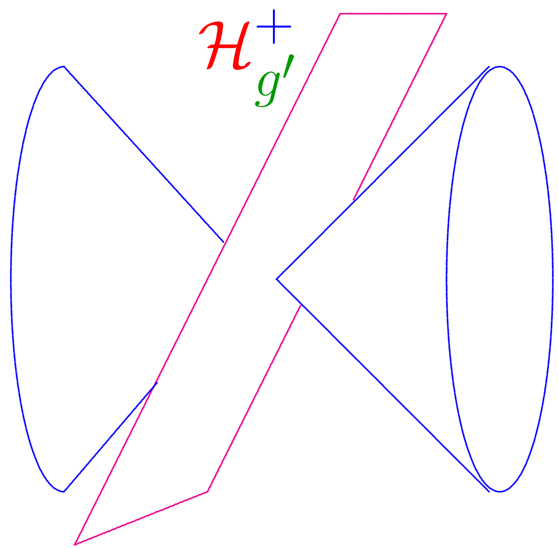

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will be called a basic class of M.

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have a solution (Φ, A) for every metric g on M.

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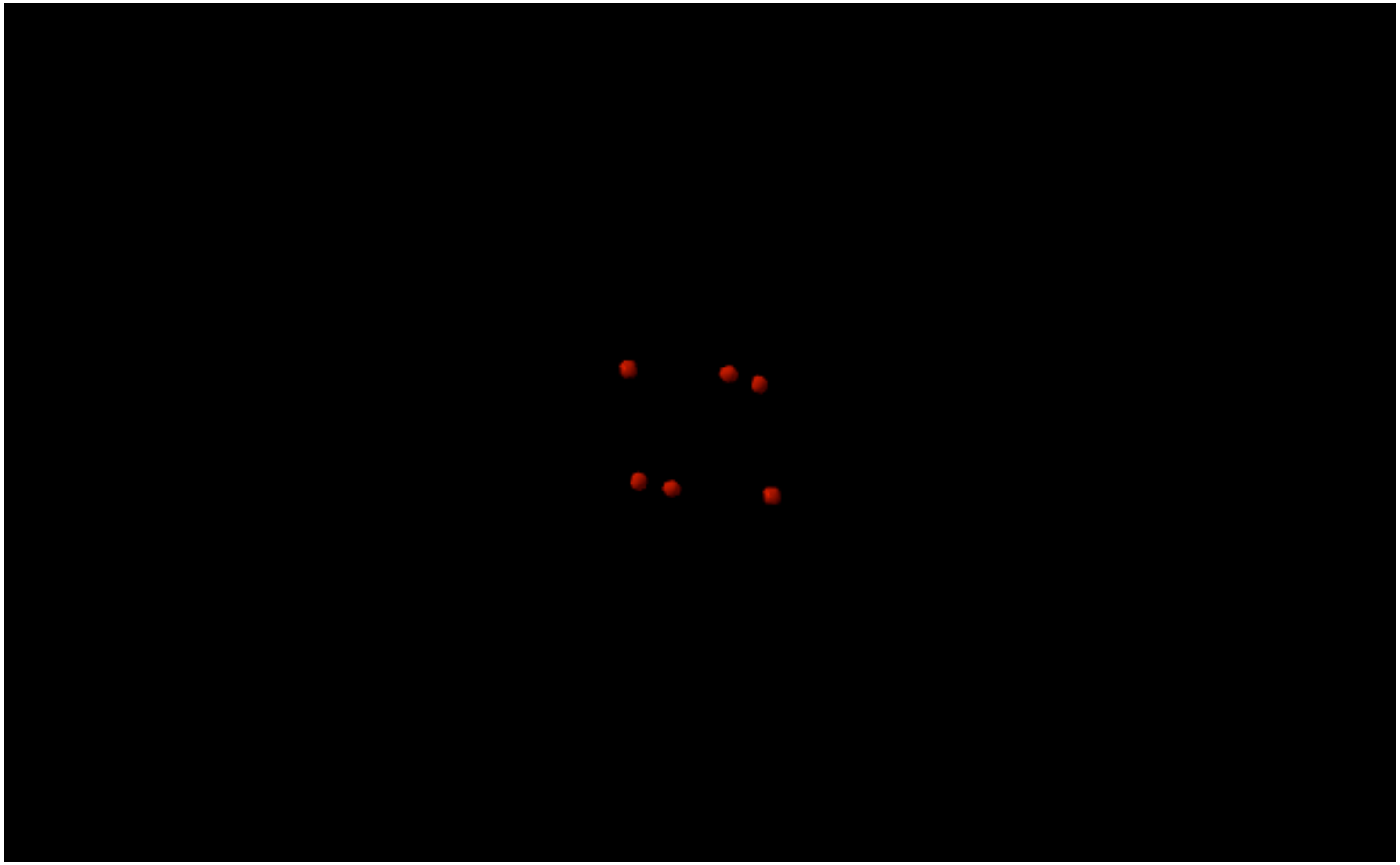

