

Einstein Constants

and

Differential Topology

Claude LeBrun

Stony Brook University

Geometry and Analysis Seminar,
Mathematical Institute, University of Oxford,
October 20, 2025

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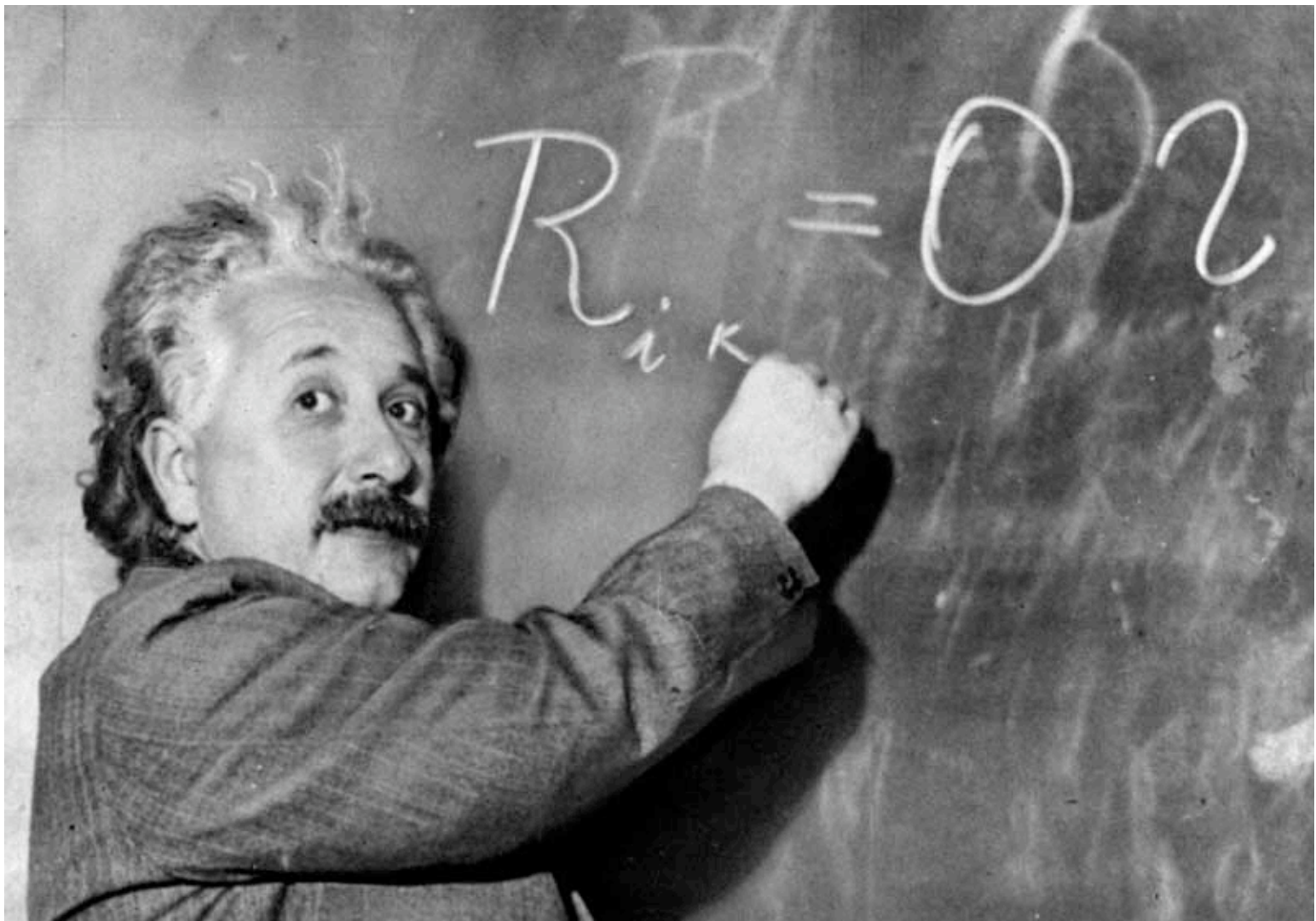
“...the greatest blunder of my life!”

— A. Einstein, to G. Gamow

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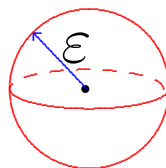
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Ergebnisse der Mathematik und ihrer Grenzgebiete

3. Folge · Band 10

A Series of Modern Surveys in Mathematics

Arthur L. Besse

Einstein Manifolds



Springer-Verlag Berlin Heidelberg GmbH

Acknowledgements

Pour rassembler les éléments un peu disparates qui constituent ce livre, j'ai dû faire appel à de nombreux amis, heureusement bien plus savants que moi. Ce sont, entre autres, Geneviève Averous, Lionel Bérard-Bergery, Marcel Berger, Jean-Pierre Bourguignon, Andrei Derdzinski, Dennis M. DeTurck, Paul Gauduchon, Nigel J. Hitchin, Josette Houillot, Hermann Karcher, Jerry L. Kazdan, Norihito Koiso, Jacques Lafontaine, Pierre Pansu, Albert Polombo, John A. Thorpe, Liane Valère.

Les institutions suivantes m'ont prêté leur concours matériel, et je les en remercie: l'UER de mathématiques de Paris 7, le Centre de Mathématiques de l'Ecole Polytechnique, Unités Associées du CNRS, l'UER de mathématiques de Chambéry et le Conseil Général de Savoie.

Enfin, qu'il me soit permis de saluer ici mon prédécesseur et homonyme Jean Besse, de Zürich, qui s'est illustré dans la théorie des fonctions d'une variable complexe (voir par exemple [Bse]).

Vôtre,

A handwritten signature in black ink, consisting of a stylized, overlapping loop structure with a horizontal line extending to the right.

Arthur Besse

Le Faux, le 15 septembre 1986



Besse en Chandesse



L'Auvergne



Marcel Berger

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But it turns out that the answer is actually **Yes!**

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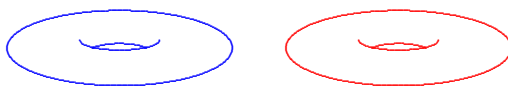
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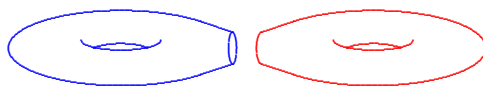
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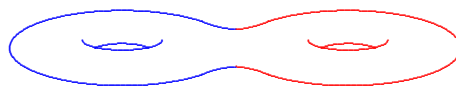
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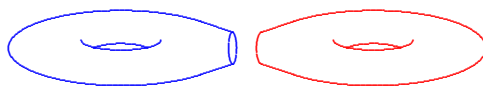
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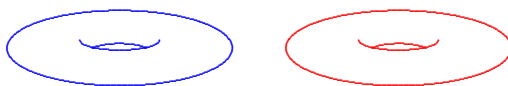
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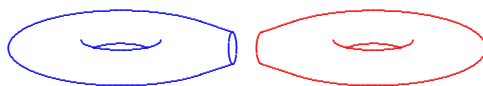
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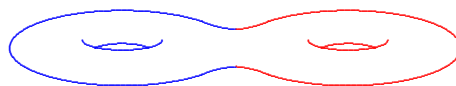
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$\iff$ they have the same Einstein constant λ .

