Gravitational Instantons,

Weyl Curvature, &

Conformally Kähler Geometry

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Relativity Seminar,
Mathematical Institute,
University of Oxford,
October 24, 2025

Joint work with

Joint work with

Olivier Biquard

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Int. Math. Res. Not. IMRN
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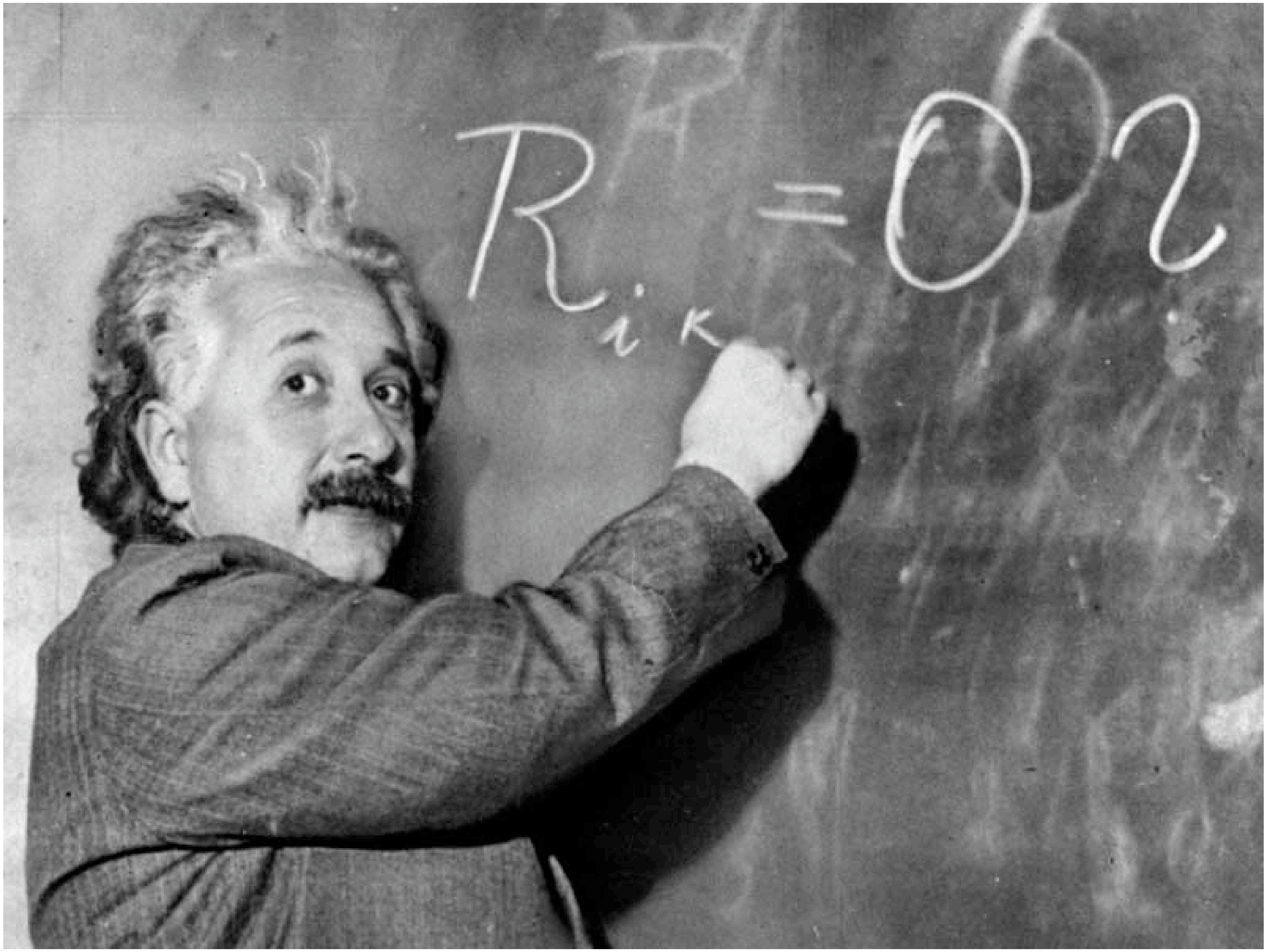

