Curvature Functionals,

Kähler Metrics, &

the Geometry of 4-Manifolds I

Claude LeBrun
Stony Brook University

IHP, December 3, 2012
Let \((M^n, g)\) be a Riemannian \(n\)-manifold, \(p \in M\).
Let \((M^n, g)\) be a Riemannian \(n\)-manifold, \(p \in M\). Metric defines locally shortest curves, called \textit{geodesics}. Following geodesics from \(p\) defines a map

\[
\exp : T_p M \rightarrow M
\]
Let \((M^n, g)\) be a Riemannian \(n\)-manifold, \(p \in M\). Metric defines locally shortest curves, called geodesics. Following geodesics from \(p\) defines a map

\[
\exp : T_p M \rightarrow M
\]

which is a diffeomorphism on a neighborhood of 0:
Let \((M^n, g)\) be a Riemannian \(n\)-manifold, \(p \in M\). Metric defines locally shortest curves, called *geodesics*. Following geodesics from \(p\) defines a map

\[
\exp : T_p M \rightarrow M
\]

which is a diffeomorphism on a neighborhood of 0:

Now choosing \(T_p M \cong \mathbb{R}^n\) via some orthonormal basis gives us special coordinates on \(M\).
In these “geodesic normal” coordinates the metric volume measure is given by

\[ d\mu_g = d\mu_{\text{Euclidean}}, \]
In these “geodesic normal” coordinates the metric volume measure is given by

\[ d\mu_g = \left[ 1 - \right] d\mu_{\text{Euclidean}}, \]
In these “geodesic normal” coordinates the metric volume measure is given by

\[
d\mu_g = \left[ 1 - \frac{1}{6} r_{jk} x^j x^k + \ldots \right] d\mu_{\text{Euclidean}},
\]
In these “geodesic normal” coordinates the metric volume measure is given by

$$d\mu_g = \left[1 - \frac{1}{6} r_{jk} x^j x^k + O(|x|^3)\right] d\mu_{\text{Euclidean}},$$
In these “geodesic normal” coordinates the metric volume measure is given by

\[
d\mu_g = \left[ 1 - \frac{1}{6} r_{jk} x^j x^k + O(|x|^3) \right] d\mu_{\text{Euclidean}},
\]

where \( r \) is the Ricci tensor.
In these “geodesic normal” coordinates the metric volume measure is given by

\[
d\mu_g = \left[ 1 - \frac{1}{6} r_{jk} x^j x^k + O(|x|^3) \right] d\mu_{\text{Euclidean}},
\]

where \( r \) is the \textit{Ricci tensor} \( r_{jk} = \mathcal{R}^i_{\ jik} \).
In these “geodesic normal” coordinates the metric volume measure is given by

\[ d\mu_g = \left[ 1 - \frac{1}{6} r_{jk} x^j x^k + O(|x|^3) \right] d\mu_{\text{Euclidean}}, \]

where \( r \) is the \textit{Ricci tensor} \( r_{jk} = \mathcal{R}^i_{\; jik} \).

The \textit{Ricci curvature}
In these “geodesic normal” coordinates the metric volume measure is given by

\[ d\mu_g = \left[ 1 - \frac{1}{6} r_{jk} x^j x^k + O(|x|^3) \right] d\mu_{\text{Euclidean}}, \]

where \( r \) is the \textit{Ricci tensor} \( r_{jk} = \mathcal{R}^i_{jik} \).

The \textit{Ricci curvature} is by definition the function on the unit tangent bundle.
In these “geodesic normal” coordinates the metric volume measure is given by

\[ d\mu_g = \left[ 1 - \frac{1}{6} r_{jk} x^j x^k + O(|x|^3) \right] d\mu_{\text{Euclidean}}, \]

where \( r \) is the \textit{Ricci tensor} \( r_{jk} = \mathcal{R}^i_{jk} \).

The \textit{Ricci curvature} is by definition the function on the unit tangent bundle

\[ STM = \{ v \in TM \mid g(v, v) = 1 \} \]
In these “geodesic normal” coordinates the metric volume measure is given by

\[ d\mu_g = \left[ 1 - \frac{1}{6} r_{jk} x^j x^k + O(|x|^3) \right] d\mu_{\text{Euclidean}}, \]

where \( r \) is the Ricci tensor \( r_{jk} = \mathcal{R}^i_{\; jik} \).

The Ricci curvature is by definition the function on the unit tangent bundle

\[ STM = \{ v \in TM \mid g(v, v) = 1 \} \]

given by

\[ v \mapsto r(v, v). \]
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature.
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$ r = \lambda g $$

for some constant $\lambda \in \mathbb{R}$. 
**Definition.** A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

“...the greatest blunder of my life!”

— A. Einstein, to G. Gamow
**Definition.** A Riemannian metric $g$ is said to be **Einstein** if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$. 
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

As punishment ...
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

$\lambda$ called Einstein constant.
**Definition.** A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

$\lambda$ called Einstein constant.

Has same sign as the scalar curvature

$$s = r_j^i = \mathcal{R}^{ij}{}_{ij}. $$
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

$\lambda$ called Einstein constant.

Has same sign as the scalar curvature

$$s = r^j_j = \mathcal{R}^{ij}_{ij}.$$ 

$$\frac{\text{vol}_g(B_\varepsilon(p))}{c_n \varepsilon^n} = 1 - s \frac{\varepsilon^2}{6(n+2)} + O(\varepsilon^4)$$
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$. 


**Definition.** A Riemannian metric $g$ is said to be *Einstein* if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.
**Definition.** A Riemannian metric $g$ is said to be *Einstein* if it has constant Ricci curvature — i.e.

$$ r = \lambda g $$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.

$n = 2, 3$: Einstein $\iff$ constant sectional
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.,

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.

$n = 2, 3$: Einstein $\iff$ constant sectional

$n \geq 4$: Einstein $\iff$, $\not\Rightarrow$ constant sectional
**Definition.** A Riemannian metric $g$ is said to be **Einstein** if it has **constant Ricci curvature** — i.e.

$$ r = \lambda g $$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.

**Determined system:**
same number of equations as unknowns.
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.

Determined system:
same number of equations as unknowns.

$g_{jk}$: $\frac{n(n+1)}{2}$ components.

$r_{jk}$: $\frac{n(n+1)}{2}$ components.
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.

Determined system:
same number of equations as unknowns.

$g_{jk}$: $\frac{n(n+1)}{2}$ components.

$r_{jk}$: $\frac{n(n+1)}{2}$ components.

$\mathcal{R}^j_{k\ell m}$: $\frac{n^2(n^2-1)}{12}$ components.
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.

Determined system: same number of equations as unknowns.

Elliptic non-linear PDE after gauge fixing.
**Definition.** A Riemannian metric $g$ is said to be **Einstein** if it has constant Ricci curvature — i.e.

$$ r = \lambda g $$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.

**Determined system:**
same number of equations as unknowns.

**Elliptic non-linear PDE after gauge fixing.**

$$ \Delta x^j = 0 \implies r_{jk} = \frac{1}{2} \Delta g_{jk} + \text{lots.} $$
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Proposition. If $n \geq 3$, a Riemannian $n$-manifold $(M^n, g)$ is Einstein iff the trace-free part of its Ricci tensor vanishes:

$$\hat{r} := r - \frac{s}{n}g = 0.$$
**Definition.** A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

---

**Proposition.** If $n \geq 3$, a Riemannian $n$-manifold $(M^n, g)$ is Einstein iff the trace-free part of its Ricci tensor vanishes:

$$\hat{r} := r - \frac{s}{n}g = 0.$$

Proof. Bianchi identity $\implies \nabla \cdot \hat{r} = (\frac{1}{2} - \frac{1}{n})ds$. 
Main Question (Yamabe). Does every smooth compact simply-connected $n$-manifold admit an Einstein metric?
Main Question (Yamabe). Does every smooth compact simply-connected $n$-manifold admit an Einstein metric?

What we know:
Main Question (Yamabe). Does every smooth compact simply-connected $n$-manifold admit an Einstein metric?

What we know:

• When $n = 2$: Yes! (Riemann)
Main Question (Yamabe). *Does every smooth compact simply-connected $n$-manifold admit an Einstein metric?*

What we know:

- When $n = 2$: Yes! (Riemann)
- When $n = 3$: $\iff$ Poincaré conjecture.
Main Question (Yamabe). Does every smooth compact simply-connected $n$-manifold admit an Einstein metric?

What we know:

- When $n = 2$: Yes! (Riemann)
- When $n = 3$: $\iff$ Poincaré conjecture. Hamilton, Perelman, . . . Yes!
Main Question (Yamabe). Does every smooth compact simply-connected $n$-manifold admit an Einstein metric?

What we know:

- When $n = 2$: Yes! (Riemann)
- When $n = 3$: $\iff$ Poincaré conjecture. Hamilton, Perelman, ... Yes!
- When $n = 4$: No! (Hitchin)
Main Question (Yamabe). Does every smooth compact simply-connected $n$-manifold admit an Einstein metric?

What we know:

- When $n = 2$: Yes! (Riemann)
- When $n = 3$: $\Longleftrightarrow$ Poincaré conjecture. Hamilton, Perelman, ... Yes!
- When $n = 4$: No! (Hitchin)
- When $n = 5$: Yes?? (Boyer-Galicki-Kollár)
Main Question (Yamabe). Does every smooth compact simply-connected $n$-manifold admit an Einstein metric?