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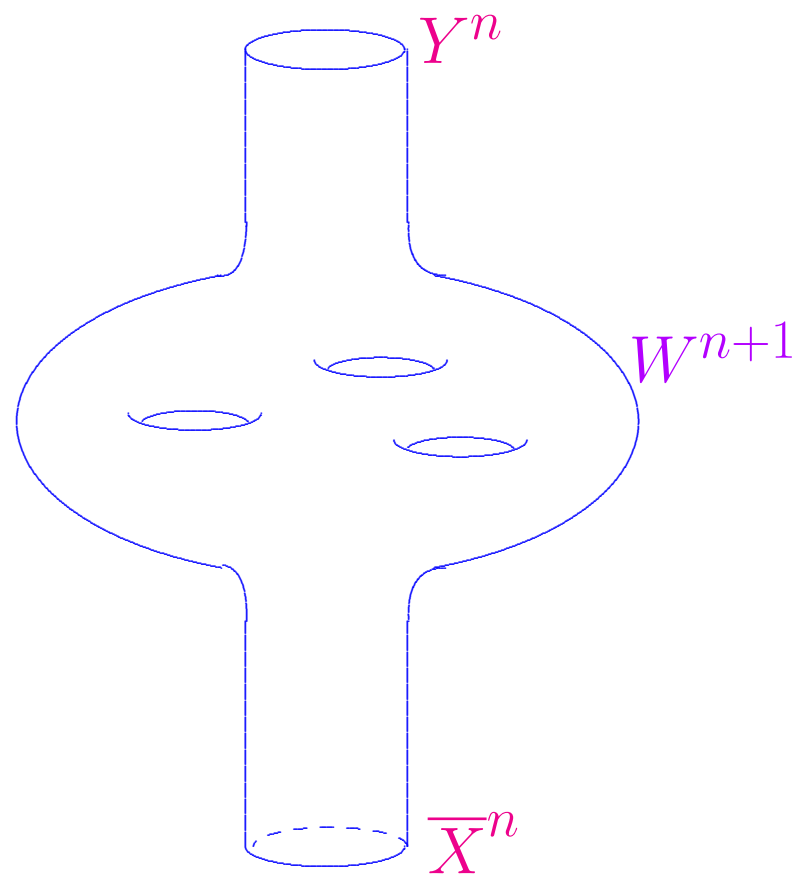
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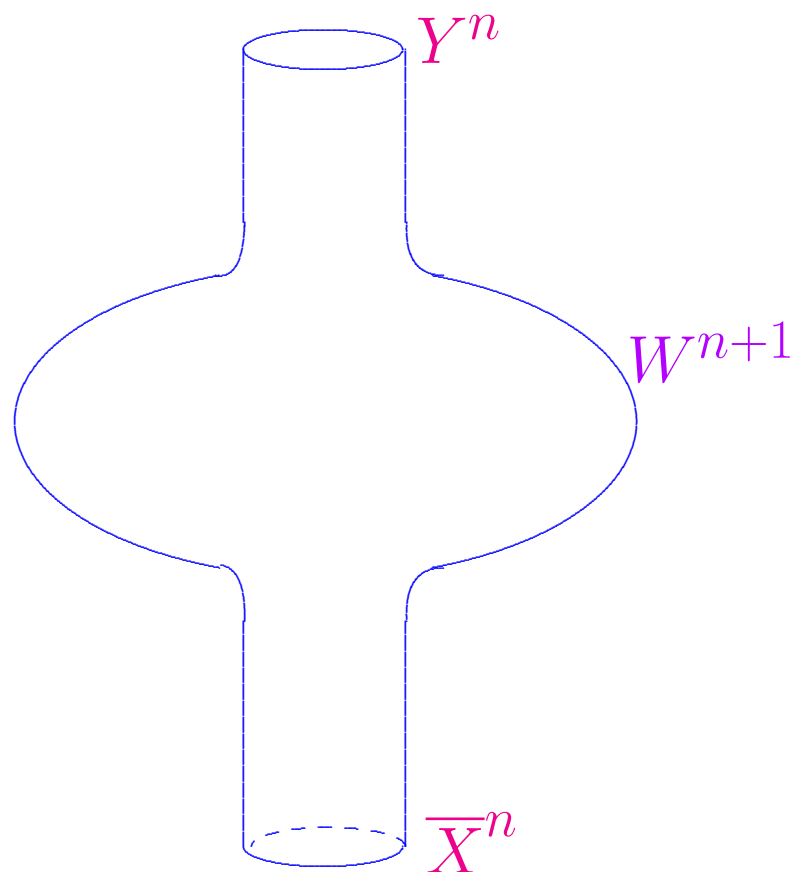
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- Smale's h -cobordism theorem.

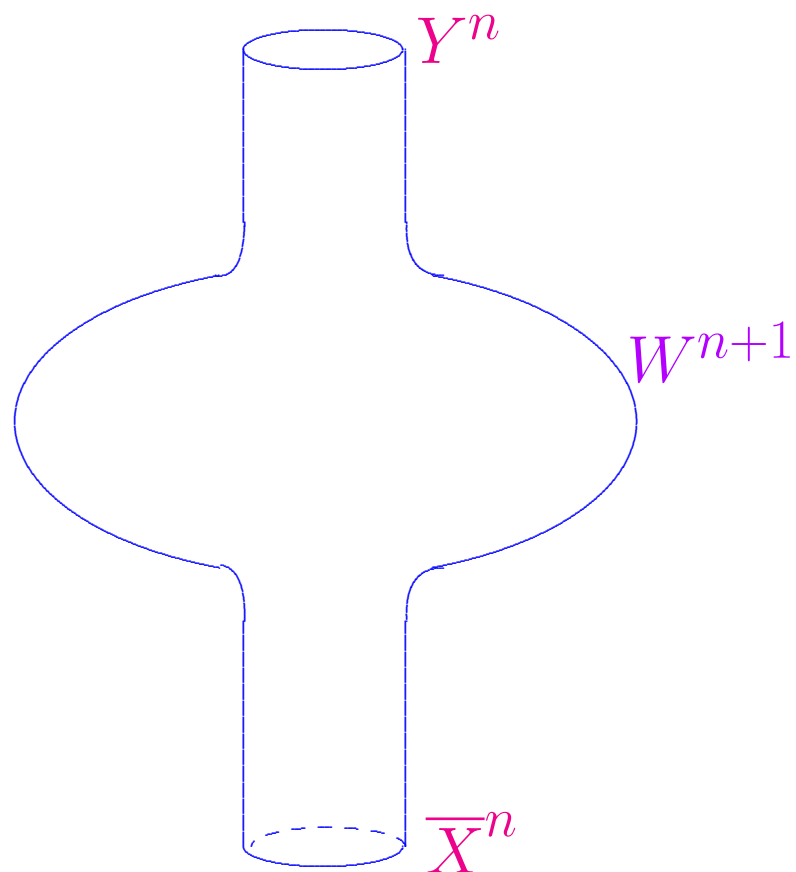


Cobordism

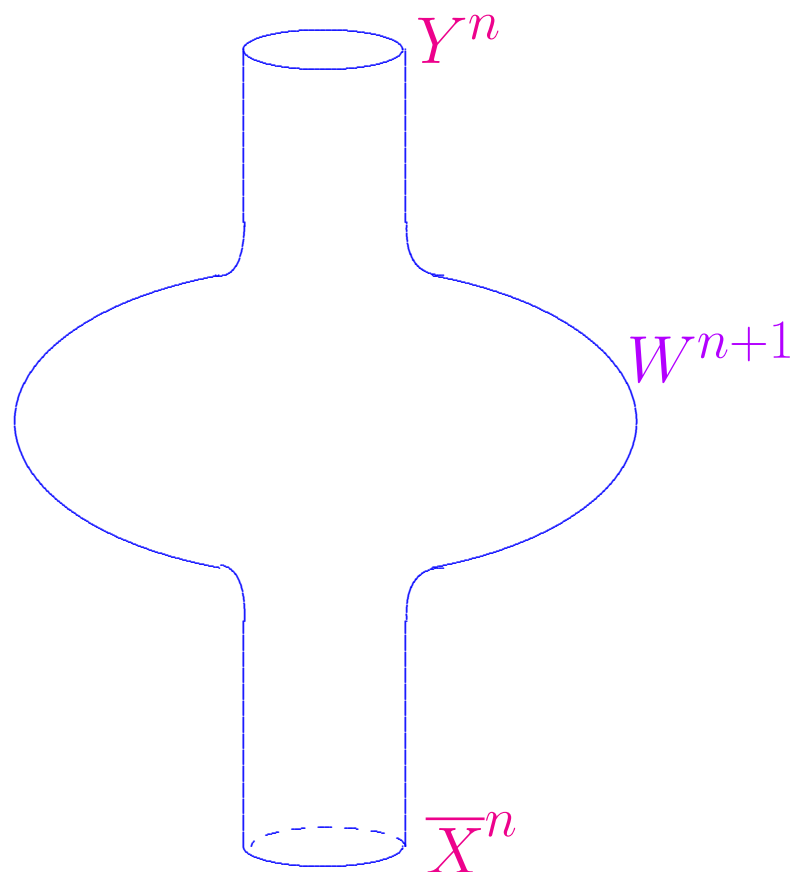


h -Cobordism

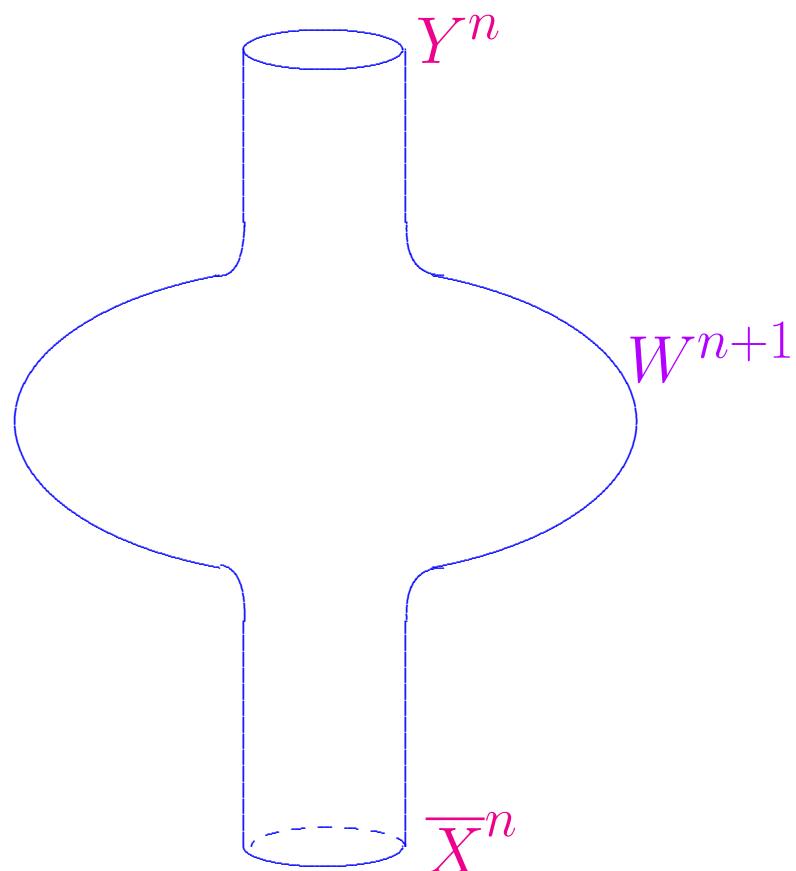
if $X \hookrightarrow W$, $Y \hookrightarrow W$ both homotopy equivalences