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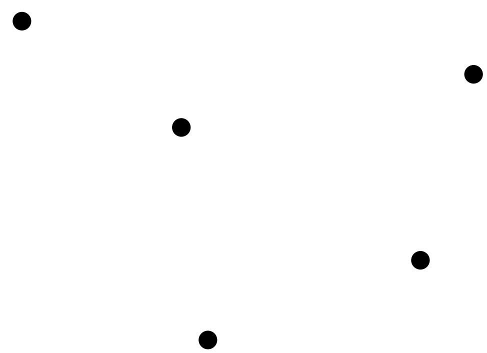

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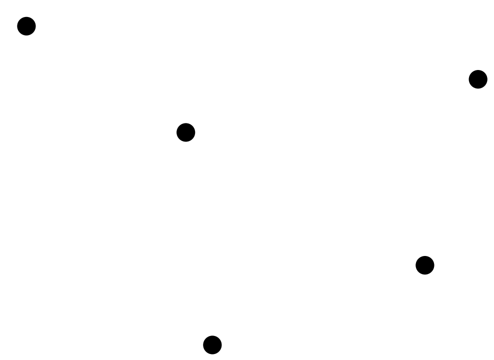

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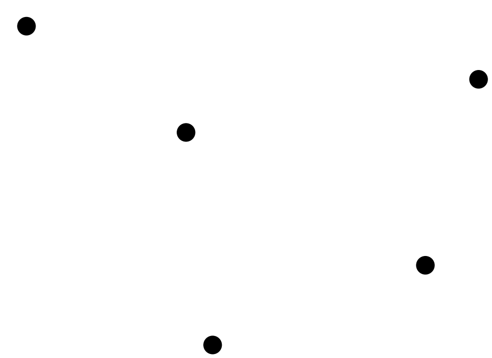

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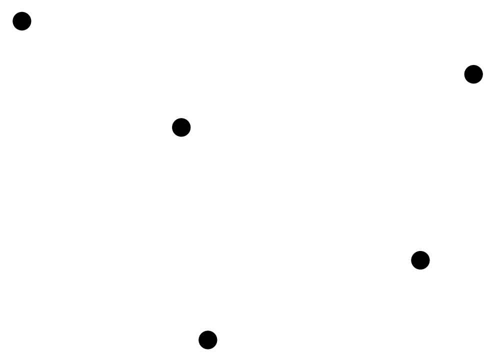

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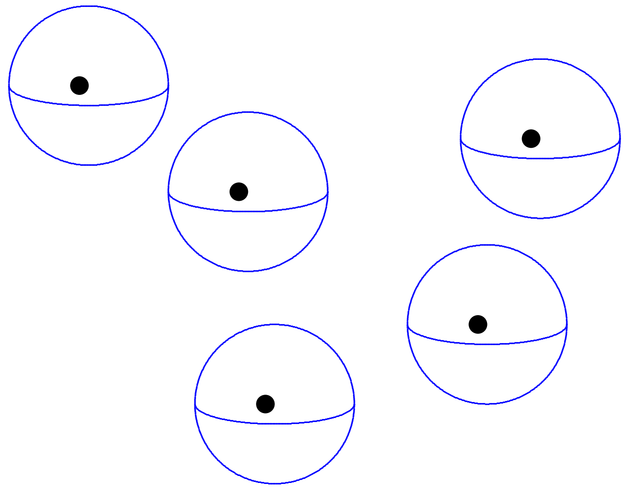

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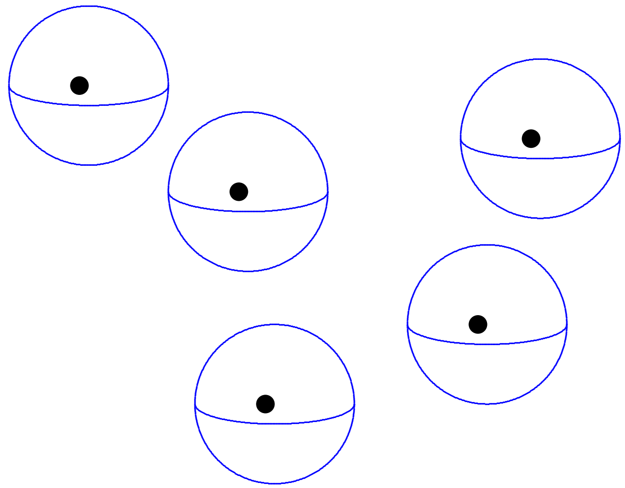

