On the Kähler Classes of Extremal Metrics

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Abstract

Let (M, J) be a compact complex manifold, and let $\mathbf{E} \subset H^{1,1}(M, \mathbf{R})$ be the set of all cohomology classes which can be represented by Kähler forms of extremal Kähler metrics, in the sense of Calabi [3]. Then $\mathbf{E}$ is an open subset.

*Supported in part by NSF grant DMS 92-04093.
†Supported in part by NSF grant DMS 90-22140.

1 Introduction

Let (M, J) be a compact complex manifold, let $[\omega] \in H^{1,1}(M, \mathbf{R}) \subset H^2(M, \mathbf{R})$ be the deRham class of a Kähler form, and let $[\omega]^+$ denote the set of all Kähler forms in this fixed cohomology class. In an attempt to represent the given class by a canonical metric, Calabi [3, 4] proposed that one should seek critical points of the functional

$$\begin{array}{ccc} [\omega]^+ & \xrightarrow{\mathcal{C}} & \mathbf{R} \\ \omega & \mapsto & \int_M \mathbf{s}_\omega^2 d\mu_\omega \end{array} , \tag{1.1}$$

where the metric associated with the form ω has scalar curvature $\mathbf{s}_\omega$ and volume form $d\mu_\omega$. He dubbed the critical points of this functional *extremal Kähler metrics*, and then observed that a Kähler metric is extremal iff the gradient of its scalar curvature $\mathbf{s}$ is a real-holomorphic vector field. In particular, a Kähler metric of constant scalar curvature is automatically extremal; and if M supports no non-trivial holomorphic vector fields, every extremal Kähler metric must conversely have constant scalar curvature. However, Calabi also produced examples [3] of compact extremal Kähler manifolds of non-constant scalar curvature; and an entire menagerie of such manifolds [6, 9] is now known.

In a previous paper [9], the authors investigated the existence of extremal Kähler metrics near a metric of constant scalar curvature. Our present aim is the generalization of those results to a strictly extremal setting. The upshot is the following:

Theorem A *Let (M, J) be a compact complex manifold, and let $H^{1,1}(M, \mathbf{R})$ denote the kernel of the natural homomorphism $H^2(M, \mathbf{R}) \to H^2(M, \mathcal{O})$. Let $\mathbf{E}$ be the set of deRham classes swept out by the Kähler forms of extremal Kähler metrics. Then $\mathbf{E}$ is an open subset of $H^{1,1}(M, \mathbf{R})$.*

It is tempting to ask whether $\mathbf{E}$ is always either the Kähler cone or the empty set. For the present, unfortunately, we must leave this question unanswered.

2 Notation and Conventions

A Riemannian $2n$-manifold (M, g) is said to be *Kähler* with respect to an almost-complex structure J on M if g is J-invariant and J is invariant with respect to the Levi-Cività parallel transport of g. These conditions are equivalent to saying that (M, J) is a complex n-manifold and that $\omega(\xi, \eta) := g(J\xi, \eta)$ defines a closed 2-form, called the *Kähler form* of (M, g, J).

Under the action of J, the complexified tangent $\mathbf{C} \otimes T^*M$ splits into a direct sum $T^{1,0} \oplus T^{0,1}$ of n-dimensional eigenspaces, and this induces a decomposition of each bundle of complex-valued tensors on M. In particular, the r-forms on M decompose as sums of forms of type (p, q):

$$\mathcal{E}^r = \bigoplus_{p+q=r} \mathcal{E}^{p,q} \ .$$

(For example, any Kähler form has type $(1, 1)$ by virtue of its J-invariance.) Because J is integrable, the exterior derivative d simply breaks up as $d = \partial + \bar{\partial}$, where $\partial : \mathcal{E}^{p,q} \to \mathcal{E}^{p+1,q}$, $\bar{\partial} : \mathcal{E}^{p,q} \to \mathcal{E}^{p,q+1}$, $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$.