Smale: Suppose that X^n is h -cobordant to Y^n . If $\pi_1 = 0$ and $n > 4$, then X is diffeomorphic to Y .

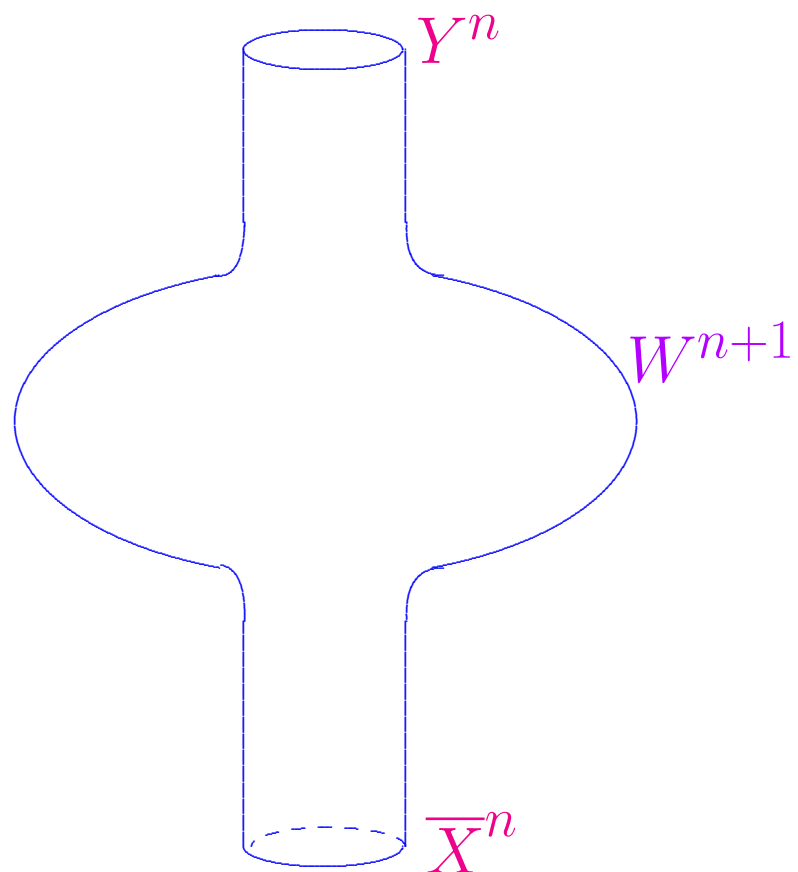


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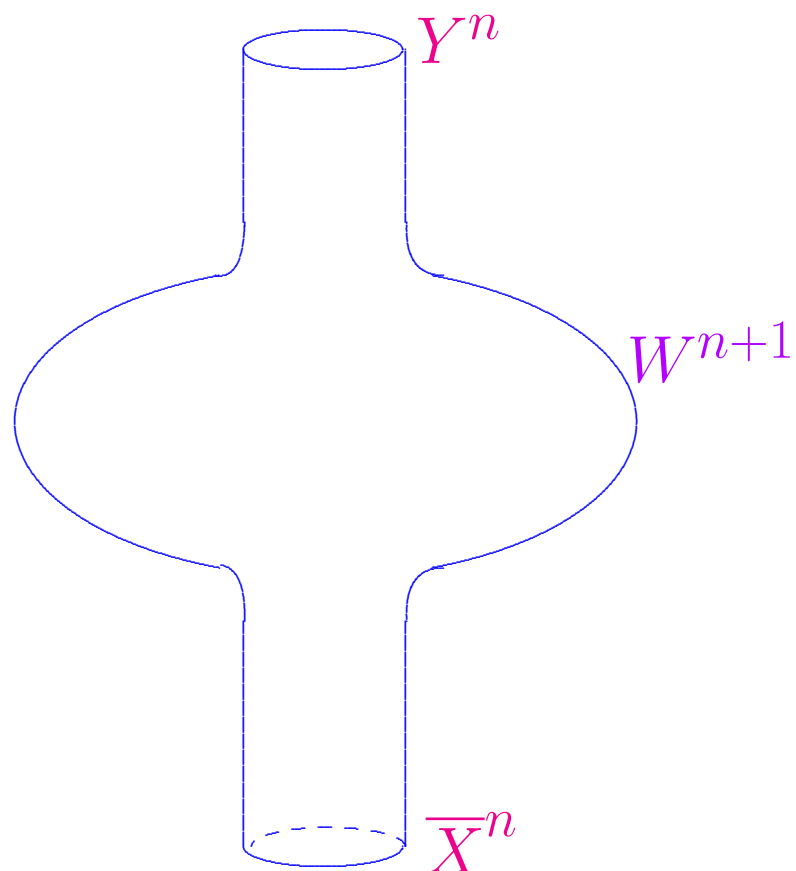


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But Smale doesn't apply when $n = 4$!