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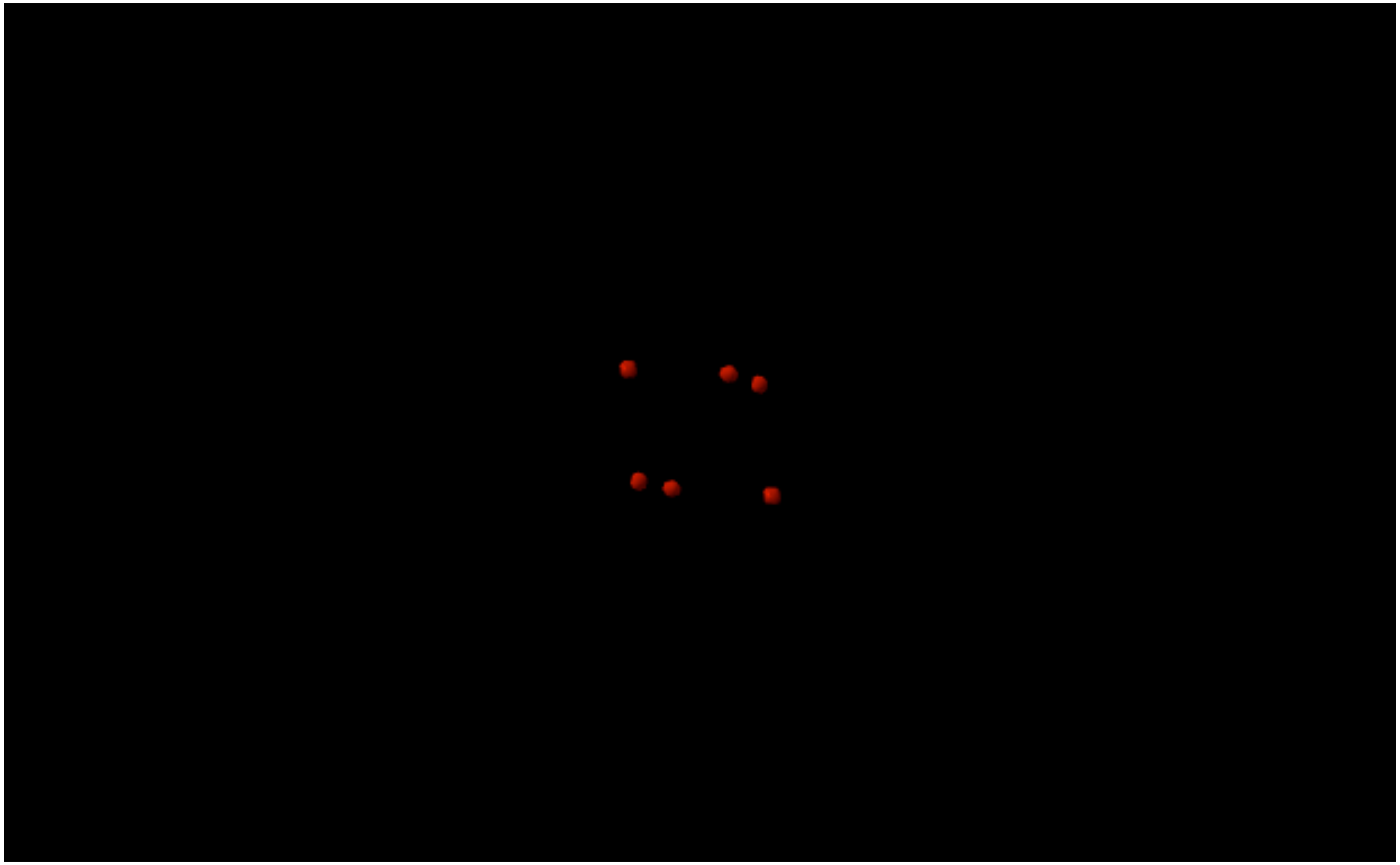