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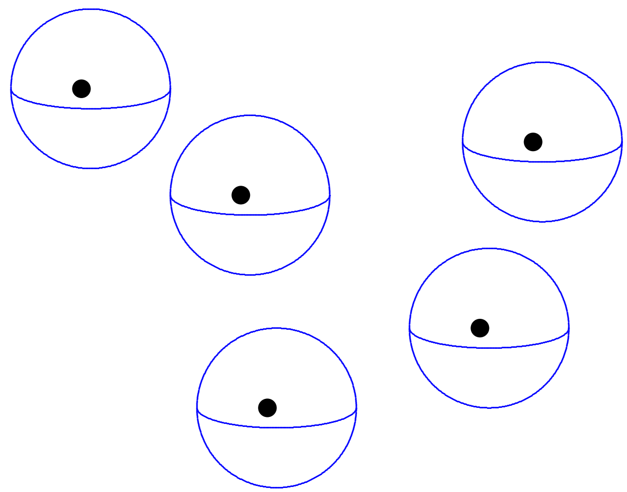

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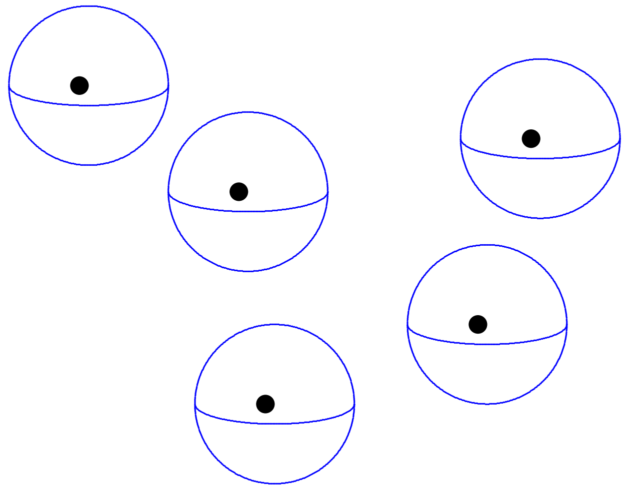

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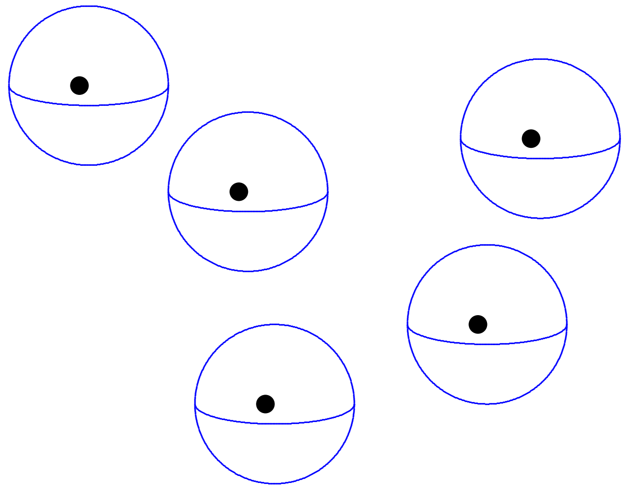

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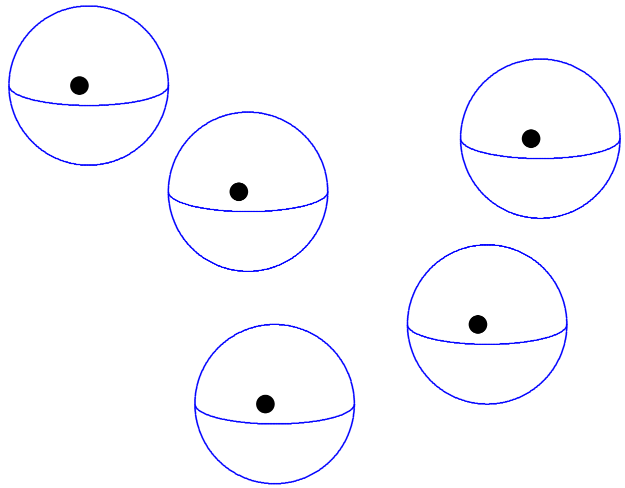