The metric g may be extended as a complex-bilinear inner product on any bundle of complex tensors over M. In addition to this bilinear extension $(\cdot, \cdot)$ of g, however, we shall also need the sesqui-linear inner product $\langle \varphi, \psi \rangle := (\varphi, \bar{\psi})$. The associated inner products of global sections will be denoted by

$$\langle \varphi, \psi \rangle_{L^2} := \int_M (\varphi, \bar{\psi}) \ d\mu \ .$$

An important geometric invariant of (M, g, J) is its *Ricci form* ρ, which may be expressed as

$$\rho(\xi, \eta) = \mathbf{r}(J\xi, \eta)$$

in terms of the Ricci tensor $\mathbf{r}$ of g. A remarkable feature of Kähler geometry is the fact that $i\rho$ is the curvature of the canonical line bundle $\kappa := \Lambda^{n,0}$. This has many implications for the *scalar curvature* $\mathbf{s}$, which is defined to be the trace of the Ricci tensor, and so can conveniently be calculated by the formula

$$\mathbf{s} \ \omega^{\wedge n} = 2n \ \rho \wedge \omega^{\wedge(n-1)}.$$

3 Holomorphic Vector Fields

If (M,J,g) be a Kähler manifold, any complex-valued smooth function gives rise to a smooth vector field of type $(1,0)$ by the rule $f \mapsto \partial^{\#} f = \partial^{\#}_g f$, where $\partial^{\#} f := (\overline{\partial} f)^{\#}$ is the type $(1,0)$ piece of the gradient of f with respect to g. This is, of course, not generally a *holomorphic* vector field— for that, we'd need to impose the equation $\overline{\partial}\partial^{\#} f = 0$. This condition, however, is equivalent to the fourth-order equation

$$(\overline{\partial}\partial^{\#})^{*}\overline{\partial}\partial^{\#} f = 0 \ , \tag{3.1}$$

since $\langle f, (\overline{\partial}\partial^{\#})^{*}\overline{\partial}\partial^{\#} f\rangle_{L^2} = \|\overline{\partial}\partial^{\#} f\|^2_{L^2}$. Every complex-valued solution f of (3.1) therefore is associated with a holomorphic vector field $\Xi = \partial^{\#} f$. The set of holomorphic vector fields arising in this way can be characterized [9] in the following simple manner:

Proposition 1 *Let (M,J,g) be a compact Kähler manifold and let Ξ be a holomorphic vector field on M. Then there exists a function $f : M \stackrel{C^\infty}{\to} \mathbf{C}$ such that $\Xi = \partial^{\#}_g f$ iff Ξ vanishes at some point of M.*

Let $\mathbf{h}(M)$ be the complex Lie algebra of holomorphic vector fields of the complex manifold (M,J); since we assume that M is compact, this is precisely the Lie algebra of the group of biholomorphism of (M,J). We denote by $\mathbf{h}_0(M) \subset \mathbf{h}(M)$ the subset of vector fields with zeroes. Proposition 1 then tells us that $\mathbf{h}_0(M)$ is— miraculously enough— a linear subspace of $\mathbf{h}(M)$. Indeed, a more detailed analysis [9] proves that $\mathbf{h}_0(M)$ is actually an ideal, and that the quotient algebra $\mathbf{h}(M)/\mathbf{h}_0(M)$ is Abelian.

The self-adjoint operator $(\overline{\partial}\partial^{\#})^{*}\overline{\partial}\partial^{\#}$ is elliptic. Indeed, for any Kähler metric one has

$$\begin{aligned}
(\overline{\partial}\partial^{\#})^{*}\overline{\partial}\partial^{\#} f &:= \nabla_j \nabla^{\overline{k}} \nabla_{\overline{k}} \nabla^j f \\
&= \nabla_j (\nabla^j \nabla^{\overline{k}} \nabla_{\overline{k}} f - \mathrm{R}^{\overline{k}j\overline{\ell}}{}_{\overline{k}} \nabla_{\overline{\ell}} f) \\
&= \nabla_j (\nabla^j \nabla^{\overline{k}} \nabla_{\overline{k}} f + \mathbf{r}^{j\overline{\ell}} \nabla_{\overline{\ell}} f) \\
&= \frac{1}{4}\Delta^2 f + \mathbf{r}^{j\overline{\ell}} \nabla_j \nabla_{\overline{\ell}} f + (\nabla_j \mathbf{r}^{j\overline{\ell}}) \nabla_{\overline{\ell}} f
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\Delta^2 f + \frac{1}{2}\mathbf{r}^{\mu\nu}\nabla_\mu\nabla_\nu f + \frac{1}{2}(\nabla^{\bar{\ell}}\mathbf{s})\nabla_{\bar{\ell}} f \\
&= \frac{1}{4}(\Delta^2 + 2\mathbf{r}\cdot\nabla\nabla + 2(\partial\mathbf{s})\lrcorner\partial^{\#})f \ , \qquad (3.2)
\end{aligned}
$$