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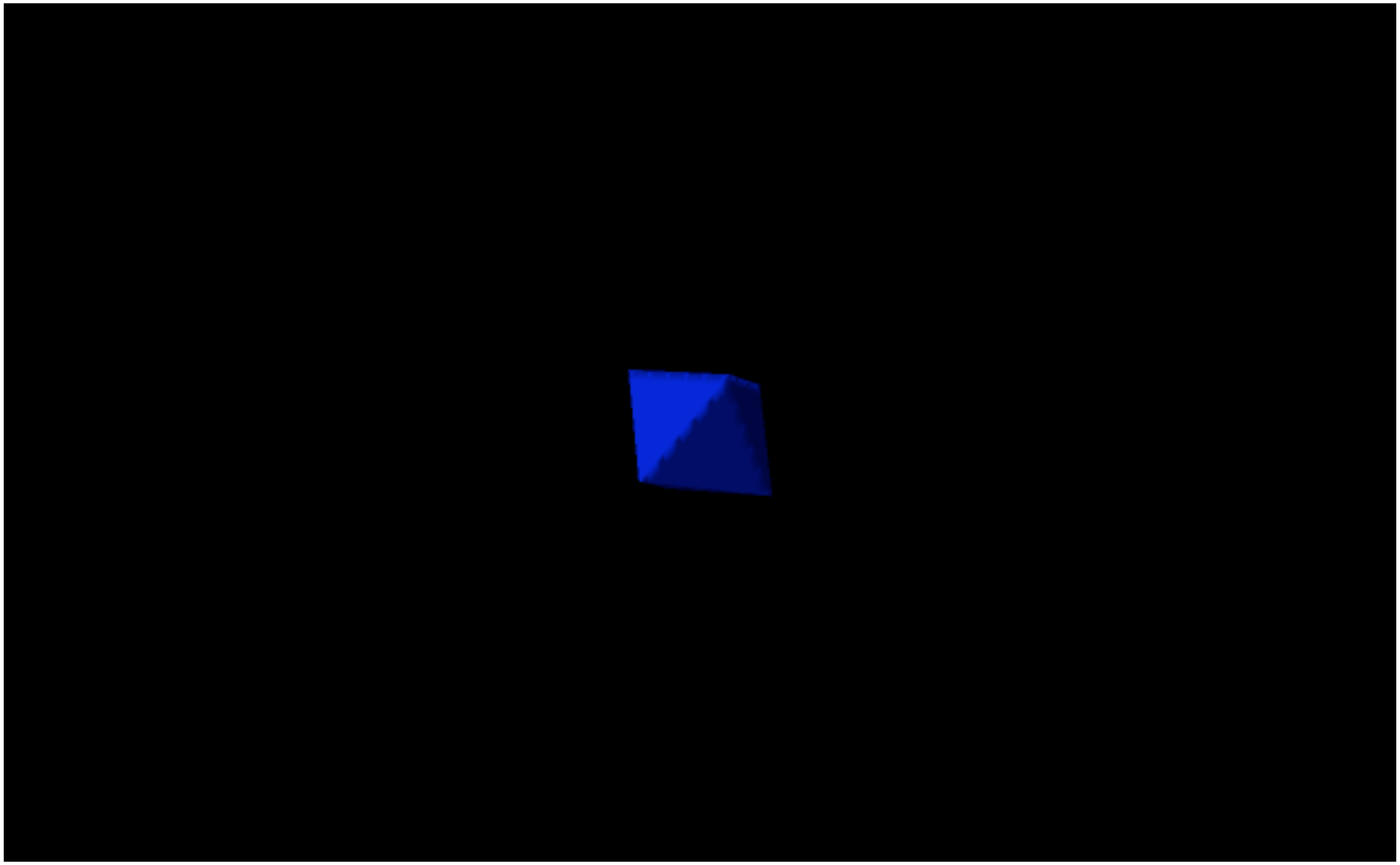