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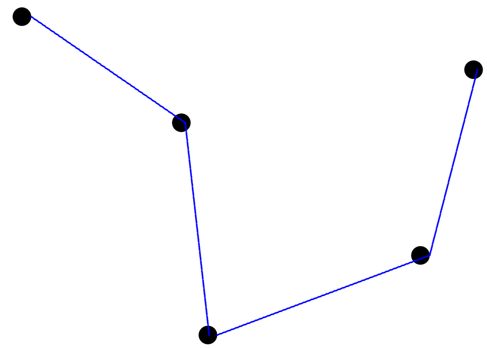

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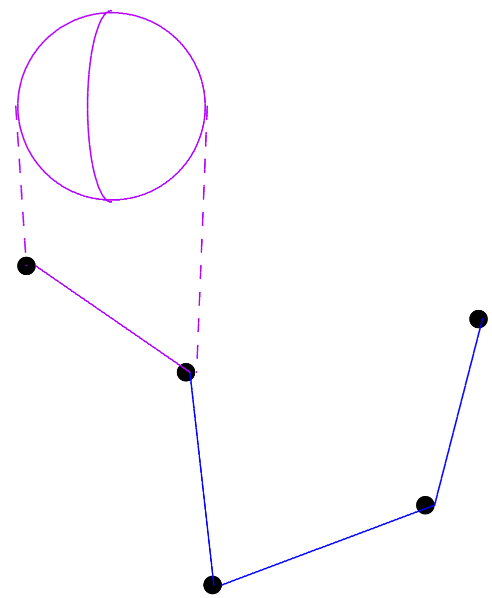

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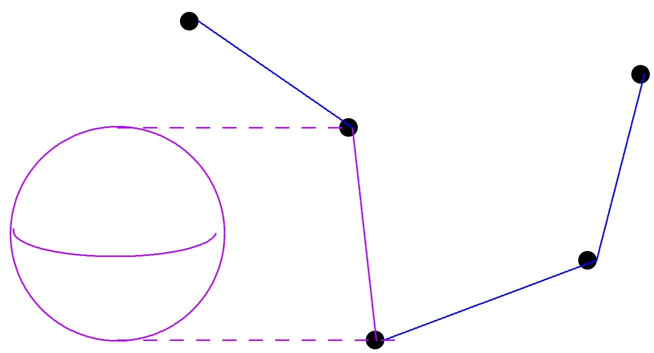

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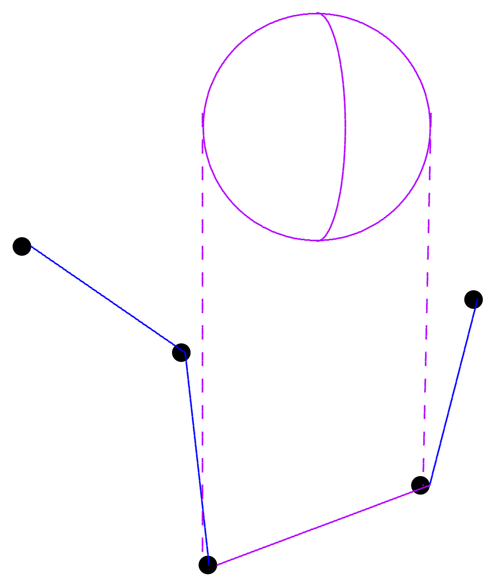

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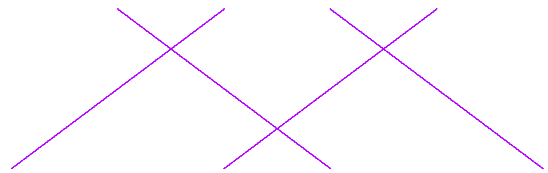

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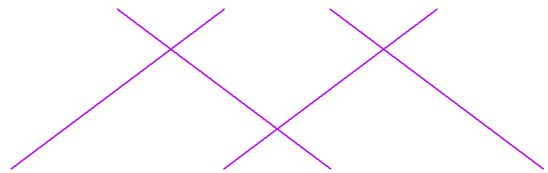