What we know:

- When $n = 2$: Yes! (Riemann)
- When $n = 3$: $\iff$ Poincaré conjecture. Hamilton, Perelman, ... Yes!
- When $n = 4$: No! (Hitchin)
- When $n = 5$: Yes?? (Boyer-Galicki-Kollár)
- When $n \geq 6$, wide open. Maybe???
Main Question (Yamabe). *Does every smooth compact simply-connected $n$-manifold admit an Einstein metric?*

What we know:

- When $n = 2$: Yes! (Riemann)
- When $n = 3$: $\iff$ Poincaré conjecture. Hamilton, Perelman, . . . Yes!
- When $n = 4$: No! (Hitchin)
- When $n = 5$: Yes?? (Boyer-Galicki-Kollár)
- When $n \geq 6$, wide open. Maybe???

Dimension 4 is exceptional . . .
Variational Problem:
Variational Problem:

If $M$ smooth compact $n$-manifold,
Variational Problem:

If $M$ smooth compact $n$-manifold, $n \geq 3$, 
Variational Problem:

If $M$ smooth compact $n$-manifold, $n \geq 3$,

$$G_M = \{ \text{smooth metrics } g \text{ on } M \}$$
Variational Problem:

If $M$ smooth compact $n$-manifold, $n \geq 3$, 

$$G_M = \{ \text{smooth metrics } g \text{ on } M \}$$

then Einstein metrics are critical points
Variational Problem:

If $M$ smooth compact $n$-manifold, $n \geq 3$,

$$\mathcal{G}_M = \{\text{smooth metrics } g \text{ on } M\}$$

then Einstein metrics are critical points of the scale-invariant action functional.
Variational Problem:

If $M$ smooth compact $n$-manifold, $n \geq 3$,

$$\mathcal{G}_M = \{ \text{smooth metrics } g \text{ on } M \}$$

then Einstein metrics are critical points of the scale-invariant action functional

$$\mathcal{G}_M \longrightarrow \mathbb{R}$$

$$g \longmapsto \int_M |s_g|^{n/2} d\mu_g$$
Variational Problem:

If $M$ smooth compact $n$-manifold, $n \geq 3$,

$$\mathcal{G}_M = \{ \text{smooth metrics } g \text{ on } M \}$$

then Einstein metrics are critical points of the scale-invariant action functional

$$\mathcal{G}_M \longrightarrow \mathbb{R}$$

$$g \longmapsto \int_M |s_g|^{n/2} d\mu_g$$

Conversely:
Variational Problem:

If $M$ smooth compact $n$-manifold, $n \geq 3$,

$$\mathcal{G}_M = \{ \text{smooth metrics } g \text{ on } M \}$$

then Einstein metrics are critical points of the scale-invariant action functional

$$\mathcal{G}_M \longrightarrow \mathbb{R}$$

$$g \longmapsto \int_M |s_g|^{n/2} d\mu_g$$

Conversely:

Critical points are Einstein
Variational Problem:

If $M$ smooth compact $n$-manifold, $n \geq 3$,

$$
\mathcal{G}_M = \{ \text{smooth metrics } g \text{ on } M \}
$$

then Einstein metrics are critical points of the scale-invariant action functional

$$
\mathcal{G}_M \longrightarrow \mathbb{R}
$$

$$
g \longmapsto \int_M |s_g|^{n/2} d\mu_g
$$

Conversely:

Critical points are Einstein or scalar-flat ($s \equiv 0$).
Variational Problem:

If $M$ smooth compact $n$-manifold, $n \geq 3$, 

$$\mathcal{G}_M = \{ \text{smooth metrics } g \text{ on } M \}$$

then Einstein metrics are critical points of the scale-invariant action functional

$$\mathcal{G}_M \rightarrow \mathbb{R}$$

$$g \mapsto \int_M |s_g|^{n/2} d\mu_g$$

Conversely:

Critical points are Einstein or scalar-flat ($s \equiv 0$).

Try to find Einstein metrics by minimizing?
A Differential-Topological Invariant:
A Differential-Topological Invariant:

\[ \int_M |s_g|^{n/2} d\mu_g \]
A Differential-Topological Invariant:

$$\inf_{g} \int_{M} |s_{g}|^{n/2} d\mu_{g}$$
A Differential-Topological Invariant:

\[ I_s(M) = \inf_g \int_M |s_g|^{n/2} d\mu_g \]
A Differential-Topological Invariant:

\[ I_s(M) = \inf_g \int_M |s_g|^{n/2} d\mu_g \]

**Theorem.** Let \( M \) be a compact simply connected \( n \)-manifold,
A Differential-Topological Invariant:

\[ I_s(M) = \inf_g \int_M |s_g|^{n/2} d\mu_g \]

**Theorem.** Let \( M \) be a compact simply connected \( n \)-manifold, \( n \geq 3 \).
A Differential-Topological Invariant:

$$\mathcal{I}_s(M) = \inf_g \int_M |s_g|^{n/2} d\mu_g$$

**Theorem.** Let $M$ be a compact simply connected $n$-manifold, $n \geq 3$. If $n \neq 4$, 
A Differential-Topological Invariant:

\[ \mathcal{I}_s(M) = \inf_g \int_M |s_g|^{n/2} d\mu_g \]

**Theorem.** Let \( M \) be a compact simply connected \( n \)-manifold, \( n \geq 3 \). If \( n \neq 4 \), \( \mathcal{I}_s(M) = 0 \).
A Differential-Topological Invariant:

\[ I_s(M) = \inf \int_M |s_g|^{n/2} d\mu_g \]

**Theorem.** Let \( M \) be a compact simply connected \( n \)-manifold, \( n \geq 3 \). If \( n \neq 4 \), \( I_s(M) = 0 \).

**Theorem.** There exist compact simply connected 4-manifolds \( M_j \)
A Differential-Topological Invariant:

\[ \mathcal{I}_s(M) = \inf_g \int_M |s_g|^{n/2} d\mu_g \]

**Theorem.** Let \( M \) be a compact simply connected \( n \)-manifold, \( n \geq 3 \). If \( n \neq 4 \), \( \mathcal{I}_s(M) = 0 \).

**Theorem.** There exist compact simply connected 4-manifolds \( M_j \) with \( \mathcal{I}_s(M_j) \to +\infty \).
A Differential-Topological Invariant:

\[
\mathcal{I}_s(M) = \inf_g \int_M |s_g|^{n/2} d\mu_g
\]

**Theorem.** Let \( M \) be a compact simply connected \( n \)-manifold, \( n \geq 3 \). If \( n \neq 4 \), \( \mathcal{I}_s(M) = 0 \).

**Theorem.** There exist compact simply connected 4-manifolds \( M_j \) with \( \mathcal{I}_s(M_j) \to +\infty \).

Moreover, can choose \( M_j \).
A Differential-Topological Invariant:

\[ \mathcal{I}_s(M) = \inf_g \int_M |s_g|^{n/2} d\mu_g \]

**Theorem.** Let \( M \) be a compact simply connected \( n \)-manifold, \( n \geq 3 \). If \( n \neq 4 \), \( \mathcal{I}_s(M) = 0 \).

**Theorem.** There exist compact simply connected 4-manifolds \( M_j \) with \( \mathcal{I}_s(M_j) \to +\infty \).

Moreover, can choose \( M_j \) such that

\[ \mathcal{I}_s(M_j) = \inf_g \int_{M_j} |s_g|^2 d\mu_g \]
A Differential-Topological Invariant:

\[ \mathcal{I}_s(M) = \inf_g \int_M |s_g|^{n/2} d\mu_g \]

**Theorem.** Let \( M \) be a compact simply connected \( n \)-manifold, \( n \geq 3 \). If \( n \neq 4 \), \( \mathcal{I}_s(M) = 0 \).

**Theorem.** There exist compact simply connected 4-manifolds \( M_j \) with \( \mathcal{I}_s(M_j) \to +\infty \).

Moreover, can choose \( M_j \) such that

\[ \mathcal{I}_s(M_j) = \inf_g \int_{M_j} |s_g|^2 d\mu_g \]

is realized by an *Einstein* metric \( g_j \) with \( \lambda < 0 \).
Four Dimensions is Exceptional
Four Dimensions is Exceptional

When $n = 4$, existence for Einstein depends delicately on smooth structure.
Four Dimensions is Exceptional

When $n = 4$, existence for Einstein depends delicately on smooth structure.

There are topological 4-manifolds which admit an Einstein metric for one smooth structure, but not for others.
Four Dimensions is Exceptional

When $n = 4$, existence for Einstein depends delicately on smooth structure.

There are topological 4-manifolds which admit an Einstein metric for one smooth structure, but not for others.

But might allow for geometrization of 4-manifolds by decomposition into Einstein and collapsed pieces.
Four Dimensions is Exceptional

When $n = 4$, existence for Einstein depends delicately on smooth structure.

There are topological 4-manifolds which admit an Einstein metric for one smooth structure, but not for others.

But might allow for geometrization of 4-manifolds by decomposition into Einstein and collapsed pieces.

Enough rigidity apparently still holds in dimension four to call this a geometrization.
Four Dimensions is Exceptional

When $n = 4$, existence for Einstein depends delicately on smooth structure.

There are topological 4-manifolds which admit an Einstein metric for one smooth structure, but not for others.

But might allow for geometrization of 4-manifolds by decomposition into Einstein and collapsed pieces.

Enough rigidity apparently still holds in dimension four to call this a geometrization.