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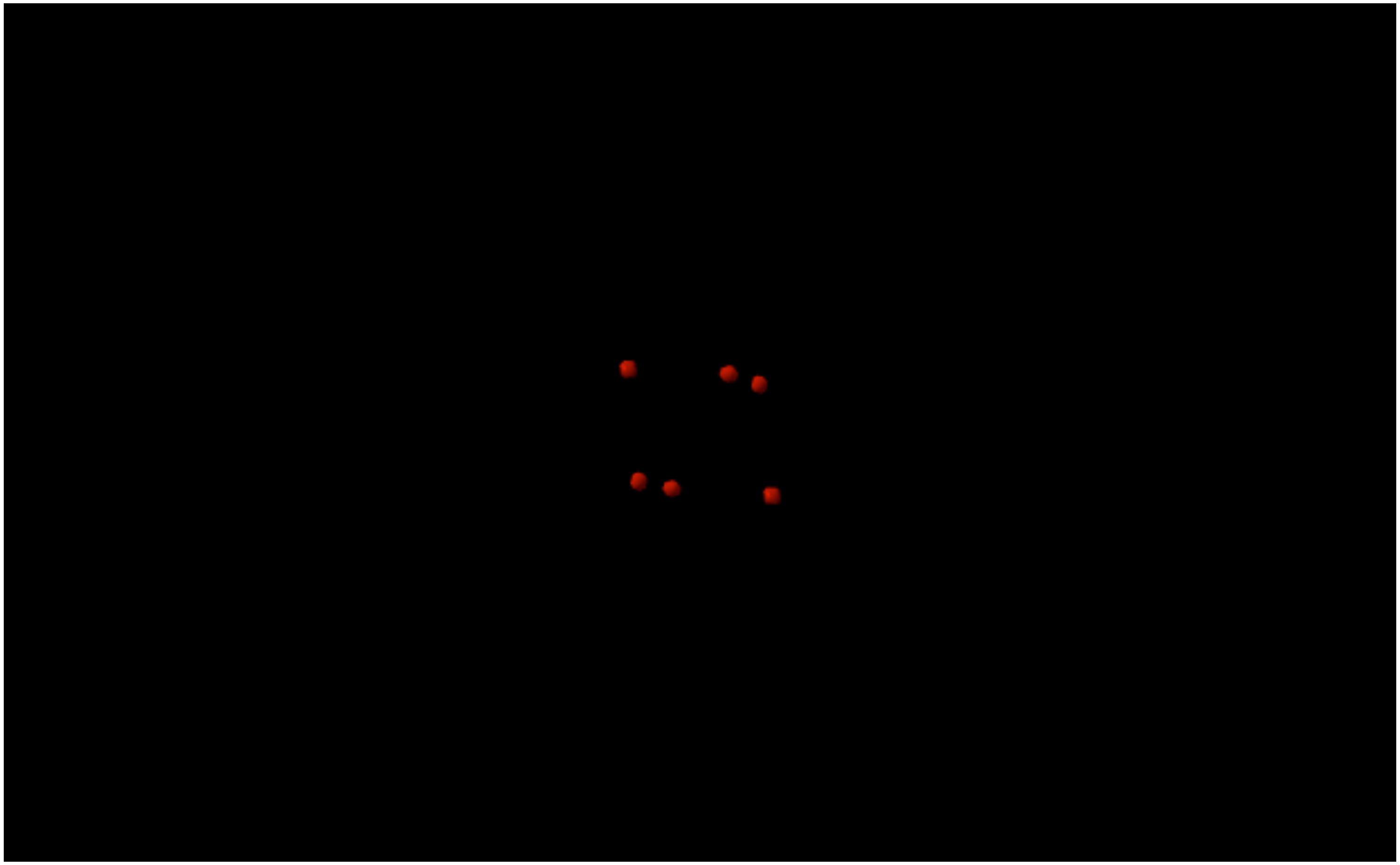