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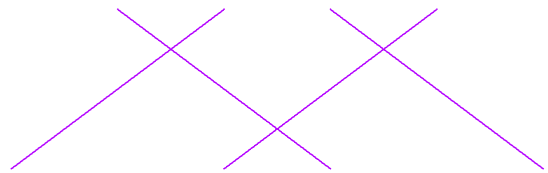

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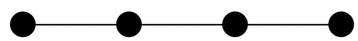

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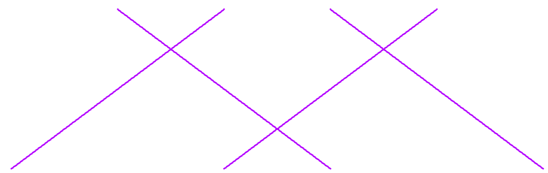

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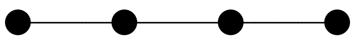

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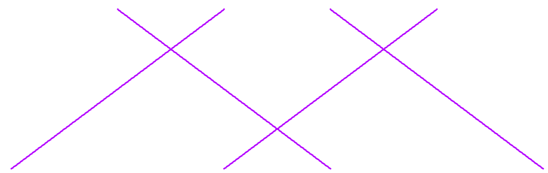

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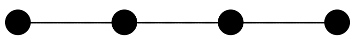

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Plumb together k copies of T^*S^2
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cf. Bishop-Gromov inequality!

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This last property distinguishes the ALF spaces from other classes of gravitational instantons:

ALG, ALH, ALG*, ALH*, ...

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Taub-NUT becomes Hermitian metric on $\mathbb{C}^2$.

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$$V = 1 + \frac{1}{2\varrho}$$

Can also write as

$$g = \frac{r+1}{4r}dr^2 + r(1+r)[\sigma_1{}^2 + \sigma_2{}^2] + \frac{r}{r+1}\sigma_3{}^2$$

for left-invariant coframe $\{\sigma_j\}$ on $S^3 = \mathbf{SU}(2)$.

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This J determines opposite orientation from the hyper-Kähler complex structures.

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Non-Kähler, but conformally Kähler!

Hawking also explored non-hyper-Kähler examples. . .

Example.

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Bach-flat Kähler metrics are extremal!

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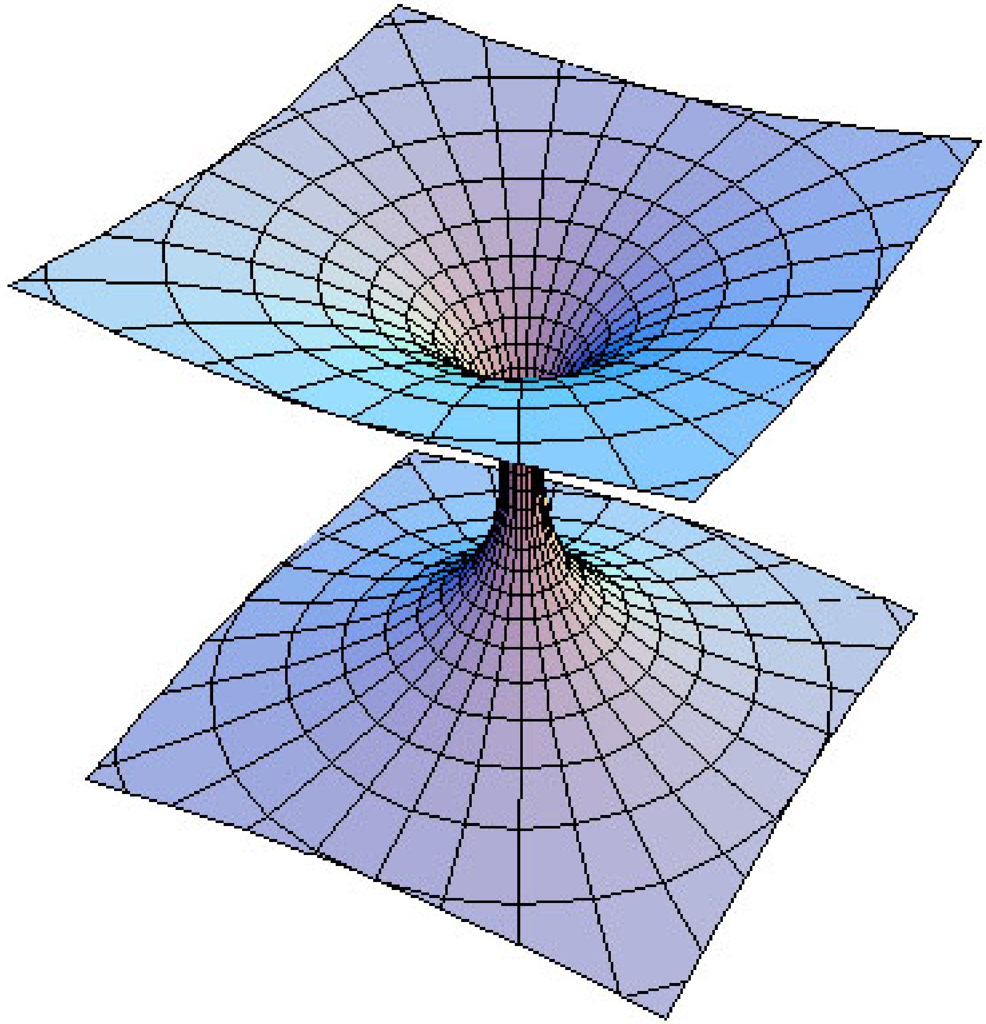
Makes h into extremal Kähler metric on $\mathbb{C} \times \mathbb{CP}_1$.