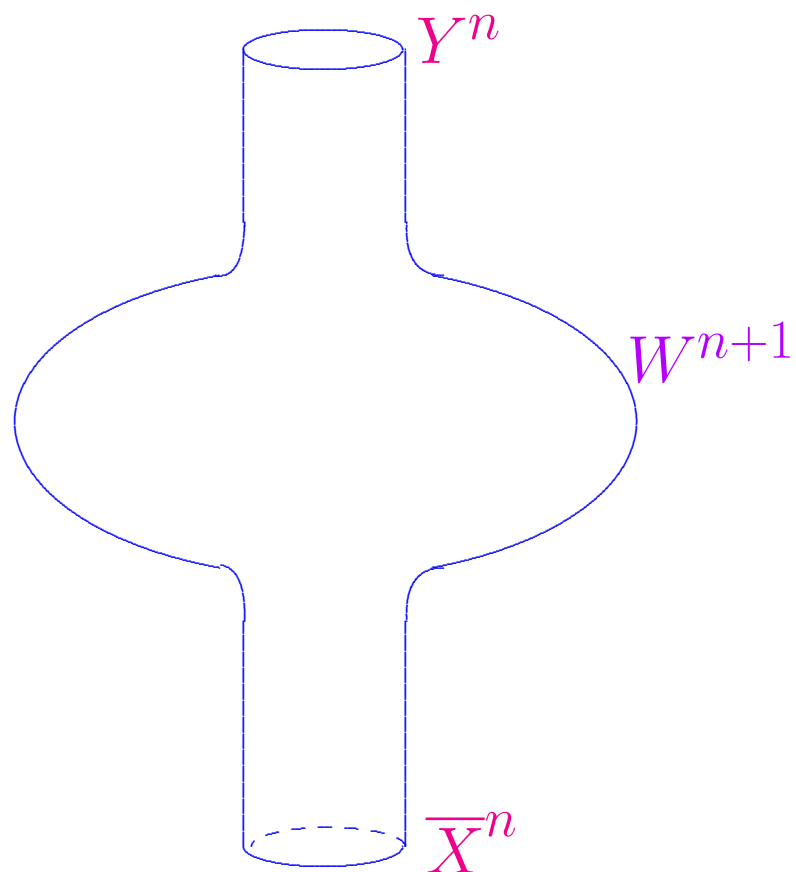


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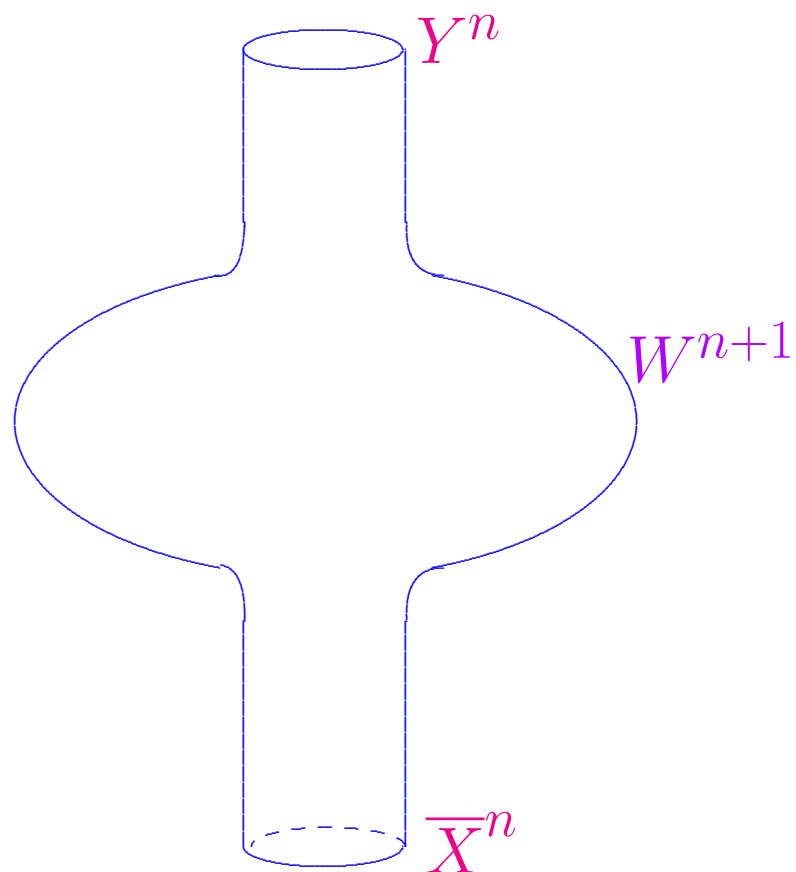
Lemma. *If X^4 and Y^4 are homotopy equivalent and simply connected, then $X \times X$ is actually diffeomorphic to $Y \times Y$.*



Indeed, if W is h -cobordism $X \sim Y$, then

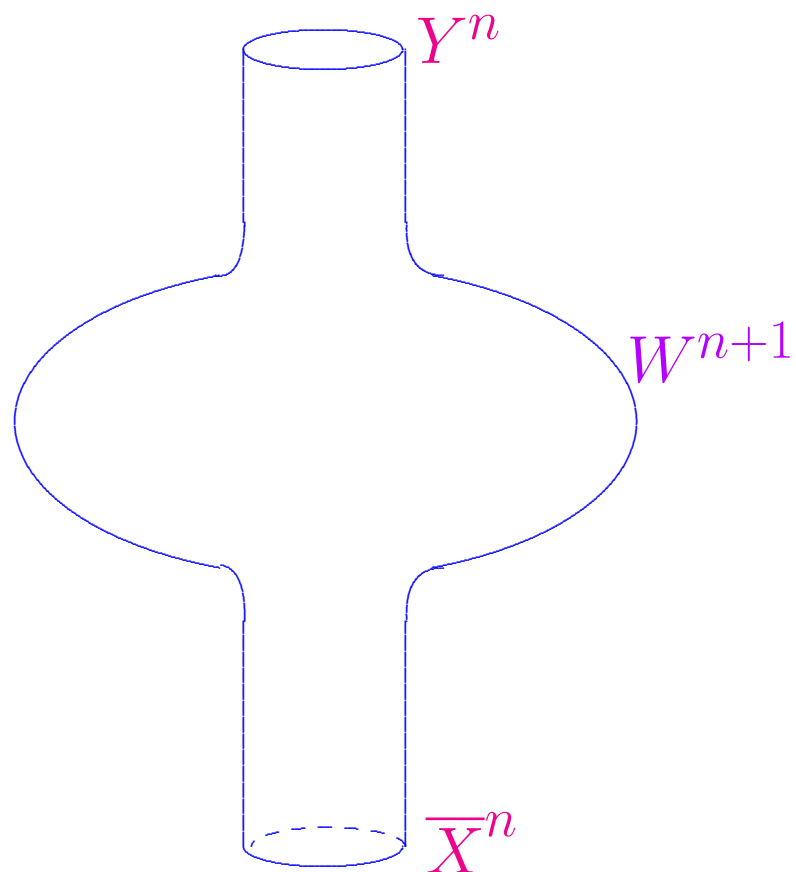
$$(X \times W) \cup_{X \times Y} (W \times Y)$$

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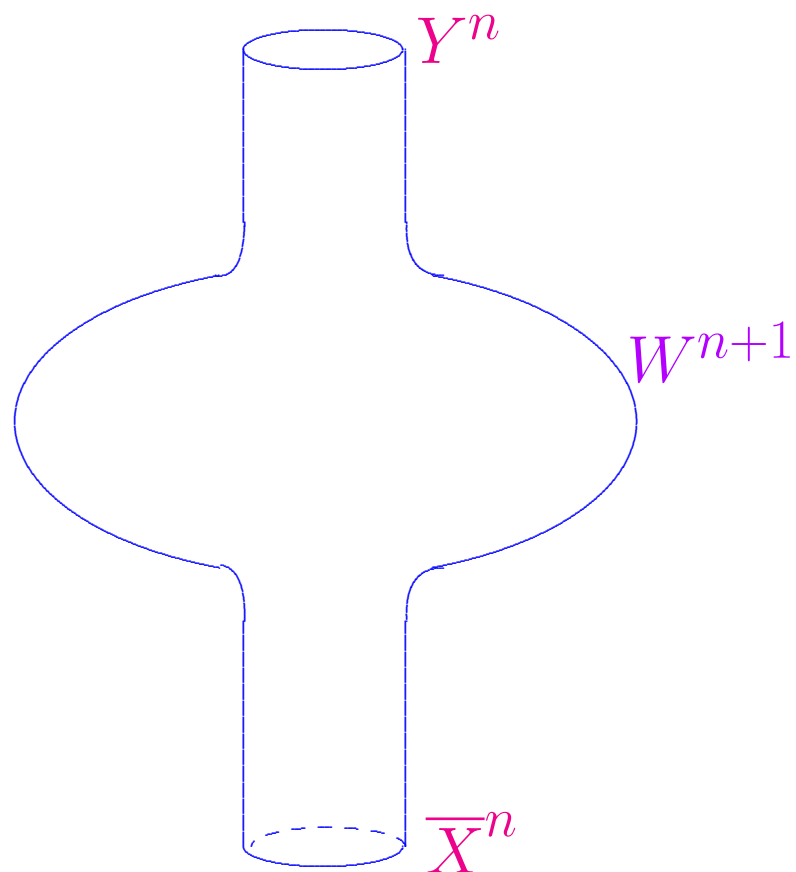
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Lemma. *If X^4 and Y^4 are simply connected, non-spin, with $\chi(X) = \chi(Y)$, $\tau(X) = \tau(Y)$, then*

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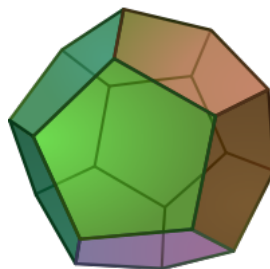
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$$\sum_{j=1}^5 z_j^5 = \frac{5}{4} \left(\sum_{j=1}^5 z_j^2 \right) \left(\sum_{j=1}^5 z_j^3 \right), \quad \sum_{j=1}^5 z_j = 0$$

is global quotient $F = S/\mathbb{Z}_2$ of smooth surface S .
 D_{10} action gen'd by (12345) and (25)(34) lifts to S .
 $Q = S/D_{10}$ is orbifold with four $\mathbb{Z}_2$ -singularities.
 Barlow surface X is minimal resolution of Q .