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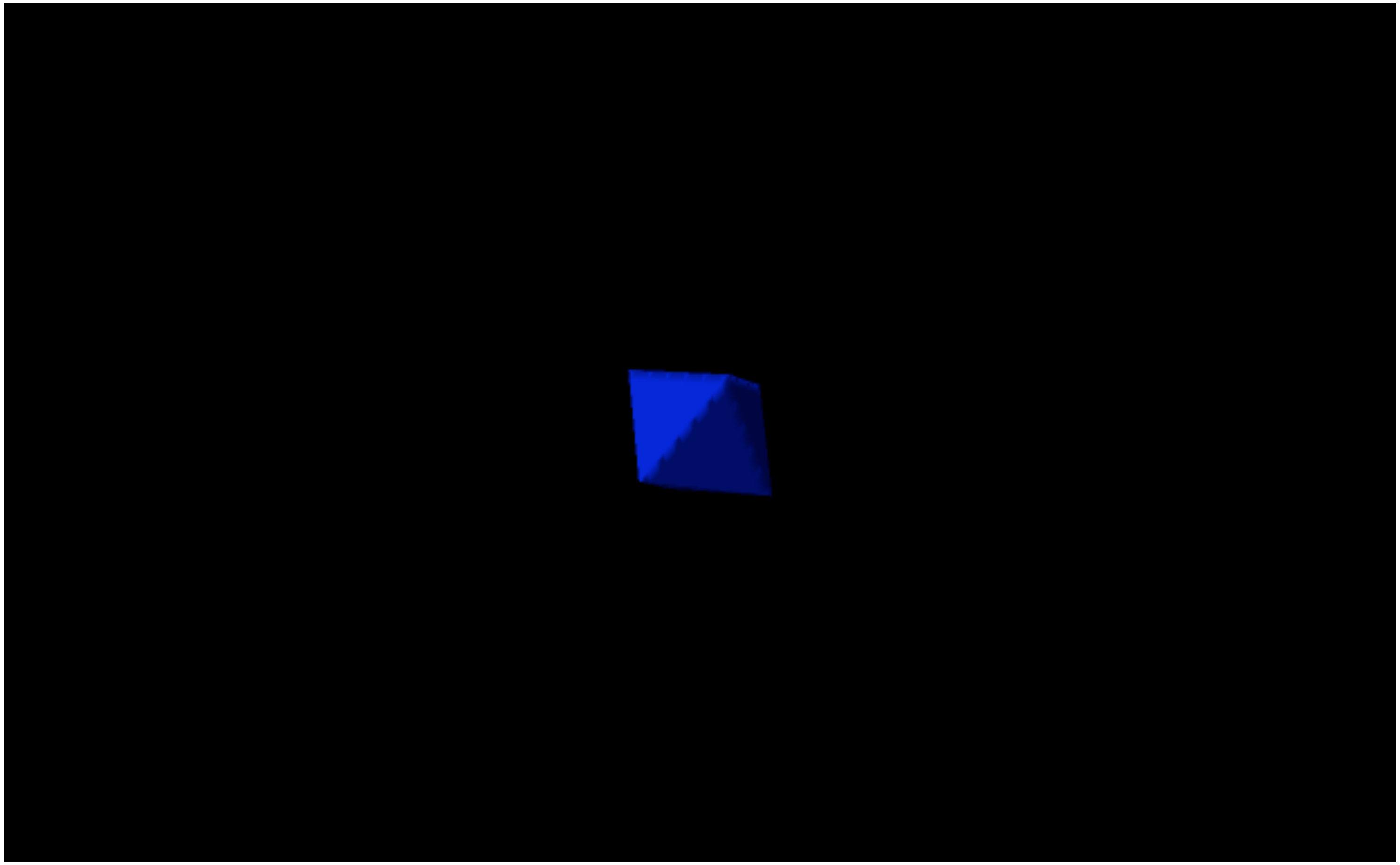

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Moreover, equality holds in either case iff $M = X$, and g is Kähler-Einstein with $\lambda < 0$.