By contrast, high-dimensional Einstein metrics too common; have little to do with geometrization.
Dimension $\leq 3$: 
Dimension $\leq 3$:

Einstein’s equations are “locally trivial.”
Dimension $\leq 3$:

Einstein’s equations are “locally trivial.”

Einstein metrics have constant sectional curvature.
Dimension $\leq 3$:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

$\implies$ If $M^3$ carries Einstein metric, $\pi_2(M) = 0$. 
Dimension $\leq 3$:

Einstein’s equations are “locally trivial:”
Einstein metrics have constant sectional curvature.

$\implies$ If $M^3$ carries Einstein metric, $\pi_2(M) = 0$.
$\implies$ Existence obstructed for connect sums $M^3 \# N^3$. 
Dimension $\leq 3$:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

$\implies$ If $M^3$ carries Einstein metric, $\pi_2(M) = 0$.

$\implies$ Existence obstructed for connect sums $M^3 \# N^3$.

Connected sum $\#$:
Dimension $\leq 3$:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

$\implies$ If $M^3$ carries Einstein metric, $\pi_2(M) = 0$.

$\implies$ Existence obstructed for connect sums $M^3 \# N^3$.

Connected sum #: 

![Connected sum diagram]
Dimension $\leq 3$:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

$\implies$ If $M^3$ carries Einstein metric, $\pi_2(M) = 0$.

$\implies$ Existence obstructed for connect sums $M^3 \# N^3$.

Connected sum $\#$:
Dimension $\leq 3$:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

$\implies$ If $M^3$ carries Einstein metric, $\pi_2(M) = 0$.

$\implies$ Existence obstructed for connect sums $M^3 \# N^3$.

Connected sum $\#$:
Dimension $\leq 3$:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

$\implies$ If $M^3$ carries Einstein metric, $\pi_2(M) = 0$.

$\implies$ Existence obstructed for connect sums $M^3 \# N^3$.

Connected sum #: 

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{blue.png} \\
\includegraphics[width=0.2\textwidth]{red.png}
\end{array}
\]
Dimension $\leq 3$:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

$\implies$ If $M^3$ carries Einstein metric, $\pi_2(M) = 0$.

$\implies$ Existence obstructed for connect sums $M^3 \# N^3$.

Connected sum $\#$:
Dimension $\leq 3$:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

$\implies$ If $M^3$ carries Einstein metric, $\pi_2(M) = 0$.

$\implies$ Existence obstructed for connect sums $M^3 \# N^3$.

Connected sum $\#$:
Dimension $\leq 3$:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

$\implies$ If $M^3$ carries Einstein metric, $\pi_2(M) = 0$.

$\implies$ Existence obstructed for connect sums $M^3 \# N^3$. 
Dimension $\leq 3$:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

$\implies$ If $M^3$ carries Einstein metric, $\pi_2(M) = 0$.

$\implies$ Existence obstructed for connect sums $M^3 \# N^3$.

Ricci flow pinches off $S^2$ necks.
Dimension $\leq 3$:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

$\implies$ If $M^3$ carries Einstein metric, $\pi_2(M) = 0$.

$\implies$ Existence obstructed for connect sums $M^3 \# N^3$.

Ricci flow pinches off $S^2$ necks.

First step in geometrization:
Dimension $\leq 3$:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

$\implies$ If $M^3$ carries Einstein metric, $\pi_2(M) = 0$.

$\implies$ Existence obstructed for connect sums $M^3 \# N^3$.

Ricci flow pinches off $S^2$ necks.

First step in geometrization:

Prime Decomposition.
Dimension $\geq 5$: 
Dimension $\geq 5$:

There are many known Einstein metrics on $S^n$, $n \geq 5$ which do not have constant curvature.
Dimension $\geq 5$:

There are many known Einstein metrics on $S^n$, $n \geq 5$ which do not have constant curvature.

The moduli space of Einstein metrics on $S^2 \times S^3$ has infinitely many connected components.
Dimension $\geq 5$:

There are many known Einstein metrics on $S^n$, $n \geq 5$ which do not have constant curvature.

The moduli space of Einstein metrics on $S^2 \times S^3$ has infinitely many connected components. Unit-volume Einstein metrics exist for sequence of $\lambda \to 0^+$. 
Dimension $\geq 5$:

There are many known Einstein metrics on $S^n$, $n \geq 5$ which do not have constant curvature.

The moduli space of Einstein metrics on $S^2 \times S^3$ has infinitely many connected components. Unit-volume Einstein metrics exist for sequence of $\lambda \to 0^+$.  

(Böhm, Wang, Ziller, et al.)
Dimension $\geq 5$:

There are many known Einstein metrics on $S^n$, $n \geq 5$ which do not have constant curvature.

The moduli space of Einstein metrics on $S^2 \times S^3$ has infinitely many connected components. Unit-volume Einstein metrics exist for sequence of $\lambda \to 0^+$.

(Böhm, Wang, Ziller, et al.)

Same behavior for connected sums

$$(S^2 \times S^3) \# \cdots \# (S^2 \times S^3).$$
Dimension \( \geq 5 \):

There are many known Einstein metrics on \( S^n, n \geq 5 \) which do not have constant curvature.

The moduli space of Einstein metrics on \( S^2 \times S^3 \) has infinitely many connected components. Unit-volume Einstein metrics exist for sequence of \( \lambda \to 0^+ \).

(Böhm, Wang, Ziller, et al.)

Same behavior for connected sums

\((S^2 \times S^3) \# \cdots \# (S^2 \times S^3)\).

Unit-volume Einstein metrics exist for sequence of \( \lambda \to 0^+ \).
Dimension $\geq 5$:

There are many known Einstein metrics on $S^n$, $n \geq 5$ which do not have constant curvature.

The moduli space of Einstein metrics on $S^2 \times S^3$ has infinitely many connected components. Unit-volume Einstein metrics exist for sequence of $\lambda \to 0^+$.

(Böhm, Wang, Ziller, et al.)

Same behavior for connected sums

$$(S^2 \times S^3) \# \cdots \# (S^2 \times S^3).$$

Unit-volume Einstein metrics exist for sequence of $\lambda \to 0^+$. (Van Coevering, Kollár)
Dimension $\geq 5$:

There are many known Einstein metrics on $S^n$, $n \geq 5$ which do not have constant curvature.

The moduli space of Einstein metrics on $S^2 \times S^3$ has infinitely many connected components. Unit-volume Einstein metrics exist for sequence of $\lambda \to 0^+$.

(Böhm, Wang, Ziller, et al.)

Same behavior for connected sums

$$(S^2 \times S^3) \# \cdots \# (S^2 \times S^3).$$

Unit-volume Einstein metrics exist for sequence of $\lambda \to 0^+$.

(Van Coevering, Kollár)

Related results for exotic 7-spheres, many other manifolds.
Dimension $\geq 5$:

There are many known Einstein metrics on $S^n$, $n \geq 5$ which do not have constant curvature.

The moduli space of Einstein metrics on $S^2 \times S^3$ has infinitely many connected components. Unit-volume Einstein metrics exist for sequence of $\lambda \to 0^+$. (Böhm, Wang, Ziller, et al.)

Same behavior for connected sums

$$(S^2 \times S^3) \# \cdots \# (S^2 \times S^3).$$

Unit-volume Einstein metrics exist for sequence of $\lambda \to 0^+$. (Van Coevering, Kollár)

Related results for exotic 7-spheres, many other manifolds. (Boyer, Galicki, Kollár, et al.)
Dimension 4:
Dimension 4:

**Theorem** (Berger). *Any Einstein metric on 4-torus $T^4$ is flat.*
Dimension 4:

**Theorem (Berger).** *Any Einstein metric on 4-torus $T^4$ is flat.*

$\implies$ Moduli space of Einstein metrics is connected.
Dimension 4:

**Theorem** (Berger). *Any Einstein metric on 4-torus $T^4$ is flat.*

$\implies$ Moduli space of Einstein metrics is connected.

**Theorem** (Hitchin). *Any Einstein metric on $K3$ is hyper-Kähler.*
\( K3 = \) Kummer-Kähler-Kodaira manifold.
$K3 = \text{Kummer-Kähler-Kodaira manifold}.$

Simply connected complex surface with $c_1 = 0.$
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Simply connected complex surface with $c_1 = 0$.

Only one deformation type.
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Simply connected complex surface with $c_1 = 0$.

Only one \textit{diffeomorphism} type.
$K3 = \text{Kummer-Kähler-Kodaira manifold}.$

Simply connected complex surface with $c_1 = 0$.

Only one diffeomorphism type.

Spin, $\chi = 24$, $\tau = -16$. 
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Diffeomorphic to quartic in $\mathbb{CP}_3$

$$x^4 + y^4 + z^4 + w^4 = 0$$
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Diffeomorphic to quartic in $\mathbb{CP}_3$

$$x^4 + y^4 + z^4 + w^4 = 0$$
Dimension 4:

**Theorem (Berger).** Any Einstein metric on $4$-torus $T^4$ is flat.

$\implies$ Moduli space of Einstein metrics is connected.

**Theorem (Hitchin).** Any Einstein metric on $K3$ is hyper-Kähler.
Dimension 4:

**Theorem** (Berger). *Any Einstein metric on 4-torus $T^4$ is flat.*

$\implies$ Moduli space of Einstein metrics is connected.

**Theorem** (Hitchin). *Any Einstein metric on K3 is hyper-Kähler.*

$\implies$ Moduli space of Einstein metrics is connected.
Dimension 4:

**Theorem (Berger).** Any Einstein metric on 4-torus $T^4$ is flat.

$\implies$ Moduli space of Einstein metrics is connected.