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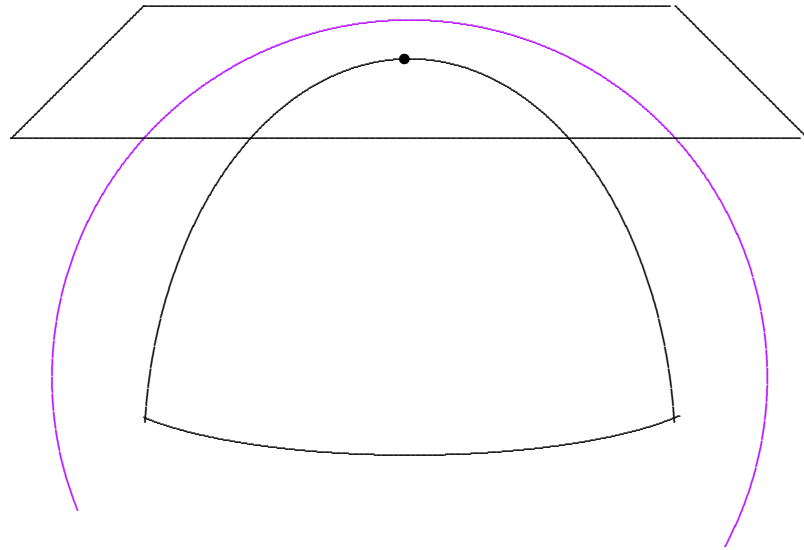
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holonomy