$$\mathbb{R} \times S^2 \subset \mathbb{R}^2 \times S^2$$

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Hitchin, Kronheimer, Cherkis-Hitchin, Minerbe, Hein, Chen-Chen, Hein-Sun-Viaclovsky-Zhang. . .

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Y. Chen & E. Teo, 2011

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The proof is actually constructive!

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By the Riemannian Goldberg-Sachs Theorem, the Hermitian assumption is equivalent to assuming that the Einstein metric g is conformally Kähler.

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Derdziński: the conformally related Kähler metric is also automatically extremal in the sense of Calabi!

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Ricci-flat case — not merely Einstein!

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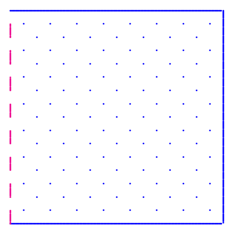

Schwarzschild

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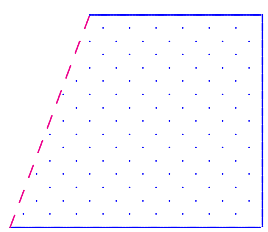

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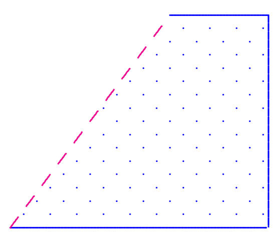

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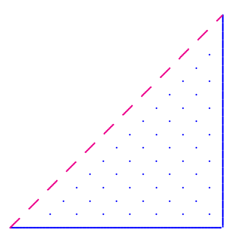

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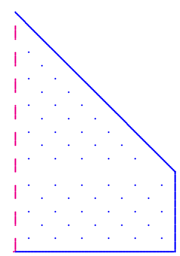

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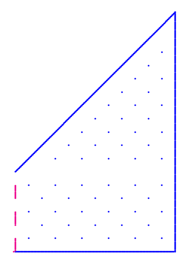

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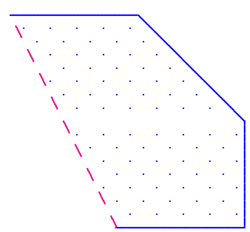

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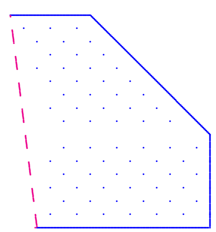

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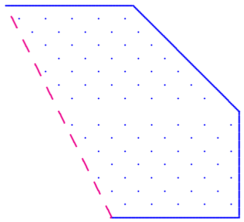

Chen-Teo

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Only depends on the conformal class

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Let $\alpha \geq \beta \geq \gamma$ be eigenvalues of W^+:

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$$2\sqrt{6}|W^+|_h + |s_h| \geq |\omega \wedge \star d\omega|^2$$

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Then $h = {\alpha_g}^{2/3} g$ is a Kähler metric.