**Theorem (Hitchin).** Any Einstein metric on $K3$ is hyper-Kähler.

$\implies$ Moduli space of Einstein metrics is connected.

(Kodaira, Yau, Siu, et al.)
Dimension 4:

**Theorem (Berger).** Any Einstein metric on 4-torus $T^4$ is flat.

$\implies$ Moduli space of Einstein metrics is connected.

**Theorem (Hitchin).** Any Einstein metric on $K3$ is hyper-Kähler.

$\implies$ Moduli space of Einstein metrics is connected.

(Kodaira, Yau, Siu, et al.)

**Theorem (Besson-Courtois-Gallot).** There is only one Einstein metric on compact hyperbolic 4-manifold $\mathcal{H}^4/\Gamma$, up to scale and diffeos.
Dimension 4:

**Theorem (Berger).** Any Einstein metric on 4-torus $T^4$ is flat.

$\Rightarrow$ Moduli space of Einstein metrics is connected.

**Theorem (Hitchin).** Any Einstein metric on $K3$ is hyper-Kähler.

$\Rightarrow$ Moduli space of Einstein metrics is connected.

(Kodaira, Yau, Siu, et al.)

**Theorem (Besson-Courtois-Gallot).** There is only one Einstein metric on compact hyperbolic 4-manifold $\mathcal{H}^4/\Gamma$, up to scale and diffeos.

**Theorem (L).** There is only one Einstein metric on compact complex-hyperbolic 4-manifold $\mathcal{CH}_2/\Gamma$, up to scale and diffeos.
Why is Dimension Four Exceptional?
Why is Dimension Four Exceptional?

The Lie group $SO(4)$ is not simple:
Why is Dimension Four Exceptional?

The Lie group $SO(4)$ is \textit{not simple}:

$$so(4) \cong so(3) \oplus so(3).$$
Why is Dimension Four Exceptional?

The Lie group $SO(4)$ is not simple:

$$so(4) \cong so(3) \oplus so(3).$$

On oriented $(M^4, g)$,
Why is Dimension Four Exceptional?

The Lie group $SO(4)$ is not simple:

$$so(4) \cong so(3) \oplus so(3).$$

On oriented $(M^4, g)$, $\implies$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$
Why is Dimension Four Exceptional?

The Lie group $SO(4)$ is not simple:

$$so(4) \cong so(3) \oplus so(3).$$

On oriented $(M^4, g)$,

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^\pm$ are $(\pm 1)$-eigenspaces of

$$\star : \Lambda^2 \to \Lambda^2,$$

$$\star^2 = 1.$$
Why is Dimension Four Exceptional?

The Lie group $SO(4)$ is not simple:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented $(M^4, g)$, $\implies$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^\pm$ are $(\pm 1)$-eigenspaces of

$$\star : \Lambda^2 \to \Lambda^2,$$

$$\star^2 = 1.$$

$\Lambda^+$ self-dual 2-forms.
Why is Dimension Four Exceptional?

The Lie group $SO(4)$ is \textit{not simple}:

$$so(4) \cong so(3) \oplus so(3).$$

On oriented $(M^4, g)$, $\implies$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^\pm$ are $(\pm 1)$-eigenspaces of

$$\star : \Lambda^2 \to \Lambda^2,$$

$$\star^2 = 1.$$

$\Lambda^+$ self-dual 2-forms.

$\Lambda^-$ anti-self-dual 2-forms.
Importance?
Importance?

Curvature is a bundle-valued 2-form!
Importance?

Curvature is a bundle-valued 2-form!

Vector-bundle-with-connection $(E, \nabla)$
Importance?

Curvature is a bundle-valued 2-form!

Vector-bundle-with-connection \((E, \nabla)\) over oriented Riemannian \((M^4, g)\)
Importance?

Curvature is a bundle-valued 2-form!

Vector-bundle-with-connection \((E, \nabla)\) over oriented Riemannian \((M^4, g)\) has curvature

\[
F_{\nabla} = F^+ + F^-
\]
Importance?

Curvature is a bundle-valued $2$-form!

Vector-bundle-with-connection $(E, \nabla)$ over oriented Riemannian $(M^4, g)$ has curvature

$$F_\nabla = F^+ + F^-$$

where $F^\pm \in \Lambda^\pm \otimes \text{End}(E)$. 
Importance?

Curvature is a bundle-valued 2-form!

Vector-bundle-with-connection \((E, \nabla)\) over oriented Riemannian \((M^4, g)\) has curvature

\[ F_{\nabla} = F^+ + F^- \]

where \( F^\pm \in \Lambda^\pm \otimes \text{End}(E) \).

If \( F^- = 0 \),
Importance?

Curvature is a bundle-valued 2-form!

Vector-bundle-with-connection \((E, \nabla)\) over oriented Riemannian \((M^4, g)\) has curvature

\[ F_{\nabla} = F^+ + F^- \]

where \( F^\pm \in \Lambda^\pm \otimes \text{End}(E) \).

If \( F^- = 0 \), so that \( F_{\nabla} = F^+ \),
Importance?

Curvature is a bundle-valued 2-form!

Vector-bundle-with-connection \((E, \nabla)\) over oriented Riemannian \((M^4, g)\) has curvature

\[ F_\nabla = F^+ + F^- \]

where \(F^\pm \in \Lambda^\pm \otimes \text{End}(E)\).

If \(F^- = 0\), so that \(F_\nabla = F^+\), \nabla is called self-dual
Importance?

Curvature is a bundle-valued 2-form!

Vector-bundle-with-connection \((E, \nabla)\) over oriented Riemannian \((M^4, g)\) has curvature

\[ F_\nabla = F^+ + F^- \]

where \(F^\pm \in \Lambda^\pm \otimes \text{End}(E)\).

If \(F^- = 0\), so that \(F_\nabla = F^+\), \nabla is called self-dual (SD).
Importance?

Curvature is a bundle-valued 2-form!

Vector-bundle-with-connection \((E, \nabla)\) over oriented Riemannian \((M^4, g)\) has curvature

\[
F_{\nabla} = F^+ + F^-
\]

where \(F^\pm \in \Lambda^\pm \otimes \text{End}(E)\).

If \(F^- = 0\), so that \(F_{\nabla} = F^+\), \(\nabla\) is called self-dual (SD).

Donaldson:
Importance?

Curvature is a bundle-valued 2-form!

Vector-bundle-with-connection \((E, \nabla)\) over oriented Riemannian \((M^4, g)\) has curvature

\[ F_\nabla = F^+ + F^- \]

where \(F^\pm \in \Lambda^\pm \otimes \text{End}(E)\).

If \(F^- = 0\), so that \(F_\nabla = F^+\), \(\nabla\) is called self-dual (SD).

Donaldson: moduli spaces of SD connections
Importance?

Curvature is a bundle-valued 2-form!

Vector-bundle-with-connection \((E, \nabla)\) over oriented Riemannian \((M^4, g)\) has curvature

\[ F_\nabla = F^+ + F^- \]

where \(F^\pm \in \Lambda^\pm \otimes \text{End}(E)\).

If \(F^- = 0\), so that \(F_\nabla = F^+\), \(\nabla\) is called self-dual (SD).

Donaldson: moduli spaces of SD connections \(\implies\) differential topological invariants of \(M^4\).
Importance?

Curvature is a bundle-valued 2-form!

Vector-bundle-with-connection \((E, \nabla)\) over oriented Riemannian \((M^4, g)\) has curvature

\[
F_\nabla = F^+ + F^-
\]

where \(F^\pm \in \Lambda^\pm \otimes \text{End}(E)\).

If \(F^- = 0\), so that \(F_\nabla = F^+\), \(\nabla\) is called self-dual (SD).

Donaldson: moduli spaces of SD connections \(\implies\) differential topological invariants of \(M^4\).

These lectures will instead emphasize Seiberg-Witten invariants.
Importance?

Curvature is a bundle-valued 2-form!

Vector-bundle-with-connection \((E, \nabla)\) over oriented Riemannian \((M^4, g)\) has curvature

\[
F_{\nabla} = F^+ + F^-
\]

where \(F^\pm \in \Lambda^\pm \otimes \text{End}(E)\).

If \(F^- = 0\), so that \(F_{\nabla} = F^+\), \(\nabla\) is called self-dual (SD).

Donaldson: moduli spaces of SD connections \(\implies\) differential topological invariants of \(M^4\).

These lectures will instead emphasize Seiberg-Witten invariants. (Interaction w/ Riemannian geometry.)
Why is Dimension Four Exceptional?