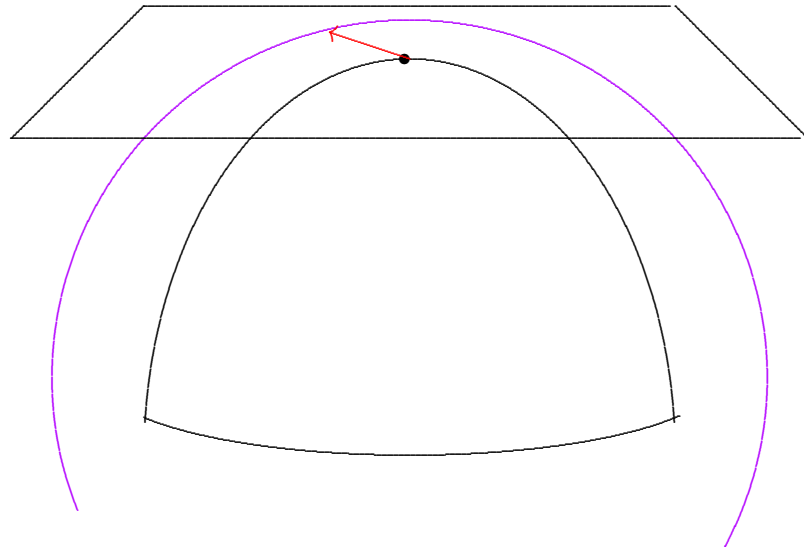
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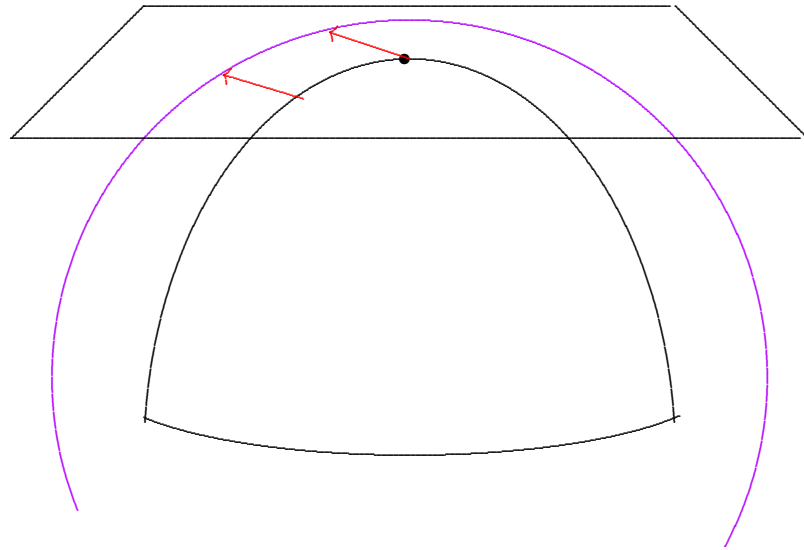
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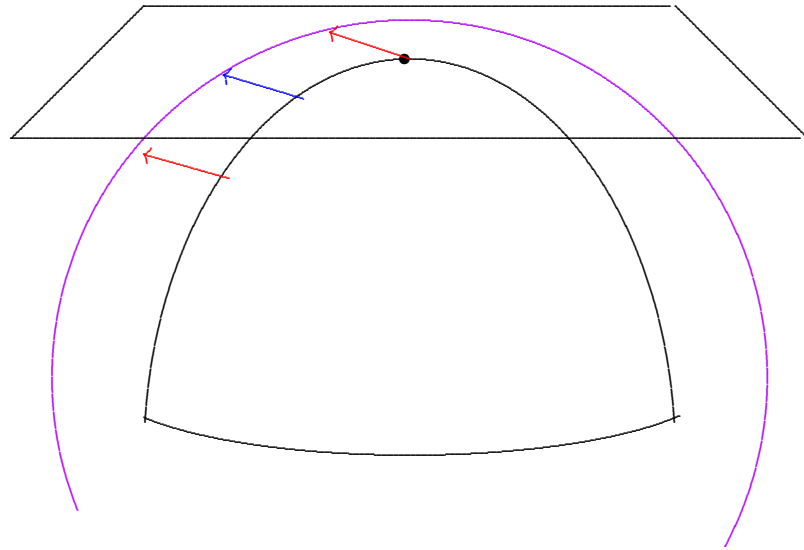
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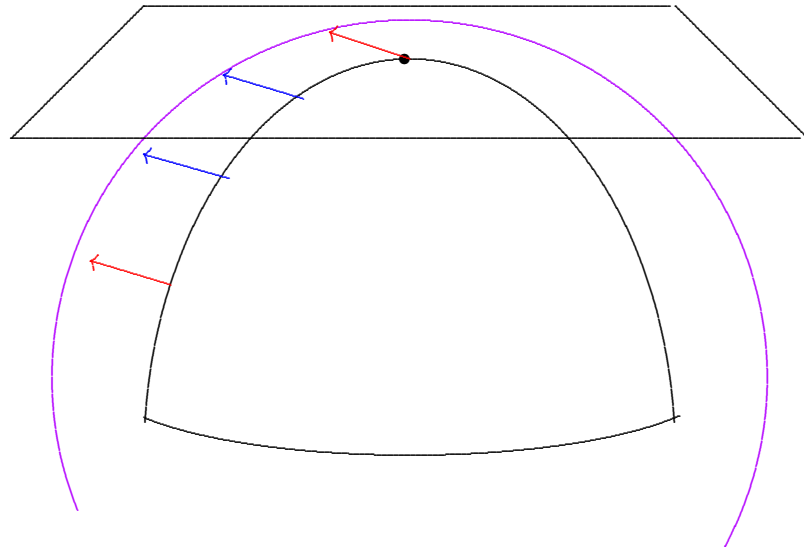
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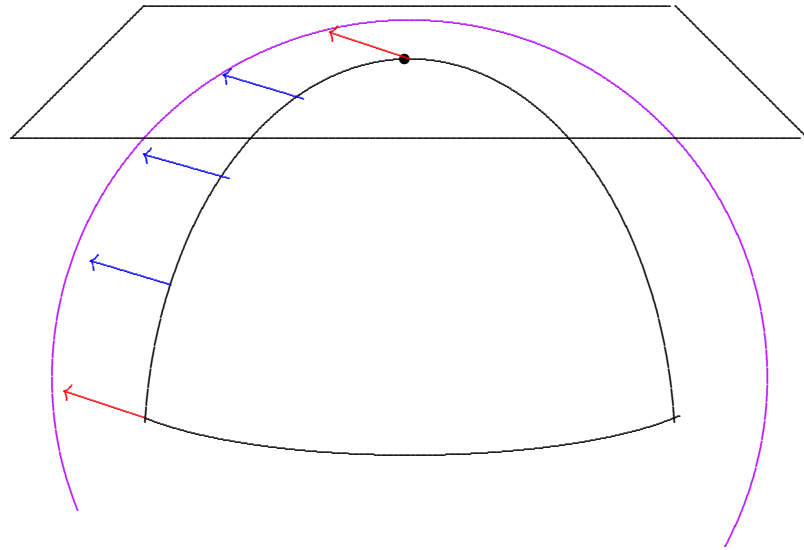
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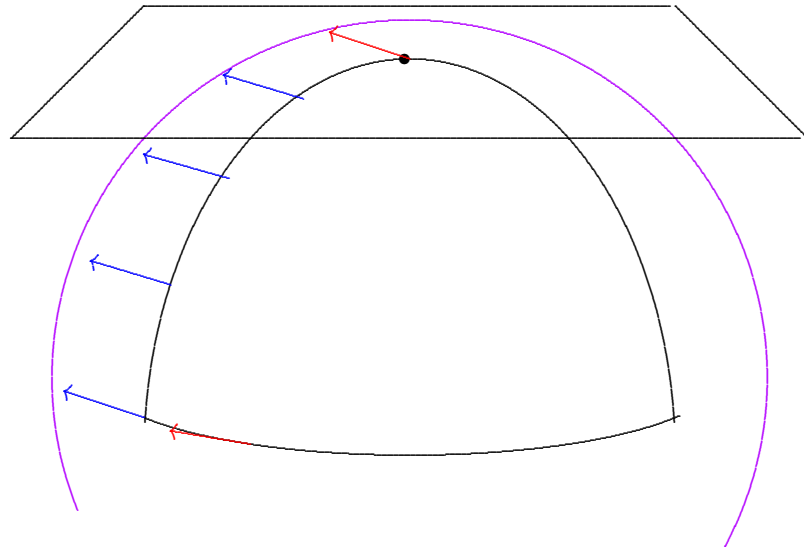
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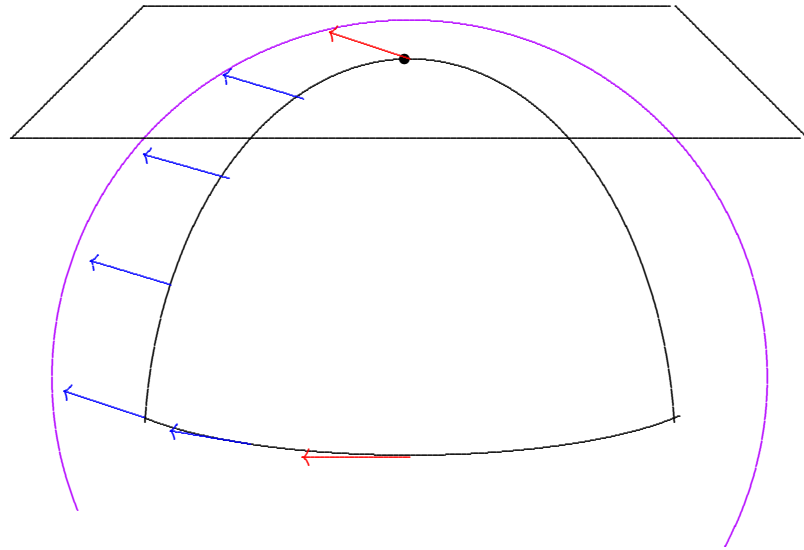
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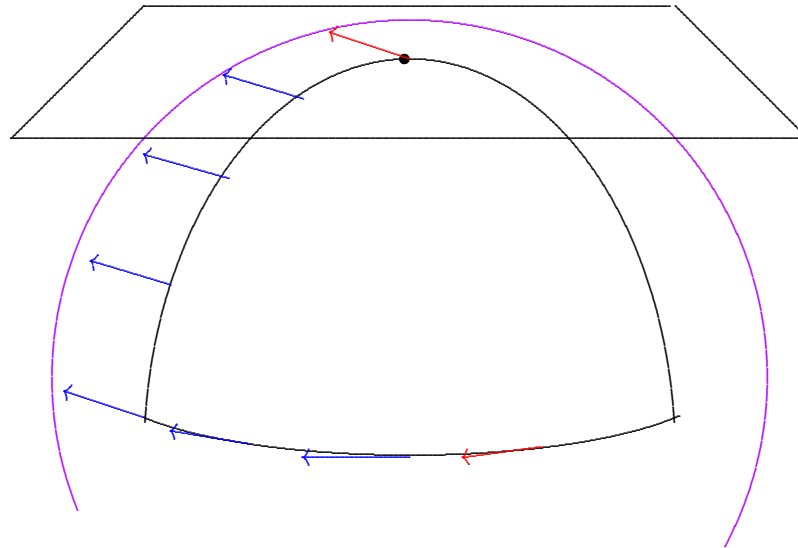
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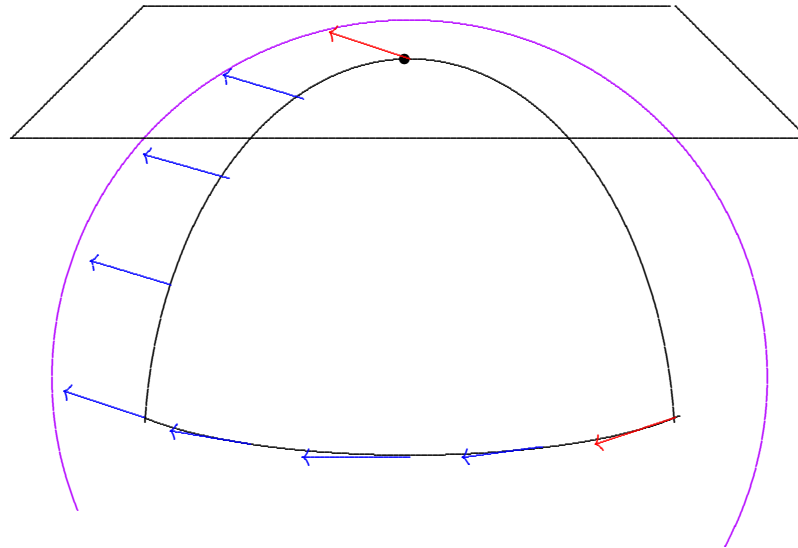
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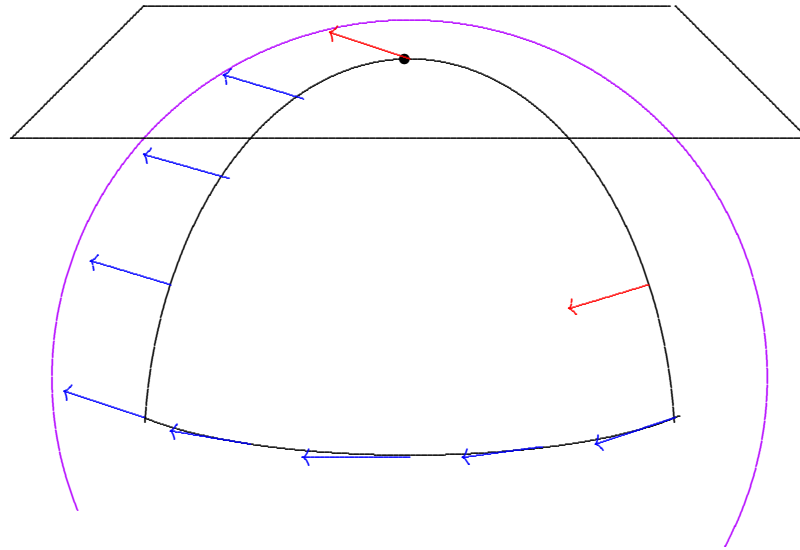
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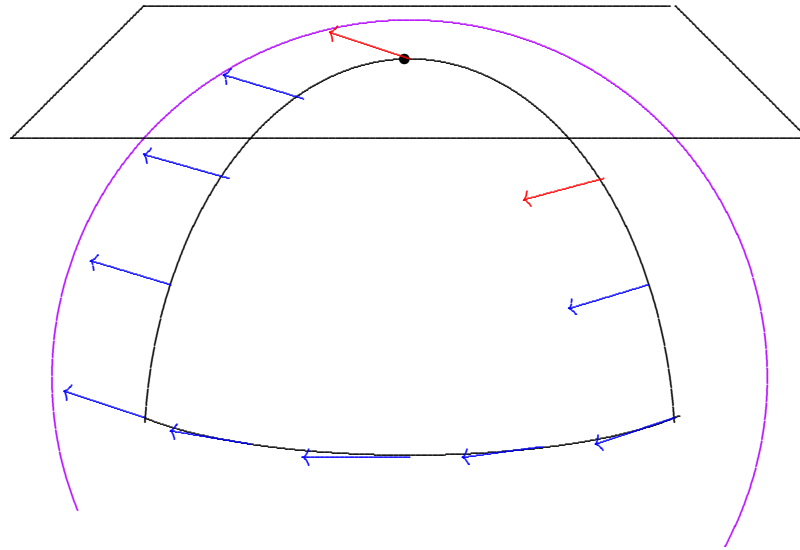
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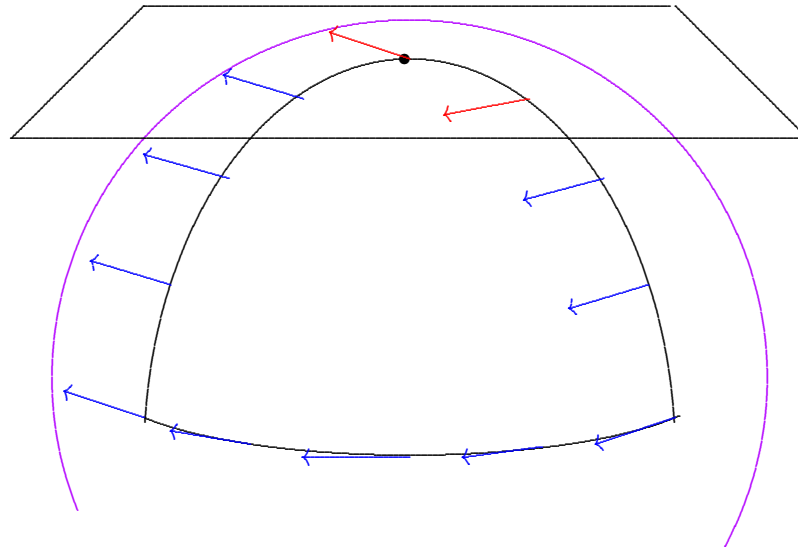
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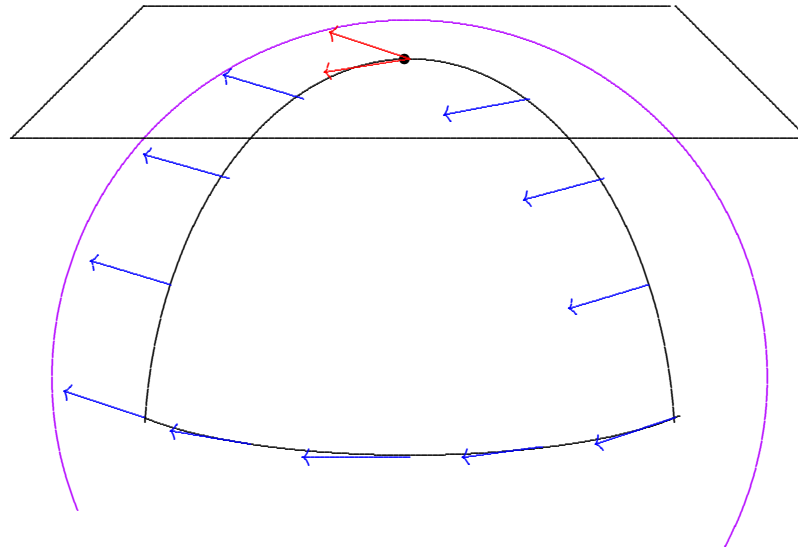
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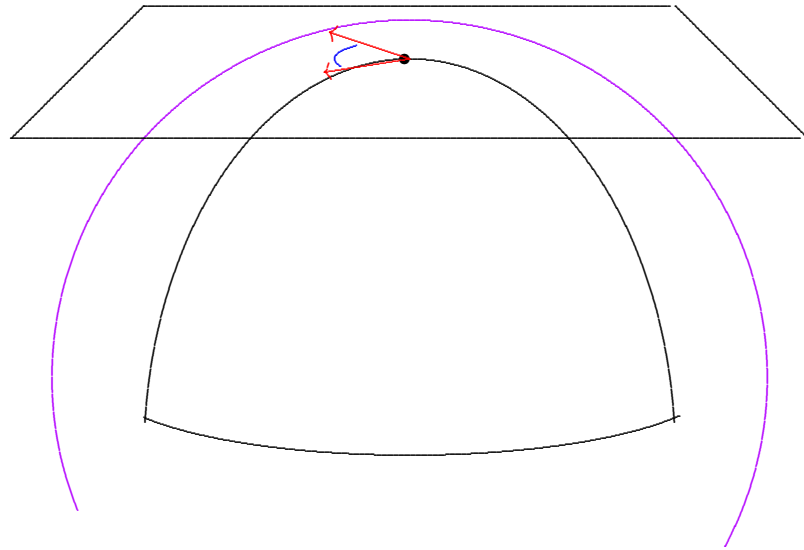
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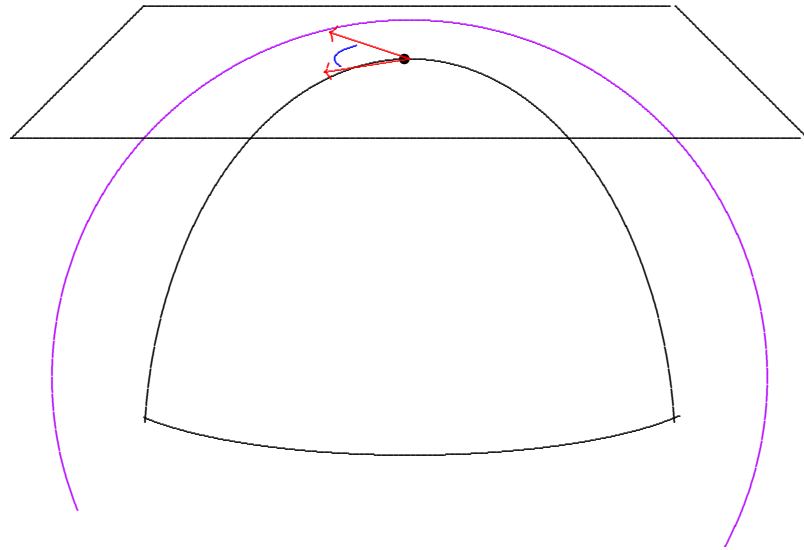
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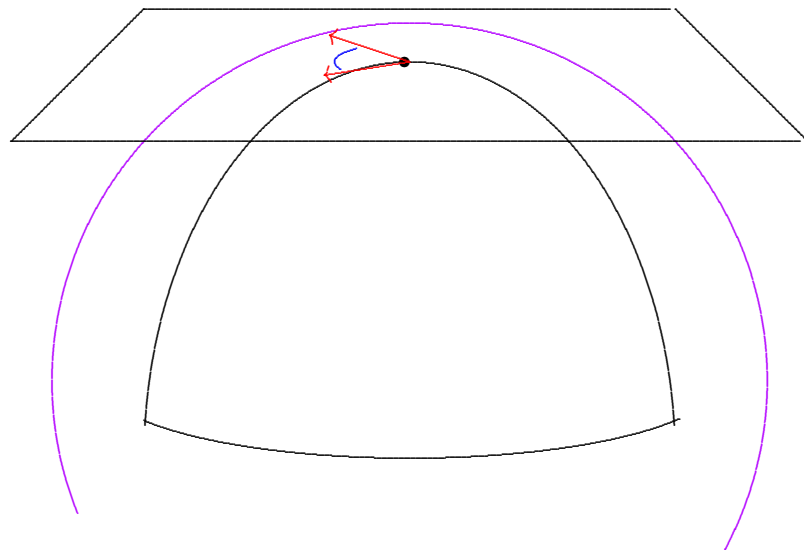
holonomy $\subset \mathbf{O}(n)$