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$$\boxed{3\, d[\omega \wedge \star d\omega] \geq \star\left(\frac{1}{2}|\nabla \omega|^2 + 3\, |d\omega|^2\right).}$$

Proposition. *Let* (M, g) *be an oriented, simply-connected Riemannian* 4*-manifold that satisfies* $\delta W^+ = 0$ *and* $\det(W^+) > 0$ *everywhere. Also suppose that* M *is a nested union* $M = \cup_j U_j$ *of compact domains*

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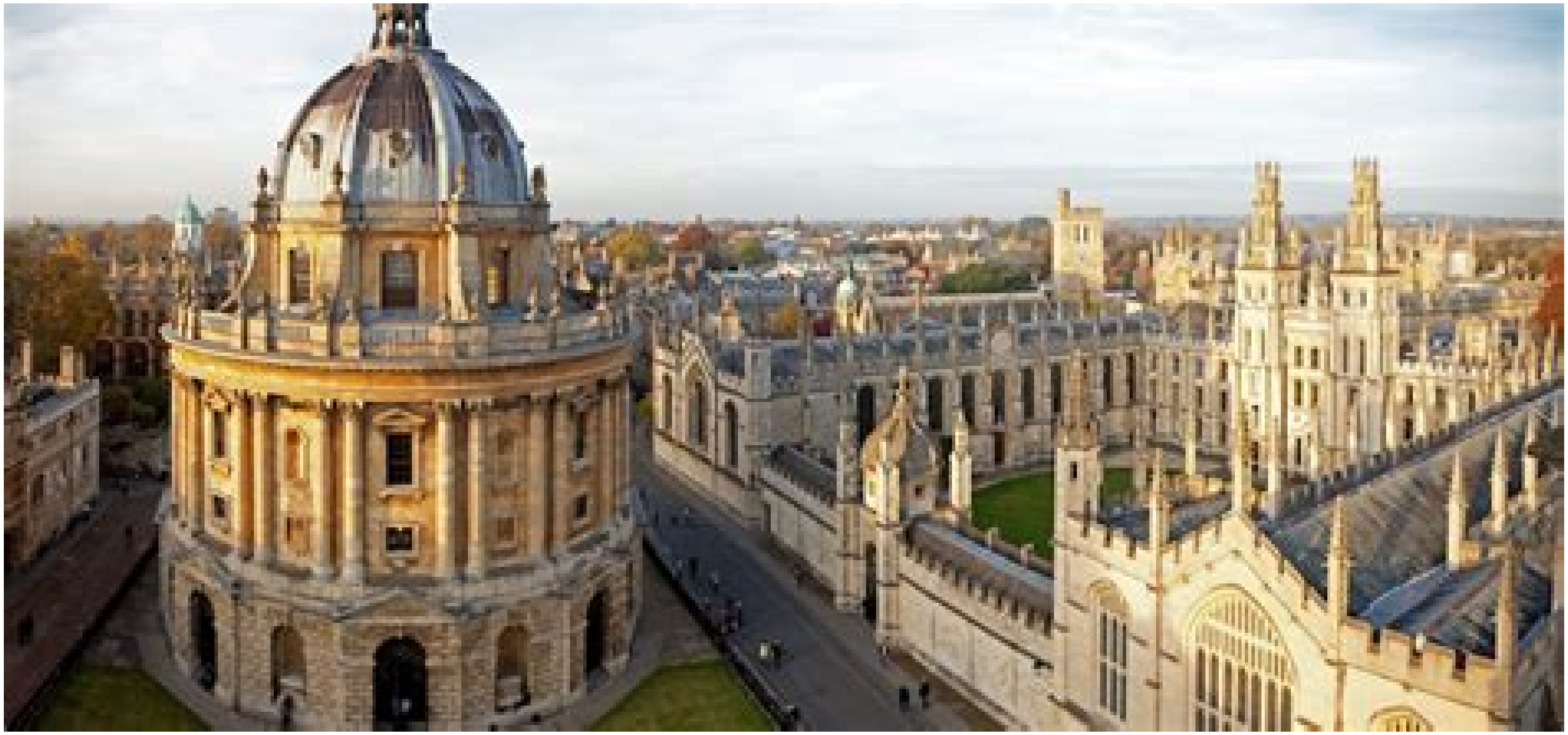

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