where we have used the contracted Bianchi identity $\nabla_\mu \mathbf{s} = 2\nabla^\nu \mathbf{r}_{\mu\nu}$. Notice, incidentally, that the operator in question is only real if our metric g happens to have *constant* scalar curvature $\mathbf{s}$.

The dimension of the space of complex-valued solutions of equation (3.1) is independent of the choice of Kähler metric. But as the operator $(\overline{\partial}\partial^{\#})^*\overline{\partial}\partial^{\#}$ is not generally real, the dimension of the space of *real* solutions of (3.1) will generally depend on g. Fortunately [9], this pathology can be completely understood in geometric terms:

Proposition 2 *Let (M, J, g) be a compact Kähler manifold. If f is a* real *solution of $(\overline{\partial}\partial^{\#}_g)^*\overline{\partial}\partial^{\#}_g f = 0$, then $\Im\mathrm{m}\partial^{\#} f$ is a Killing field of g. Moreover, a Killing field arises this way iff it has a zero.*

Given a Kähler metric g and a vector field $\Xi \in \mathbf{h}_0(M)$ which vanishes at some point of M, the Proposition 1 guarantees the existence of a function $f_{g,\Xi}$ such that $\Xi = \partial^{\#}_g f_{g,\Xi}$. The dependence of this function on the metric may be described [9] as follows:

Proposition 3 *Let $L^2_{k+1,\mathbf{C}}$ denote the Sobolev space of complex-valued functions on M whose first $k+1$ distributional derivatives are in L^2, with the usual Hilbert space inner product induced by some fixed but arbitrary Kähler metric g on (M, J), and let $\mathcal{F}^{1,1}_k$ be similarly the space of $(1,1)$-forms on (M, J) of Sobolev class L^2_k. Let $\mathbf{G}$ be the Green's operator of the Laplacian of g. Then the bounded $\mathbf{C}$-bilinear map*

$$
\begin{array}{rcl}
\mathbf{h}_0(M) \times \mathcal{F}^{1,1}_k & \stackrel{\mathbf{P}}{\longrightarrow} & L^2_{k+1,\mathbf{C}} \\
(\Xi, \chi) & \mapsto & 2i\mathbf{G}\overline{\partial}^*_g(\Xi\lrcorner\chi)
\end{array}
$$

has the property that

$$\partial^{\#}_{\tilde{g}}\mathbf{P}(\Xi, \tilde{\omega}) = \Xi$$

whenever $\tilde{\omega}$ is the Kähler form of a Kähler metric $\tilde{g}$ (and so is real, closed, and positive).

In this article, we will be interested in a very special class of Kähler metrics:

Definition 1 *A C^2 Kähler metric g is said to be* extremal *if its scalar curvature* $\mathbf{s}$ *satisfies* $\overline{\partial}\partial^{\#}_{g}\mathbf{s} = 0$.

Of course, this is equivalent to saying that $\mathbf{s}$ satisfies (3.1), and it is in this form [3] that the extremal condition arises from the Euler-Lagrange equations of the variational problem (1.1). The above definition, however, makes the proof of the following regularity result particularly simple:

Proposition 4 *Let g be an extremal Kähler metric on a compact complex manifold. Then g is smooth with respect to the complex atlas.*

Proof. By assumption, the metric is of Hölder class $C^{1,\alpha}$ for any $\alpha \in (0,1)$. We will now show by induction that it is of class $C^{\ell,\alpha}$ for all $\ell \in \mathbf{N}$.