The Lie group $SO(4)$ is not simple:

$$so(4) \cong so(3) \oplus so(3).$$

On oriented $(M^4, g)$, $\implies$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$
Riemann curvature of $g$

$\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$
Riemann curvature of $g$

$$\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$$

splits into 4 irreducible pieces:
Riemann curvature of $g$ 

$\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$

splits into 4 irreducible pieces:

$$
\mathcal{R} = \begin{pmatrix}
W_+ + \frac{s}{12} & \hat{r} \\
\hat{r} & W_- + \frac{s}{12}
\end{pmatrix}
$$
Riemann curvature of $g$

$$\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$$

splits into 4 irreducible pieces:

<table>
<thead>
<tr>
<th>$\Lambda^+$</th>
<th>$\Lambda^+$</th>
<th>$\Lambda^-$</th>
<th>$\Lambda^-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_+ + \frac{s}{12}$</td>
<td>$\hat{r}$</td>
<td>$\hat{r}$</td>
<td>$W_- + \frac{s}{12}$</td>
</tr>
</tbody>
</table>
Riemann curvature of $g$

\[ \mathcal{R} : \Lambda^2 \rightarrow \Lambda^2 \]

splits into 4 irreducible pieces:

\[
\begin{array}{cc}
\Lambda^+ & \Lambda^{**} \\
W_+ + \frac{s}{12} & \cdot \\
\Lambda^- & \cdot \\
& W_- + \frac{s}{12}
\end{array}
\]

where

- $s = \text{scalar curvature}$
- $\cdot = \text{trace-free Ricci curvature}$
- $W_+ = \text{self-dual Weyl curvature}$
- $W_- = \text{anti-self-dual Weyl curvature}$
Riemann curvature of $g$

$$\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$$

splits into 4 irreducible pieces:

$$\begin{array}{cc}
\Lambda^+ & \Lambda^{++} \\
W_+ + \frac{s}{12} & \hat{\rho} \\
\Lambda^- & \hat{\rho} \\
& W_- + \frac{s}{12}
\end{array}$$

where

$s$ = scalar curvature

$\hat{\rho}$ = trace-free Ricci curvature

$W_+$ = self-dual Weyl curvature ($conformally$ $invariant$)

$W_-$ = anti-self-dual Weyl curvature
Thus $(M^4, g)$ Einstein $\iff$
Thus \((M^4, g)\) Einstein \iff \\
\(\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2\)
Thus \((M^4, g)\) Einstein \iff \(\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2\)

commutes with
Thus \((M^4, g)\) Einstein \iff \(\mathcal{R} : \Lambda^2 \to \Lambda^2\) commutes with \(\star : \Lambda^2 \to \Lambda^2\):
Thus \((M^4, g)\) Einstein \iff 

\[
\mathcal{R} : \Lambda^2 \to \Lambda^2
\]

commutes with

\[
\ast : \Lambda^2 \to \Lambda^2 :
\]

\[
\mathcal{R} = \begin{pmatrix}
W_+ + \frac{s}{12} & \hat{r} \\
\hat{r} & W_- + \frac{s}{12}
\end{pmatrix}
\]
Thus \((M^4, g)\) Einstein \(\iff\)

\[ \mathcal{R} : \Lambda^2 \rightarrow \Lambda^2 \]

commutes with

\[ \star : \Lambda^2 \rightarrow \Lambda^2 : \]

\[
\mathcal{R} = \begin{pmatrix}
W_+ + \frac{s}{12} & 0 \\
0 & W_- + \frac{s}{12}
\end{pmatrix}.
\]
Thus \( (M^4, g) \) Einstein ⇐⇒

\[
\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2
\]

commutes with

\[
\star : \Lambda^2 \rightarrow \Lambda^2 :
\]

\[
\mathcal{R} = \begin{pmatrix}
W_+ + \frac{s}{12} & \hat{r} \\
\hat{r} & W_- + \frac{s}{12}
\end{pmatrix}.
\]
Thus $(M^4, g)$ Einstein $\iff$

\[ \mathcal{R} : \Lambda^2 \rightarrow \Lambda^2 \]

commutes with

\[ \star : \Lambda^2 \rightarrow \Lambda^2 : \]

\[
\mathcal{R} = \begin{pmatrix}
W_+ + \frac{s}{12} & 0 \\
0 & W_- + \frac{s}{12}
\end{pmatrix}.
\]
Thus \((M^4, g)\) Einstein \iff \(\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2\) commutes with

\[\star : \Lambda^2 \rightarrow \Lambda^2 :\]

\[
\mathcal{R} = \begin{pmatrix}
W + + \frac{s}{12} & \mathring{r} \\
\mathring{r} & W - + \frac{s}{12}
\end{pmatrix}.
\]
Thus \((M^4, g)\) Einstein \iff \(\mathcal{R} : \Lambda^2 \to \Lambda^2\) commutes with
\[
\star : \Lambda^2 \to \Lambda^2
\]
\[
\mathcal{R} = \begin{pmatrix}
W_+ + \frac{s}{12} & 0 \\
0 & W_- + \frac{s}{12}
\end{pmatrix}.
\]
Thus \((M^4, g)\) Einstein \iff \(R : \Lambda^2 \to \Lambda^2\) commutes with \(\star : \Lambda^2 \to \Lambda^2\) :

\[
R = \begin{pmatrix}
W_+ + \frac{s}{12} & \hat{r} \\
\hat{r} & W_- + \frac{s}{12}
\end{pmatrix}.
\]
Thus \((M^4, g)\) Einstein \iff\n
\[ \mathcal{R} : \Lambda^2 \rightarrow \Lambda^2 \]

commutes with \n
\[ \star : \Lambda^2 \rightarrow \Lambda^2 : \]

\[
\mathcal{R} = \begin{pmatrix}
W_+ + \frac{s}{12} & 0 \\
0 & W_- + \frac{s}{12}
\end{pmatrix}.
\]
Corollary. A Riemannian 4-manifold \((M, g)\) is Einstein \iff\ section curvatures are equal for any pair of perpendicular 2-planes.
Corollary. A Riemannian 4-manifold \((M, g)\) is Einstein \iff sectional curvatures are equal for any pair of perpendicular 2-planes.
Corollary. A Riemannian 4-manifold $(M, g)$ is Einstein $\iff$ sectional curvatures are equal for any pair of perpendicular 2-planes.
**Corollary.** A Riemannian 4-manifold \((M, g)\) is Einstein \iff sectional curvatures are equal for any pair of perpendicular 2-planes.

\[ K(P) = K(P^\perp) \]
We’ve seen that it is interesting to consider

\[ \mathcal{G}_M \longrightarrow \mathbb{R} \]

\[ g \longmapsto \int_M |s_g|^2 d\mu_g \]

for metrics on \( M^4 \).
We’ve seen that it is interesting to consider

\[ G_M \rightarrow \mathbb{R} \]

\[ g \mapsto \int_M |s_g|^2 d\mu_g \]

for metrics on \( M^4 \).

But also natural and interesting to consider

\[ g \mapsto \int_M |r|^2_g d\mu_g \]

or

\[ g \mapsto \int_M |\mathcal{R}|^2_g d\mu_g \]
Polynomial Curvature Functionals
Polynomial Curvature Functionals

\[ G_M \rightarrow \mathbb{R} \]

\[ g \mapsto \int_M P(\mathcal{R}_g) d\mu_g \]
Polynomial Curvature Functionals

\[ G_M \longrightarrow \mathbb{R} \]

\[ g \longmapsto \int_M P(\mathcal{R}_g) d\mu_g \]

where \( P(\mathcal{R}) \) is \( SO(4) \)-invariant polynomial function of curvature.
Polynomial Curvature Functionals

\[ \mathcal{G}_M \longrightarrow \mathbb{R} \]

\[ g \longmapsto \int_M P(\mathcal{R}_g) \, d\mu_g \]

where \( P(\mathcal{R}) \) is \( SO(4) \)-invariant polynomial function of curvature.

Scale invariance \( \iff \) \( P \) quadratic.
Polynomial Curvature Functionals

\[ G_M \rightarrow \mathbb{R} \]

\[ g \longmapsto \int_M P(\mathcal{R}_g) d\mu_g \]

where \( P(\mathcal{R}) \) is \( SO(4) \)-invariant polynomial function of curvature.

Scale invariance \( \implies P \) quadratic.

Any such \( P(\mathcal{R}) \) is linear combinations of
Polynomial Curvature Functionals

\[ G_M \longrightarrow \mathbb{R} \]

\[ g \longmapsto \int_M P(\mathcal{R}_g) d\mu_g \]

where \( P(\mathcal{R}) \) is \( SO(4) \)-invariant polynomial function of curvature.

Scale invariance \( \implies \) \( P \) quadratic.

Any such \( P(\mathcal{R}) \) is linear combinations of

\[ s^2, \quad \vert \dot{r} \vert^2, \quad \vert W_+ \vert^2, \quad \vert W_- \vert^2. \]
Polynomial Curvature Functionals

\[ G_M \longrightarrow \mathbb{R} \]
\[ g \longmapsto \int_M P(\mathcal{R}_g) d\mu_g \]

where \( P(\mathcal{R}) \) is \( SO(4) \)-invariant polynomial function of curvature.

Scale invariance \( \implies P \) quadratic.