Kähler metrics:

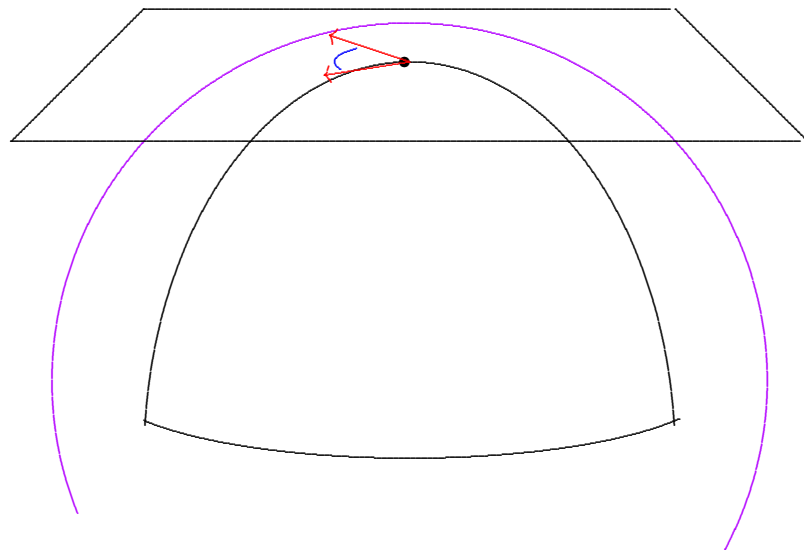
(M^{2m}, g) :