The gradient of the scalar curvature $\mathbf{s}$ is, by assumption, the real part of a holomorphic vector field, and so real-analytic. If the metric g is of class $C^{\ell,\alpha}$, it therefore follows that $d\mathbf{s}$ is of class $C^{\ell,\alpha}$, and $\mathbf{s}$ is itself therefore of class $C^{\ell+1,\alpha}$. But the equation that the scalar curvature is $\mathbf{s}$ may be rewritten in a complex coordinate chart as the pair of equations

$$-2g^{j\bar{k}}\frac{\partial^2}{\partial z^j \partial \bar{z}^k}u = \mathbf{s} \tag{3.3}$$

$$\det\left(g_{j\bar{k}}\right) = e^u \; . \tag{3.4}$$

From (3.3) and the regularity theory of linear elliptic operators [8] it follows that u is of class $C^{\ell+2,\alpha}$; but, since $\ell+2>0$, the regularity theory [1] for the Monge-Ampère equation (3.4) in turn guarantees that g is of class $C^{\ell+2,\alpha}$. The claim thus follows by induction on ℓ. ∎

The isometry group of any compact Kähler manifold is always a compact subgroup of the complex biholomorphism group. But if the Kähler metric is extremal, much more is true. Indeed, generalizing earlier work of Matsushima and Lichnerowicz, Calabi [4] proved the following result, which will be critical importance for us:

Proposition 5 (Calabi) *Let g be an extremal Kähler metric on a compact complex manifold (M, J). Let G denote the identity component of the isometry group of (M, g), and let $\mathbf{H}$ denote the identity component of the biholomorphism group of (M, J). Then $G \subset \mathbf{H}$ is a maximal compact subgroup.*

Corollary 1 *Let g and $\tilde{g}$ be two extremal Kähler metrics on a compact complex manifold (M, J). Then there is a biholomorphism Φ of M such that the identity components of the isometry groups of g and $\Phi^*\tilde{g}$ coincide.*

Proof. Let G and $\tilde{G}$ denote the identity components of the isometry groups of g and $\tilde{g}$, respectively. By Proposition 5, G and $\tilde{G}$ are maximal compact subgroups of the connected Lie group $\mathbf{H}$, and so, by a theorem of Iwasawa ([7], p.530), they are conjugate: there is an element $\Phi \in \mathbf{H}$ such that $\tilde{G} = \Phi G \Phi^{-1}$. But since every element of $\Phi^{-1}\tilde{G}\Phi$ automatically preserves $\Phi^*\tilde{g}$, we conclude that G is the identity component of the isometry group of the extremal Kähler metric $\Phi^*\tilde{g}$. $\blacksquare$

Thus, modulo biholomorphisms, the search for extremal Kähler metrics is completely equivalent to the search for extremal metrics among the Kähler metrics which are invariant under the action of a fixed maximal compact subgroup $G \subset \mathbf{H}$ of the connected biholomorphism group. However, for a host of technical reasons, the latter problem will turn out, for our purposes, to be much more tractable.

4 Scalar Curvature as an Operator

If (M, J) is a complex manifold, we will use $H^{1,1}(M, \mathbf{R})$ to denote the kernel of the natural homomorphism $H^2(M, \mathbf{R}) \to H^2(M, \mathcal{O})$ induced by the inclusion $\mathbf{R} \hookrightarrow \mathcal{O}$. Equivalently, $H^{1,1}(M, \mathbf{R})$ is the linear subspace of $H^2(M, \mathbf{R})$ consisting of those deRham classes which are representable by real closed (1,1)-forms. This equivalence is an immediate consequence of the fact that we have a commutative diagram

$$\begin{array}{ccccccc}
0 & \rightarrow & \mathbf{R} & \rightarrow & \mathcal{E}^0_{\mathbf{R}} & \stackrel{d}{\rightarrow} & \mathcal{E}^\bullet_{\mathbf{R}} \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \mathcal{O} & \rightarrow & \mathcal{E}^{0,0} & \stackrel{\overline{\partial}}{\rightarrow} & \mathcal{E}^{0,\bullet}
\end{array}$$

of fine resolutions. Indeed, the kernel of $H^2(M, \mathbf{R}) \rightarrow H^2(M, \mathcal{O})$ is thereby identified with

$$\{\text{real 2-forms } \gamma \ \mid \ d\gamma = 0, \gamma^{0,2} = \overline{\partial}\nu^{0,1}\}/\{\text{exact forms}\}.$$

But any such class $[\gamma]$ can also be represented by the real closed $(1,1)$-form $\gamma - d(\nu + \overline{\nu})$.