Any such \( P(\mathcal{R}) \) is linear combinations of

\[ s^2, \quad |\hat{r}|^2, \quad |W_+|^2, \quad |W_-|^2. \]

Integrals give four scale-invariant functionals.
Four Basic Quadratic Curvature Functionals
Four Basic Quadratic Curvature Functionals

\[ \mathcal{G}_M \rightarrow \mathbb{R} \]

\[ g \rightarrow \left\{ \begin{array}{l} \int_M s^2 d\mu_g \\ \int_M |\mathring{\mathcal{R}}|^2 d\mu_g \\ \int_M |W_+|^2 d\mu_g \\ \int_M |W_-|^2 d\mu_g \end{array} \right\} \]
Four Basic Quadratic Curvature Functionals

\[ \mathcal{G}_M \rightarrow \mathbb{R} \]

\[ g \rightarrow \left\{ \begin{array}{l}
\int_M s^2 d\mu_g \\
\int_M |\dot{r}|^2 d\mu_g \\
\int_M |W_+|^2 d\mu_g \\
\int_M |W_-|^2 d\mu_g 
\end{array} \right. \]

However, these are not independent!
(M, \, g) \text{ compact oriented Riemannian.}

4-dimensional Gauss-Bonnet formula

\[ \chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + \right) d\mu \]
$(M, g)$ compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

$$
\chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 \right) d\mu
$$
$(M, g)$ compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

$$
\chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 \right) d\mu
$$
$(M, g)$ compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{\hat{r}^2}{2} \right) d\mu$$
\((M, g)\) compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

\[
\chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu
\]

for Euler-characteristic \(\chi(M) = \sum_j (-1)^j b_j(M)\).
4-dimensional signature formula

\[ \tau(M) = \frac{1}{12\pi^2} \int_M (|W_+|^2) \, d\mu \]
4-dimensional signature formula

\[ \tau(M) = \frac{1}{12\pi^2} \int_M \left( |W_+|^2 - |W_-|^2 \right) d\mu \]
4-dimensional signature formula

$$
\tau(M) = \frac{1}{12\pi^2} \int_M \left( |W_+|^2 - |W_-|^2 \right) d\mu
$$

for signature $$\tau(M) = b_+(M) - b_-(M)$$.
4-dimensional signature formula

\[
\tau(M) = \frac{1}{12\pi^2} \int_M \left( |W_+|^2 - |W_-|^2 \right) d\mu
\]

for signature \( \tau(M) = b_+(M) - b_-(M) \).

Here \( b_\pm(M) = \text{max dim subspaces} \subset H^2(M, \mathbb{R}) \) on which intersection pairing

\[
H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) \rightarrow \mathbb{R}
\]

\( ( [\varphi], [\psi] ) \mapsto \int_M \varphi \wedge \psi \)

is positive (resp. negative) definite.
Thus any quadratic curvature functional expressible in terms of
Thus any quadratic curvature functional expressible in terms of

\[ \int_M s^2 d\mu_g \quad \text{and} \quad \int_M |W_+|^2 d\mu_g. \]
Thus any quadratic curvature functional expressible in terms of e.g.

$$\int_M s^2 d\mu_g \quad \text{and} \quad \int_M |W_+|^2 d\mu_g .$$
Thus any quadratic curvature functional expressible in terms of e.g.

\[ \int_M s^2 \, d\mu_g \quad \text{and} \quad \int_M |W_+|^2 \, d\mu_g . \]

**Examples.**

\[ \int_M |W|^2_g \, d\mu_g = -12\pi^2 \tau(M) + 2 \int_M |W_+|^2 \, d\mu_g . \]
Thus any quadratic curvature functional expressible in terms of e.g.
\[ \int_M s^2 d\mu_g \quad \text{and} \quad \int_M |W_+|^2 d\mu_g. \]

Examples.

\[ \int_M |W|^2_g d\mu_g = -12\pi^2 \tau(M) + 2 \int_M |W_+|^2 d\mu_g \]

\[ \int_M |\mathcal{R}|^2_g d\mu_g = -8\pi^2 (\chi + 3\tau)(M) + 2 \int_M \left( \frac{s^2}{24} + 2|W_+|^2 \right) d\mu_g \]
Thus any quadratic curvature functional expressible in terms of e.g.

\[ \int_M s^2 d\mu_g \quad \text{and} \quad \int_M |W|^2 d\mu_g. \]

**Examples.**

\[ \int_M |W|^2_g d\mu_g = -12\pi^2 \tau(M) + 2 \int_M |W_+|^2 d\mu_g \]

\[ \int_M |\mathcal{R}|^2_g d\mu_g = -8\pi^2 (\chi + 3\tau)(M) + 2 \int_M \left( \frac{s^2}{24} + 2|W_+|^2 \right) d\mu_g \]

\[ \int_M |r|^2_g d\mu_g = -8\pi^2 (2\chi + 3\tau)(M) + 8 \int_M \left( \frac{s^2}{24} + \frac{1}{2}|W_+|^2 \right) d\mu_g \]
Thus any quadratic curvature functional expressible in terms of e.g.

\[ \int_M s^2 d\mu_g \quad \text{and} \quad \int_M |W_+|^2 d\mu_g . \]
Thus any quadratic curvature functional expressible in terms of e.g.
\[
\int_M s^2 d\mu_g \quad \text{and} \quad \int_M |W_+|^2 d\mu_g.
\]

Note that \( \int_M |W_+|^2 d\mu_g \) is conformally invariant:
Thus any quadratic curvature functional expressible in terms of e.g. 

\[ \int_M s^2 d\mu_g \quad \text{and} \quad \int_M |W_+|^2 d\mu_g . \]

Note that \( \int_M |W_+|^2 d\mu_g \) is conformally invariant: unchanged by \( g \sim u^2 g \).
Thus any quadratic curvature functional expressible in terms of e.g.

\[ \int_M s^2 d\mu_g \quad \text{and} \quad \int_M |W_+|^2 d\mu_g. \]

Note that \( \int_M |W_+|^2 d\mu_g \) is conformally invariant: unchanged by \( g \sim u^2 g \).

By contrast, \( \int_M s^2 d\mu_g \) varies on any conformal class.
Thus any quadratic curvature functional expressible in terms of e.g.

\[ \int_M s^2 d\mu_g \quad \text{and} \quad \int_M |W_+|^2 d\mu_g . \]

Note that \( \int_M |W_+|^2 d\mu_g \) is conformally invariant: unchanged by \( g \mapsto u^2 g \).

By contrast, \( \int_M s^2 d\mu_g \) varies on any conformal class.

\[ g \mapsto u^2 g \quad \implies \quad \int_M s^2 d\mu_g \mapsto \int_M (s+6u^{-1}\Delta u)^2 d\mu_g \]
Thus any quadratic curvature functional expressible in terms of e.g.

$$\int_M s^2 d\mu_g \quad \text{and} \quad \int_M |W_+|^2 d\mu_g.$$  

Note that $\int_M |W_+|^2 d\mu_g$ is conformally invariant: unchanged by $g \rightsquigarrow u^2 g$.

By contrast, $\int_M s^2 d\mu_g$ varies on any conformal class.

$$g \rightsquigarrow u^2 g \implies \int_M s^2 d\mu_g \rightsquigarrow \int_M (s + 6u^{-1}\Delta u)^2 d\mu_g$$

Critical in $[g] \iff s = \text{constant}$. 
Thus any quadratic curvature functional expressible in terms of e.g.

\[ \int_M s^2 d\mu_g \quad \text{and} \quad \int_M |W_+|^2 d\mu_g. \]

Note that \( \int_M |W_+|^2 d\mu_g \) is conformally invariant: unchanged by \( g \leadsto u^2 g \).

By contrast, \( \int_M s^2 d\mu_g \) varies on any conformal class.

\[ g \leadsto u^2 g \quad \Rightarrow \quad \int_M s^2 d\mu_g \leadsto \int_M (s + 6u^{-1} \Delta u)^2 d\mu_g. \]

Critical in \([g]\) \( \iff \) \( s = \) constant.

Minimizer \( \iff \) minimizes \( V^{-1/2} \int s \, d\mu \) (in \([g]\))
Thus any quadratic curvature functional expressible in terms of e.g.

\[ \int_M s^2 d\mu_g \quad \text{and} \quad \int_M |W_+|^2 d\mu_g. \]

Note that \( \int_M |W_+|^2 d\mu_g \) is conformally invariant: unchanged by \( g \mapsto u^2 g \).

By contrast, \( \int_M s^2 d\mu_g \) varies on any conformal class.

\[ g \mapsto u^2 g \implies \int_M s^2 d\mu_g \mapsto \int_M (s+6u^{-1}\Delta u)^2 d\mu_g \]

Critical in \([g]\) \iff \( s = \text{constant} \).

Minimizer in \([g]\) \iff \( g \) is “Yamabe metric.”
Thus any quadratic curvature functional expressible in terms of e.g.

\[ \int_M s^2 d\mu_g \quad \text{and} \quad \int_M |W_+|^2 d\mu_g. \]

Consequence.
Thus any quadratic curvature functional expressible in terms of e.g.

\[ \int_M s^2 d\mu_g \quad \text{and} \quad \int_M |W_+|^2 d\mu_g. \]

Consequence. Any critical metric for
Thus any quadratic curvature functional expressible in terms of e.g.

\[ \int_M s^2 d\mu_g \quad \text{and} \quad \int_M |W_+|^2 d\mu_g. \]

Consequence. Any critical metric for

\[ \int_M |\mathcal{R}|^2_g d\mu_g \quad \text{or} \quad \int_M |r|^2_g d\mu_g \]
Thus any quadratic curvature functional expressible in terms of e.g.

\[ \int_M s^2 d\mu_g \quad \text{and} \quad \int_M |W_+|^2 d\mu_g. \]

Consequence. Any critical metric for

\[ \int_M |\mathcal{R}|_g^2 d\mu_g \quad \text{or} \quad \int_M |r|^2_g d\mu_g \]

must have constant scalar curvature.
Thus any quadratic curvature functional expressible in terms of e.g.

\[ \int_M s^2 d\mu_g \quad \text{and} \quad \int_M |W_+|^2 d\mu_g . \]

Consequence. Any critical metric for

\[ \int_M |\mathcal{R}|^2_g d\mu_g \quad \text{or} \quad \int_M |r|^2_g d\mu_g \]

must have constant scalar curvature.