Cauchy-Schwarz argument $\Longrightarrow$

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Einstein $\Longrightarrow$ $\quad \frac{1}{3} {c_1}^2(X) > k$

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So being "very" non-minimal is an obstruction.

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When $n = 2m = 4$, such M are the minimal complex surfaces of general type such that

$$\nexists \mathbb{CP}_1 \overset{\mathcal{O}}{\hookrightarrow} M$$

of homological self-intersection -2.

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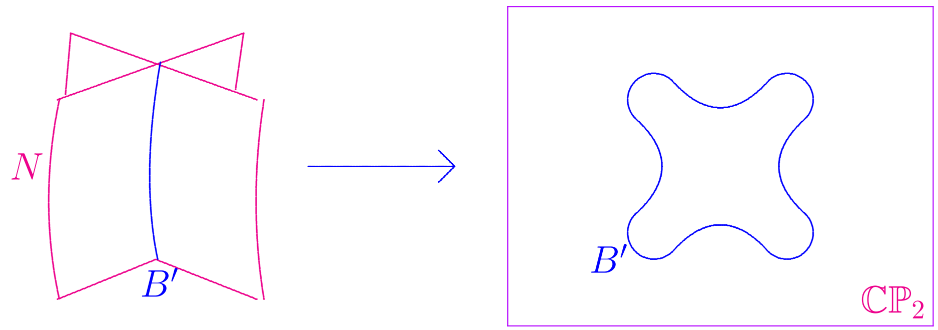

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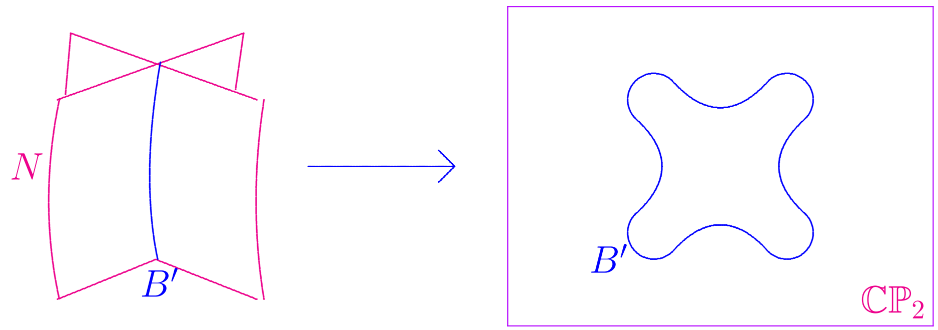
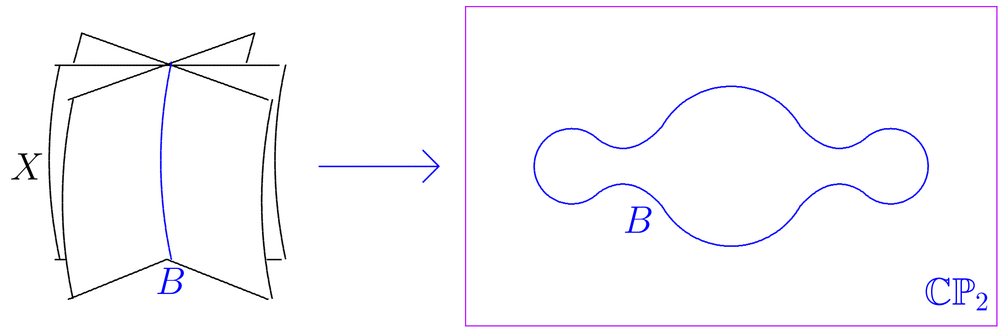


Aubin/Yau $\Longrightarrow$ N carries Einstein metric.