If M is now compact and we are given a Kähler metric g on M, Hodge theory [5] tells us that $H^{1,1}(M, \mathbf{R})$ may be identified with the space $\mathcal{H}^{1,1}(M)$ of real-valued g-harmonic (1,1)-forms on M. We will now assume that g is smooth, so that elliptic regularity implies that $\mathcal{H}^{1,1}(M)$ consists entirely of smooth forms, and we will denote the Kähler form of g by ω.

Let $k > n$, and let $\mathcal{U} \subset \mathcal{H}^{1,1}(M) \times L^2_{k+4}(M)$ be the open neighborhood of $(0,0)$ consisting of pairs (α, φ) such that $\tilde{\omega} = \omega + \alpha + i\partial\overline{\partial}\varphi$ is the Kähler form of a C^2 Kähler metric. We may then consider the map

$$\begin{array}{rccc}
\mathcal{H}^{1,1}(M) \times L^2_{k+4}(M) \supset & \mathcal{U} & \stackrel{\mathcal{S}}{\longrightarrow} & L^2_k(M) \\
 & (\alpha, \varphi) & \mapsto & \mathbf{s}(\tilde{\omega}) \ ,
\end{array} \tag{4.1}$$

where $\mathbf{s}(\tilde{\omega}) = \mathbf{s}(\omega + \alpha + i\partial\overline{\partial}\varphi)$ is the scalar curvature of the metric with the Kähler from $\omega + \alpha + i\partial\overline{\partial}\varphi$. We have observed elsewhere [9] that

Proposition 6 *For $k > n$ the map $\mathcal{S}$ in* (4.1) *is well-defined and C^1. Moreover, its Fréchet derivative at the origin is given by*

$$D\mathcal{S}_{(0,0)} = \left[-2(\rho, \ \cdot \ ) \quad -\frac{1}{2}(\Delta^2 + 2\mathbf{r} \cdot \nabla\nabla) \right] \ , \tag{4.2}$$

where $\mathbf{r}\cdot$ denotes full contraction with the Ricci tensor of g.

Notice that the fourth-order operator $\mathcal{L} = -\frac{1}{2}(\Delta^2 + 2\mathbf{r} \cdot \nabla\nabla)$ occurring in the above result also appeared in (3.2), which we may now rewrite as

$$\mathcal{L} = -2(\overline{\partial}\partial^\#)^*\overline{\partial}\partial^\# + \partial\mathbf{s} \lrcorner \partial^\# \ . \tag{4.3}$$

5 Openness

Let (M, J, g) be a compact complex manifold with C^2 extremal Kähler metric. Proposition 4 then informs us that g is actually smooth, while Proposition 5 tells us that the connected component G of the isometry group of g is a maximal compact Lie group of the the biholomorphism group $\mathbf{H}$.

Let $L^2_{k,G}$ denote the real Hilbert space of G-invariant real-valued functions of class L^2_k. Now every g-harmonic form is invariant under G, since the connected isometry group G obviously sends every harmonic form to a harmonic form in the same cohomology class. Setting $\mathcal{V} = \mathcal{U} \cap (\mathcal{H}^{1,1}(M) \times L^2_{k+4,G})$, it then follows that for $(\alpha, \varphi) \in \mathcal{V}$ the Kähler metric $\tilde{g}$ with Kähler form

$$\tilde{\omega} = \omega + \alpha + i\partial\overline{\partial}\varphi$$

is G-invariant, and hence its scalar curvature $\tilde{\mathbf{s}}$ is G-invariant, too. For $k > n$, Proposition 6 therefore tells us that

$$\begin{array}{rccl} \mathcal{H}^{1,1}(M) \times L^2_{k+4,G} \supset & \mathcal{V} & \xrightarrow{\mathcal{S}_G} & L^2_{k,G} \\ & (\alpha, \varphi) & \mapsto & \mathbf{s}(\omega + \alpha + i\partial\overline{\partial}\varphi) \end{array} \qquad (5.1)$$

is a C^1 map whose Fréchet derivative at $(0, 0)$ is just the restriction of (4.2) to $\mathcal{H}^{1,1}(M) \times L^2_{k+4,G}$.