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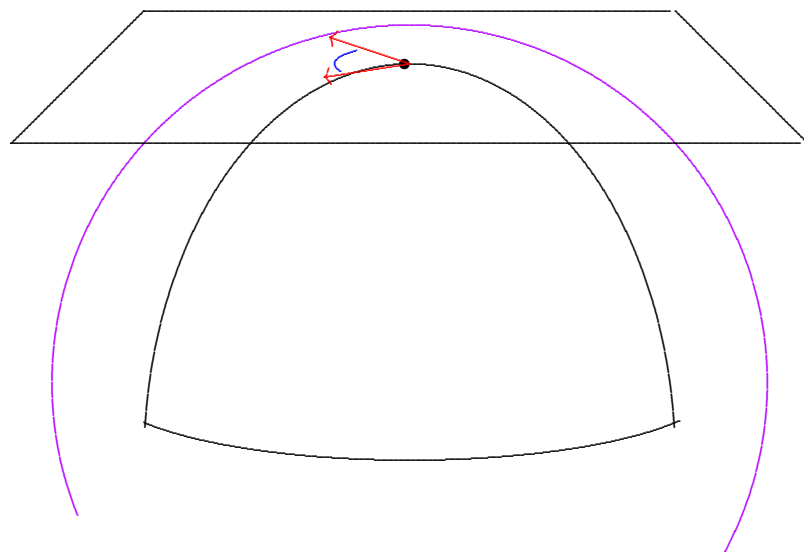
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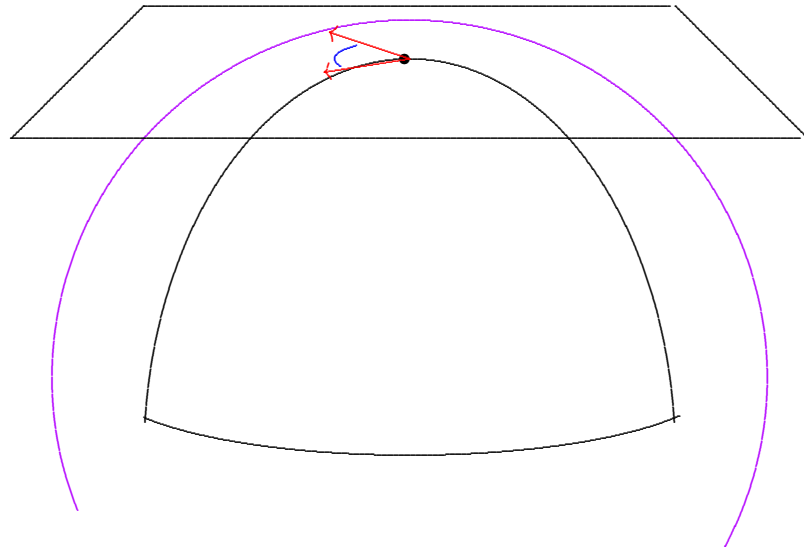
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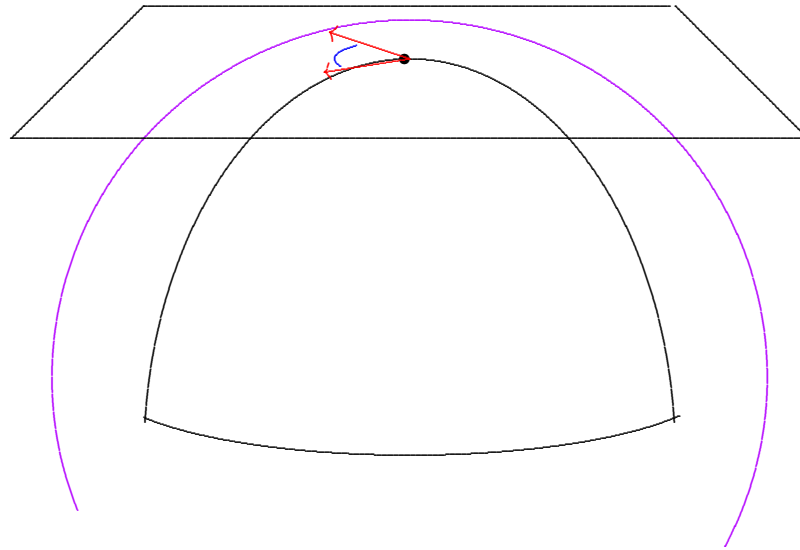
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What are the possible holonomy groups?

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$2m$	$\mathbf{U}(m)$	Kähler	$\mathbb{C}$

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dimension n	$\text{Hol}(M^n, g)$	geometry	relevant field
n	$\mathbf{SO}(n)$	generic	$\mathbb{R}$
$2m$	$\mathbf{U}(m)$	Kähler	$\mathbb{C}$
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n	$\mathbf{SO}(n)$	generic	?
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The latter seems to be especially delicate!

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Gromov-Lawson: $\widetilde{M}^7$ admits metrics with $s > 0$.

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Can such metrics coexist?

Alas, no!

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Theorem (L '25).

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To make this plausible,

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To make this plausible, will first illustrate assertion for prototypical examples due to S. Kobayashi '63.

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Basic forms:

$$\mathcal{E}_B^p(M, \mathfrak{F}) = \{\alpha \in \mathcal{E}^p(M) \mid \xi \lrcorner \alpha = 0, \xi \lrcorner d\alpha = 0\}$$

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Sub-complex of deRham complex:

$$\cdots \xrightarrow{d} \mathcal{E}_B^{p-1}(M, \mathfrak{F}) \xrightarrow{d} \mathcal{E}_B^p(M, \mathfrak{F}) \xrightarrow{d} \mathcal{E}_B^{p+1}(M, \mathfrak{F}) \xrightarrow{d} \cdots$$

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$$H_B^p(M, \mathfrak{F}) := \frac{\ker \left[d : \mathcal{E}_B^p(M, \mathfrak{F}) \rightarrow \mathcal{E}_B^{p+1}(M, \mathfrak{F}) \right]}{\text{image} \left[d : \mathcal{E}_B^{p-1}(M, \mathfrak{F}) \rightarrow \mathcal{E}_B^p(M, \mathfrak{F}) \right]} .$$

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Replace Hard Lefschetz on $H^*(X, \mathbb{R})$ with transverse version on $H_B^*(M, \mathfrak{F})$ due to El Kacimi-Alaoui.

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Proposition. *If smooth compact M^7 carries a Sasaki-Einstein metric g_1 , then $p_1(M) \in H^4(M, \mathbb{Z})$ is a torsion class.*

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Theorem. *No smooth compact M^7 can admit both a Sasaki-Einstein metric g_1 and a metric g_2 with holonomy $\subset \mathbf{G}_2$.*

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CTC Wall: Set of invariants that determines when a given 6-manifold is diffeomorphic to one of these.

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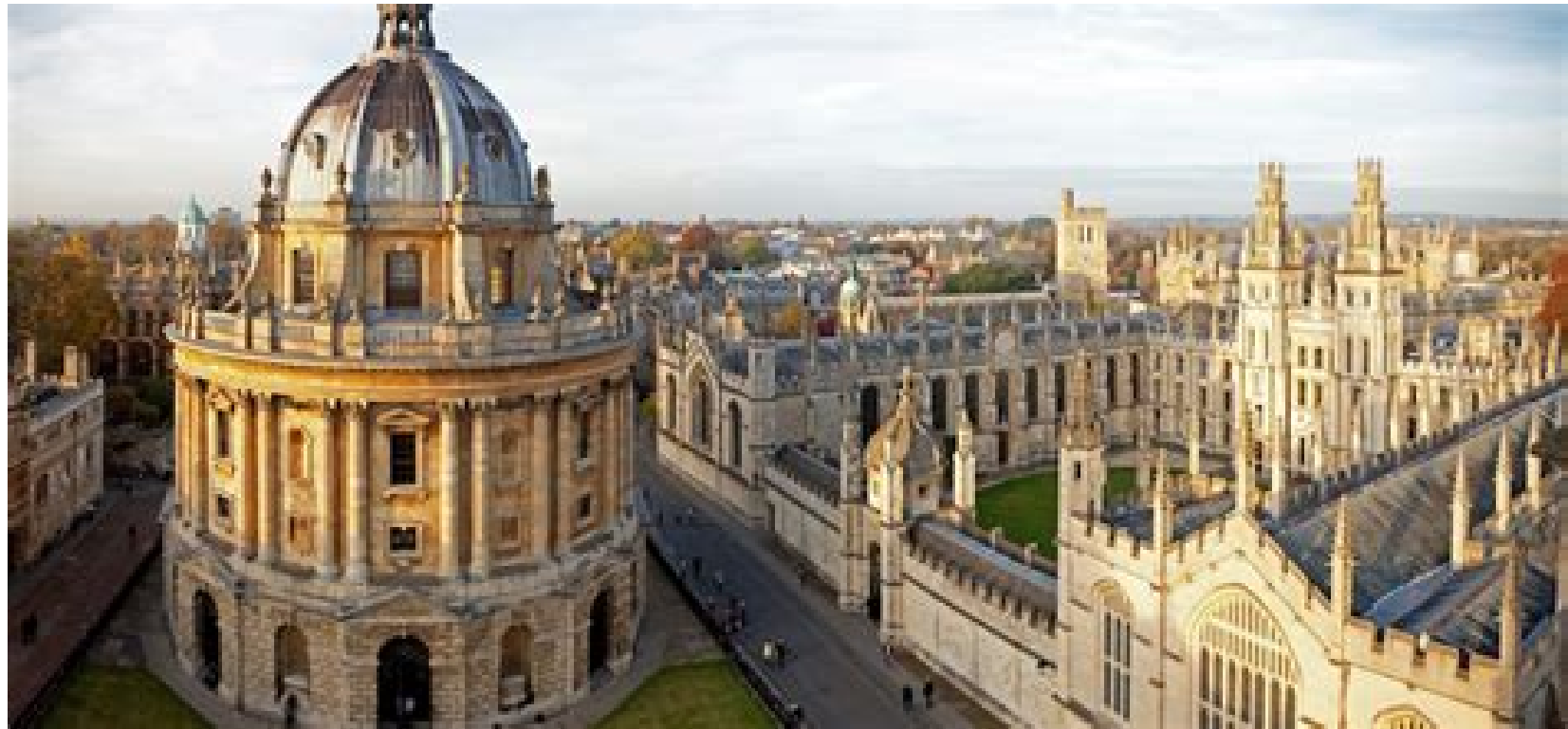
But no one has yet discovered a Calabi-Yau partner for any of these Fano manifolds!

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