Now let X be a triple cyclic cover $\mathbb{CP}_2$, ramified at a smooth sextic



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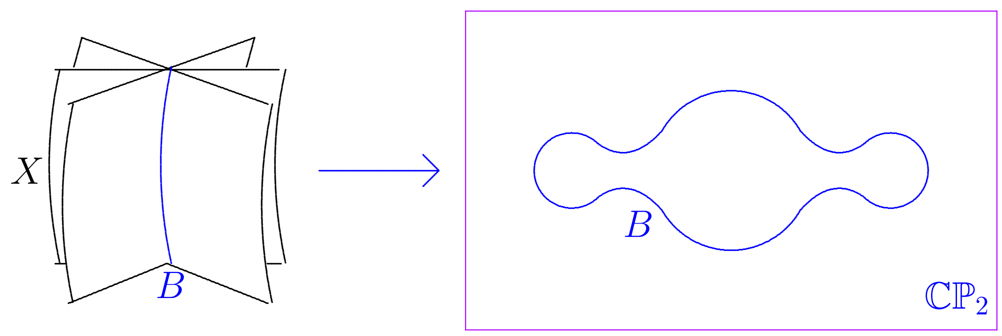

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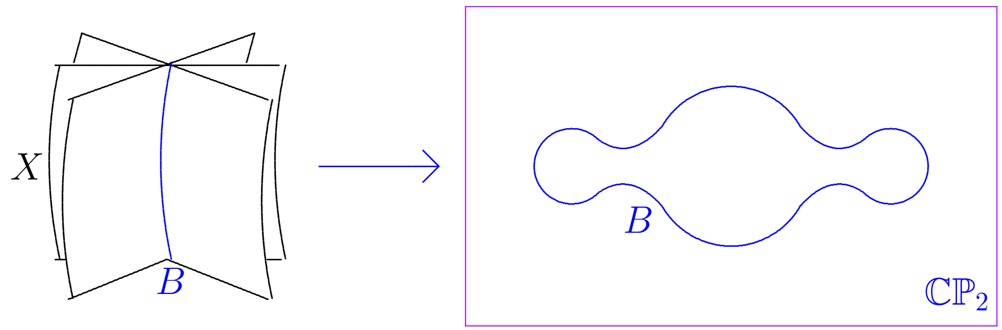

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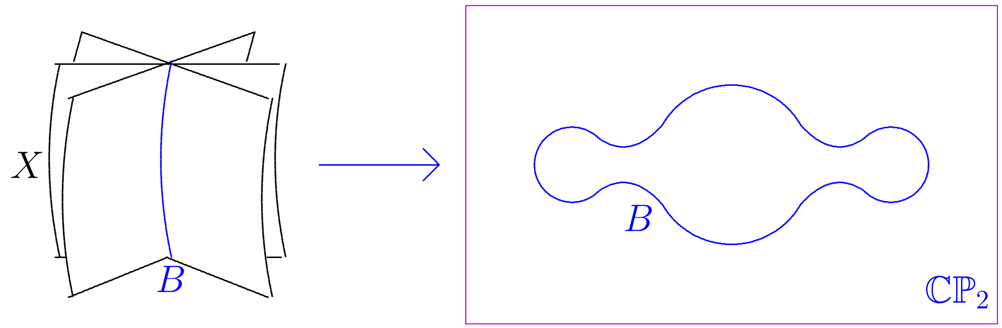
In example:

$${c_1}^2(X) = 3$$

$$k = 1 = {c_1}^2(X)/3$$

X is triple cover $\mathbb{CP}_2$ ramified at sextic



$$M = X \# \overline{\mathbb{CP}}_2 .$$

Theorem B $\Longrightarrow$ *no* Einstein metric on M.

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Moral: Existence depends on diffeotype!

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End, Part III