Let $\mathbf{z} \subset \mathbf{g}$ denote the center of $\mathbf{g}$, and let $\mathbf{z}_0 = \mathbf{z} \cap \mathbf{g}_0$, where $\mathbf{g}_0 \subset \mathbf{g}$ is the ideal of Killing fields which have zeroes. If $\tilde{g}$ is any G-invariant Kähler metric on (M, J), then, as a consequence of Proposition 2, each element of $\mathbf{z}_0$ is of the form $J \operatorname{grad} f$ for a real-valued solution of $(\overline{\partial}\partial^{\#}_{\tilde{g}})^* \overline{\partial}\partial_{\tilde{g}} f = 0$. Moreover, $\mathbf{z}_0$ thereby precisely corresponds to the set of real solutions f which are *invariant under* G, since

$$\partial^{\#} : \ker[(\overline{\partial}\partial^{\#}_{\tilde{g}})^* \overline{\partial}\partial_{\tilde{g}}] \to \mathbf{h}_0$$

is a homomorphism of G-modules.

These observations now allow us to show that the kernel of the restriction of $(\overline{\partial}\partial^{\#}_{\tilde{g}})^* \overline{\partial}\partial_{\tilde{g}}$ to $L^2_{k+4,G}$ depends smoothly on the G-invariant metric $\tilde{g}$. Indeed, choose a basis $\{\xi_1, \ldots, \xi_m\}$ for $\mathbf{z}_0$, and, for each $(1, 1)$-form χ on (M, J), set $p_0(\chi) = 1$ and

$$p_j(\chi) = 2i\mathbf{G}\overline{\partial}^*_g((J\xi + i\xi) \lrcorner \chi), \quad j = 1, \ldots, m.$$

If $\tilde{\omega}$ is the Kähler form of a G-invariant metric $\tilde{g}$, then, by Theorem 5 and Propositions 2 and 3, the $p_j(\tilde{\omega})$ are real-valued and form a basis for $\ker(\overline{\partial}\partial^{\#}_{\tilde{g}})^*\overline{\partial}\partial_{\tilde{g}}$. Moreover, $(\alpha, \varphi) \mapsto p_j(\omega+\alpha+i\partial\overline{\partial}\varphi)$ is, for each j, a bounded linear map $\mathcal{H}^{1,1}(M) \times L^2_{k+4,G} \to L^2_{k+3,G}$.

With respect to the fixed L^2 inner product, let $\{f^0_{\tilde{\omega}}, \ldots, f^m_{\tilde{\omega}}\}$ be the orthonormal set extracted from $\{p_j(\tilde{\omega})\}$ by the Gram-Schmidt procedure. We then let

$$\begin{aligned} \Pi_{\tilde{\omega}} : L^2_{k,G} &\to L^2_{k,G} \\ u &\mapsto \sum_{j=0}^{m} \langle f^j_{\tilde{\omega}}, u\rangle_{L^2} f^j_{\tilde{\omega}} \end{aligned} \tag{5.2}$$

denote the associated projector. Thus $(\alpha, \varphi) \mapsto \Pi_{\tilde{\omega}}$ defines a smooth map from $\mathcal{V} \subset \mathcal{H}^{1,1}(M) \times L^2_{k+4,G}$ to the real Hilbert space $\mathrm{End}(L^2_{k,G}) \cong \bigotimes^2 L^2_{k,G}$.