Similarly for any quadratic curvature functional which is not conformally invariant.
Thus any quadratic curvature functional expressible in terms of e.g.

\[ \int_M s^2 d\mu_g \quad \text{and} \quad \int_M |W_+|^2 d\mu_g. \]
But any quadratic curvature functional also expressible in terms of

\[
\int_M s^2 d\mu_g \quad \text{and} \quad \int_M |\hat{\sigma}|^2 d\mu_g .
\]
But any quadratic curvature functional also expressible in terms of

\[ \int_M s^2 d\mu_g \quad \text{and} \quad \int_M |\hat{\rho}|^2 d\mu_g. \]

Einstein metrics are critical for both.
But any quadratic curvature functional also expressible in terms of

\[ \int_M s^2 d\mu_g \quad \text{and} \quad \int_M |\mathring{r}|^2 d\mu_g . \]

Einstein metrics are critical for both.

\[ \therefore \text{Einstein metrics critical } \forall \text{ quadratic functionals!} \]
But any quadratic curvature functional also expressible in terms of

\[ \int_M s^2 d\mu_g \quad \text{and} \quad \int_M \left| \mathring{r} \right|^2 d\mu_g. \]

Einstein metrics are critical for both.

\[ \therefore \text{Einstein metrics critical } \forall \text{ quadratic functionals!} \]

e.g. critical for conformally invariant functional

\[ g \mapsto \int_M \left| W \right|_g^2 d\mu_g. \]
But any quadratic curvature functional also expressible in terms of

\[ \int_M s^2 d\mu_g \quad \text{and} \quad \int_M |\hat{\rho}|^2 d\mu_g. \]

Einstein metrics are critical for both.

\[ \therefore \text{Einstein metrics critical } \forall \text{ quadratic functionals!} \]

e.g. critical for conformally invariant functional

\[ g \mapsto -\int_M |W|^2 g d\mu_g \]

Euler-Lagrange equations \( B = 0, \)
But any quadratic curvature functional also expressible in terms of
\[
\int_M s^2 d\mu_g \quad \text{and} \quad \int_M |\dot{r}|^2 d\mu_g .
\]

Einstein metrics are critical for both.

∴ Einstein metrics critical ∀ quadratic functionals!

e.g. critical for conformally invariant functional

\[
g \longmapsto \int_M |W|^2_g d\mu_g
\]

Euler-Lagrange equations \( B = 0 \), where

\[
B_{ab} := (\nabla^c \nabla^d + \frac{1}{2} \dot{r}^{cd}) W_{acbd} .
\]
But any quadratic curvature functional also expressible in terms of

\[ \int_M s^2 d\mu_g \quad \text{and} \quad \int_M |\overset{\circ}{r}|^2 d\mu_g . \]

Einstein metrics are critical for both.

\[ \therefore \text{Einstein metrics critical } \forall \text{ quadratic functionals!} \]

e.g. critical for conformally invariant functional

\[ g \mapsto \int_M |W|^2_g d\mu_g \]

Euler-Lagrange equations \( B = 0 \), where

\[ B_{ab} := (\nabla^c \nabla^d + \frac{1}{2} \overset{\circ}{r}_{cd}) W_{acbd} . \]

“Bach-flat” metrics.
But any quadratic curvature functional also expressible in terms of

\[ \int_M s^2 d\mu_g \quad \text{and} \quad \int_M |\mathring{r}|^2 d\mu_g. \]

Einstein metrics are critical for both.

\[ \therefore \text{Einstein metrics critical } \forall \text{ quadratic functionals!} \]

e.g. critical for conformally invariant functional

\[ g \rightarrow \int_M |W|^2_g d\mu_g \]

Euler-Lagrange equations \( B = 0 \), where

\[ B_{ab} := (\nabla^c \nabla^d + \frac{1}{2} \mathring{r}^{cd}) W_{acbd}. \]

“Bach-flat” metrics. Conformally invariant!
Main Question. Which smooth compact 4-manifolds $M^4$ admit Einstein metrics?
Main Question. Which smooth compact 4-manifolds $M^4$ admit Einstein metrics?

Kähler geometry provides rich source of examples.
Main Question. *Which smooth compact 4-manifolds $M^4$ admit Einstein metrics?*

Kähler geometry provides rich source of examples.

On suitable 4-manifolds, Seiberg-Witten theory allows one to mimic Kähler geometry when treating non-Kähler metrics.
Main Question. Which smooth compact 4-manifolds $M^4$ admit Einstein metrics?

Kähler geometry provides rich source of examples.

On suitable 4-manifolds, Seiberg-Witten theory allows one to mimic Kähler geometry when treating non-Kähler metrics.

Our Focus. If $(M^4, J)$ is a compact complex surface, when does $M^4$ admit an Einstein metric $g$ (unrelated to $J$)?
\((M^n, g)\): holonomy
Kähler metrics: 

\((M^n, g)\): Kähler ⇐⇒ holonomy
$(M^n, g)$: holonomy
\((M^n, g)\): holonomy
$(M^n, g)$: \text{holonomy}
$(M^n, g)$: holonomy
$(M^n, g)$: holonomy
\( (M^n, g) \): holonomy
\[(M^n, g)\]: holonomy
$(M^n, g)$:

holonomy
$(M^n, g)$: holonomy
$(M^n, g)$: holonomy
$(M^n, g)$: holonomy
$(M^n, g)$: \text{holonomy}
$(M^n, g)$: holonomy
$(M^n, g)$: \quad holonomy \subset O(n)$
Kähler metrics:

\((M^{2m}, g)\): holonomy
Kähler metrics:

\((M^{2m}, g) \text{ Kähler } \iff \text{ holonomy } \subset U(m)\)
Kähler metrics:

\[(M^4, g) \text{ Kähler } \iff \text{ holonomy } \subset \mathbf{U}(2)\]
Kähler metrics:

\((M^4, g)\) Kähler ⇐⇒ holonomy \(\subset \mathbf{U}(2)\)
Kähler metrics:

\((M^4, g)\) Kähler \iff \text{holonomy} \subset U(2)

\iff \exists\ \text{almost complex-structure } J \text{ with } \nabla J = 0
\text{ and } g(J\cdot, J\cdot) = g.
Kähler metrics:

\((M^4, g)\) Kähler \iff \text{holonomy} \subset \textbf{U}(2) \iff \exists \text{almost complex-structure } J \text{ with } \nabla J = 0 \text{ and } g(J\cdot, J\cdot) = g.

\iff (M^4, J) \text{ is a complex surface and } \exists J\text{-invariant closed 2-form } \omega \text{ such that } g = \omega(\cdot, J\cdot).
Kähler metrics:

\((M^4, g)\) Kähler \iff\text{holonomy } \subset U(2)

\iff\exists\text{ almost complex-structure } J \text{ with } \nabla J = 0 \text{ and } g(J \cdot, J \cdot) = g.

\iff (M^4, J) \text{ is a complex surface and } \exists J\text{-invariant closed 2-form } \omega \text{ such that } g = \omega(\cdot, J \cdot).

Kähler magic:
Kähler metrics:

$$(M^4, g) \text{ Kähler} \iff \text{holonomy } \subset \text{U}(2)$$

$$\iff \exists \text{ almost complex-structure } J \text{ with } \nabla J = 0$$

and $g(J\cdot, J\cdot) = g$.

$$\iff (M^4, J) \text{ is a complex surface and } \exists J\text{-invariant}$$

closed 2-form $\omega$ such that $g = \omega(\cdot, J\cdot)$.

Kähler magic:

The 2-form

$$i\pi(J\cdot, \cdot)$$
Kähler metrics:

\((M^4, g) \text{ Kähler} \iff \text{holonomy } \subset U(2)\)

\iff \exists \text{ almost complex-structure } J \text{ with } \nabla J = 0 \text{ and } g(J\cdot, J\cdot) = g.

\iff (M^4, J) \text{ is a complex surface and } \exists J\text{-invariant closed 2-form } \omega \text{ such that } g = \omega(\cdot, J\cdot).\)

Kähler magic:

The 2-form

\(i\tau(J\cdot, \cdot)\)

is curvature of canonical line bundle \(K = \Lambda^{m,0}\).
Kähler metrics:

\((M^4, g)\) Kähler \iff \text{holonomy} \subset U(2) \iff \exists \text{almost complex-structure } J \text{ with } \nabla J = 0 \text{ and } g(J\cdot, J\cdot) = g. \iff (M^4, J) \text{ is a complex surface and } \exists J\text{-invariant closed 2-form } \omega \text{ such that } g = \omega(\cdot, J\cdot). \)

Kähler magic:

The 2-form

\[ i\text{r}(J\cdot, \cdot) \]

is curvature of canonical line bundle \( K = \Lambda^{2,0}. \)
\[(M^n, g): \quad \text{holonomy}\]
\((M^n, g)\): holonomy
$(M^n, g)$: \quad holonomy \subset O(n)$
Hyper-Kähler metrics:

\((M^{4\ell}, g)\) Hyper-Kähler \iff \text{holonomy} \subset \text{Sp}(\ell)\)
Hyper-Kähler metrics:

\[(M^{4\ell}, g) \text{ Hyper-Kähler } \iff \text{holonomy } \subset \text{Sp}(\ell)\]

Kähler, for many different \(J\)'s.

\[\text{Sp}(\ell) = \text{O}(4\ell) \cap \text{GL}(\ell, \mathbb{H})\]
Hyper-Kähler metrics:

\((M^4, g)\) Hyper-Kähler \iff \text{holonomy} \subset \text{Sp}(1)\)
Hyper-Kähler metrics:

$$(M^4, g) \text{ Hyper-Kähler} \iff \text{holonomy} \subset SU(2)$$
Hyper-Kähler metrics:

\((M^4, g)\) Hyper-Kähler \iff \text{holonomy} \subset SU(2) \iff (\Lambda^+, \nabla) \text{ flat, trivial.}\)
Hyper-Kähler metrics:

$$(M^4, g) \text{ Hyper-Kähler} \iff \text{holonomy} \subset SU(2)$$

$$\iff (\Lambda^+, \nabla) \text{ flat, trivial.}$$

$$0 \to \mathbb{Z}_2 \to SO(4) \to SO(3) \times SO(3) \to 0$$
Hyper-Kähler metrics:

\[(M^4, g) \text{ Hyper-Kähler } \iff \text{holonomy } \subset SU(2)\]

\[\iff (\Lambda^+, \nabla) \text{ flat, trivial.}\]

\[0 \to \mathbb{Z}_2 \to SO(4) \to SO(3) \times SO(3) \to 0\]

\[\Lambda^+ \quad \oplus \quad \Lambda^-\]
Hyper-Kähler metrics:

\[(M^4, g) \text{ Hyper-Kähler } \iff \text{holonomy } \subset SU(2) \]

\[\iff (\Lambda^+, \nabla) \text{ flat, trivial.} \]

\[0 \to \mathbb{Z}_2 \to SO(4) \to SO(3) \times SO(3) \to 0\]

\[0 \to \mathbb{Z}_2 \to U(2) \to SO(2) \times SO(3) \to 0\]
Hyper-Kähler metrics:

$$(M^4, g) \text{ Hyper-Kähler } \iff \text{ holonomy } \subset SU(2)$$

$$\iff (\Lambda^+, \nabla) \text{ flat, trivial.}$$

$$\implies \text{ Kähler and Ricci-flat}$$
Hyper-Kähler metrics:

\((M^4, g)\) Hyper-Kähler ⇔ holonomy \(\subset SU(2)\)

⇔ \((\Lambda^+, \nabla)\) flat, trivial.

⇒ Kähler and Ricci-flat

Calabi-Yau metrics
Hyper-Kähler metrics:

\((M^4, g)\) Hyper-Kähler \iff \text{holonomy} \subset SU(2) \iff (\Lambda^+, \nabla) \text{ flat, trivial.}

If \(\pi_1(M) = 0\), \iff \text{Kähler and Ricci-flat}

Calabi-Yau metrics
$K3 = \text{Kummer-Kähler-Kodaira manifold}.$

Simply connected complex surface with $c_1 = 0$. 
\( K3 = \) Kummer-Kähler-Kodaira manifold.

Diffeomorphic to quartic in \( \mathbb{CP}_3 \)

\[ x^4 + y^4 + z^4 + w^4 = 0 \]
Hyper-Kähler metrics:

\[(M^4, g) \text{ Hyper-Kähler } \iff \text{holonomy } \subset \text{SU}(2) \iff (\Lambda^+, \nabla) \text{ flat, trivial.}\]

If \(\pi_1(M) = 0\), \(\iff\) Kähler and Ricci-flat

Calabi-Yau metrics

Any simply connected, compact hyper-Kähler is \(K3\).
Hyper-Kähler metrics:

\[(M^4, g) \text{ Hyper-Kähler } \iff \text{holonomy } \subset SU(2) \]
\[\iff (\Lambda^+, \nabla) \text{ flat, trivial.} \]

If \(\pi_1(M) = 0\), \(\iff\) Kähler and Ricci-flat Calabi-Yau metrics

Any simply connected, compact hyper-Kähler is K3.

Yau: Conversely, any K3 admits such metrics.
Berger’s Inequality:

\[ \chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\dot{r}|^2}{2} \right) d\mu_g \]
Berger’s Inequality:

\[ \chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{\hat{r}^2}{2} \right) d\mu_g \]

Einstein \( \Rightarrow \) \[ = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 \right) d\mu_g \]
Berger’s Inequality:

\[ \chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu_g \]

Einstein \Rightarrow \quad = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 \right) d\mu_g

**Theorem** (Berger Inequality). *If smooth compact\( M^4 \) admits Einstein\( g \), then \( \chi(M) \geq 0 \), with equality only if (\( M, g \)) flat, and finitely covered by \( T^4 = \mathbb{R}^4/\Lambda \).*
Hitchin-Thorpe Inequality:

\[
(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W|^2 - \frac{|\hat{\rho}|^2}{2} \right) d\mu_g
\]
Hitchin-Thorpe Inequality:

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 - \frac{\hat{r}^2}{2} \right) d\mu_g$$

Einstein $\Rightarrow$ $= \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 \right) d\mu_g$
Hitchin-Thorpe Inequality:

\[(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 - \frac{\hat{r}^2}{2} \right) d\mu_g \]

Einstein \(\Rightarrow\) \[= \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 \right) d\mu_g \]

\[\mathcal{R} = \begin{pmatrix}
W_+ + \frac{s}{12} & \hat{r} \\
\hat{r} & W_- + \frac{s}{12}
\end{pmatrix}\]
Hitchin-Thorpe Inequality:

\[(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 - \frac{\dot{\hat{r}}^2}{2} \right) d\mu_g \]

Einstein \implies \quad = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 \right) d\mu_g

\[
\begin{pmatrix}
W_+ + \frac{s}{12} & \dot{\hat{r}} \\
\dot{\hat{r}} & W_- + \frac{s}{12}
\end{pmatrix}
\]

Curvature \(\Lambda^+\)    Curvature \(\Lambda^-\)
Hitchin-Thorpe Inequality:

\[(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W + |^2 - \frac{|\hat{\nabla}|^2}{2} \right) d\mu_g \]

Einstein \Rightarrow \quad = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W + |^2 \right) d\mu_g

**Theorem (Hitchin-Thorpe Inequality).** If smooth compact oriented \( M^4 \) admits Einstein \( g \), then

\[(2\chi + 3\tau)(M) \geq 0,\]

with equality only if \((M, g)\) is locally hyper-Kähler. The latter case happens only if \( M \) finitely covered by flat \( T^4 \) or \( K3 \).
If $M$ any smooth oriented 4-manifold,
If $M$ any smooth oriented 4-manifold,

$$(2\chi + 3\tau)(M \# \overline{\mathbb{CP}^2} \# \cdots \# \overline{\mathbb{CP}^2}) = (2\chi + 3\tau)(M) - k$$
If $M$ any smooth oriented 4-manifold,

$$(2\chi + 3\tau)(M \# \overline{\mathbb{CP}^2} \# \cdots \# \overline{\mathbb{CP}^2}) = (2\chi + 3\tau)(M) - k$$

so Hitchin-Thorpe $\implies$ not Einstein if $k \gg 0$. 
If $M$ any smooth oriented 4-manifold,

$$(2\chi + 3\tau)(M \# \overline{\mathbb{CP}^2} \# \cdots \# \overline{\mathbb{CP}^2}) = (2\chi + 3\tau)(M) - k$$

so Hitchin-Thorpe $\implies$ not Einstein if $k \gg 0$.

$M$ simply connected $\leadsto$ simply connected examples.
Hitchin-Thorpe Inequality:

\[(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 - \frac{|\hat{r}|^2}{2} \right) d\mu_g \]

Einstein \Rightarrow \quad = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 \right) d\mu_g
Hitchin-Thorpe Inequality:

\[(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 - \frac{\mathring{r}^2}{2} \right) d\mu_g \]

Einstein \Rightarrow \quad = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 \right) d\mu_g

Certainly \geq 0.
Hitchin-Thorpe Inequality:

\[(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 - \frac{r^2}{2} \right) d\mu_g \]

Einstein \implies \quad = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 \right) d\mu_g

Certainly \geq 0.

But if we could get better lower bounds for
Hitchin-Thorpe Inequality:

\[(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 - \frac{\mathring{r}^2}{2} \right) d\mu_g\]

Einstein \(\Rightarrow\) \[\frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 \right) d\mu_g\]

Certainly \(\geq 0\).

But if we could get better lower bounds for

\[\int s^2 d\mu_g \quad \text{and} \quad \int |W_+|^2 d\mu_g\]
Hitchin-Thorpe Inequality:

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu_g$$

Einstein $\Rightarrow$ $\quad = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 \right) d\mu_g$

Certainly $\geq 0$.

But if we could get better lower bounds for

$$\int s^2 d\mu_g \quad \text{and} \quad \int |W_+|^2 d\mu_g$$

we would obtain a better result.
Hitchin-Thorpe Inequality:

\[(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu_g \]

Einstein \implies = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 \right) d\mu_g

Certainly \geq 0.

But if we could get better lower bounds for

\[\int s^2 d\mu_g \text{ and } \int |W_+|^2 d\mu_g\]

we would obtain a better result.

Next lecture:
Hitchin-Thorpe Inequality:

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 - \frac{|\hat{r}|^2}{2}\right) d\mu_g$$

Einstein $\Rightarrow$ $= \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2\right) d\mu_g$

Certainly $\geq 0$.

But if we could get better lower bounds for

$$\int s^2 d\mu_g \quad \text{and} \quad \int |W_+|^2 d\mu_g$$

we would obtain a better result.

Next lecture: Obtaining such estimates, using Seiberg-Witten theory.
End, Part I