Since the expressions $\langle f^j_{\omega}, f^{\ell}_{\tilde{\omega}}\rangle_{L^2}$ are continuous functions on $\mathcal{V}$, $(0,0)$ has an open neighborhood $\mathcal{V}_0 \subset \mathcal{V}$ such that

$$\det\left[\langle f^j_{\omega}, f^{\ell}_{\tilde{\omega}}\rangle_{L^2}\right] \neq 0$$

for all $(\alpha, \varphi) \in \mathcal{V}_0$. This has the useful consequence that

$$\ker\ (1-\Pi_{\omega})(1-\Pi_{\tilde{\omega}}) = \ker\ (1-\Pi_{\tilde{\omega}}) \tag{5.3}$$

whenever $\tilde{\omega} = \omega + \alpha + i\partial\overline{\partial}\varphi$ for some $(\alpha, \varphi) \in \mathcal{V}_0$.

For any integer ℓ, we let $I_\ell \subset L^2_{\ell,G}$ denote the orthogonal complement of the kernel of $(\overline{\partial}\partial^{\#}_g)^*\overline{\partial}\partial_g$, and set $\mathcal{W} = \mathcal{V}_0 \cap (\mathcal{H}^{1,1}(M) \times I_{k+4})$. Since a Kähler form $\tilde{\omega}$ is extremal iff its scalar curvature is killed by $1 - \Pi_{\tilde{\omega}}$, it is quite natural to use these function spaces in order to find G-invariant extremal Kähler metrics. For obvious technical reasons, though, we'd like to construct a surjective map between fixed Hilbert spaces. In light of (5.3), it is thus reasonable to introduce the map

$$\mathcal{H}^{1,1}(M) \times I_{k+4} \supset \quad \mathcal{W} \xrightarrow{\mathbf{S}} \mathcal{H}^{1,1}(M) \times I_k$$

defined by

$$\mathbf{S}(\alpha, \varphi) = (\alpha, (1-\Pi_{\omega})(1-\Pi_{\omega+\alpha+i\partial\overline{\partial}\varphi})\ \mathcal{S}_G(\alpha, \varphi)\ ) \tag{5.4}$$

where $\mathcal{S}_G(\alpha,\varphi)$ is defined in (5.1) and $\Pi_{\omega+\alpha+i\partial\overline{\partial}\varphi} = \Pi_{\tilde{\omega}}$ is defined in (5.2).

We would like to compute the Fréchet derivative at the origin of the map (5.4), assuming that the metric we begin with is extremal. This derivative in the direction of (α,φ) is given by the evaluation at $t=0$ of

$$\frac{d}{dt}(1-\Pi_\omega)(1-\Pi_{\tilde{\omega}_t})\ \mathcal{S}_G(t\alpha,t\varphi)\,,$$

where $\tilde{\omega}_t = \omega + t(\alpha + i\partial\overline{\partial}\varphi)$. Using the Leibniz rule, this equals

$$\frac{d}{dt}(1-\Pi_\omega)(1-\Pi_{\tilde{\omega}_t})\ \mathcal{S}_G(t\alpha,t\varphi)\Big|_{t=0} = (1-\Pi_\omega)(D_{(0,0)}\mathcal{S}_G)\binom{\alpha}{\varphi} - (1-\Pi_\omega)\ (\frac{d}{dt}\Pi_{\tilde{w}_t})\Big|_{t=0}\mathbf{s}\,, \tag{5.5}$$

where $\mathbf{s}$ denotes the scalar curvature of the original metric g. Since

$$D\mathcal{S}_{G(0,0)} = [-2(\rho,\ \cdot\ )\quad \mathcal{L}]$$

is already known, we now just need to compute $(1-\Pi_\omega)(\frac{d}{dt}\Pi_{\tilde{w}_t})|_{t=0}\mathbf{s}$.

Now $\Pi_{\tilde{w}_t}\mathbf{s} = \sum_{j=0}^m \langle f^j_{\tilde{\omega}_t},\mathbf{s}\rangle_{L^2} f^j_{\tilde{\omega}_t}$, where $\{f^0_{\tilde{\omega}_t},\ldots,f^m_{\tilde{\omega}_t}\}$ is the orthonormal set constructed above. Consequently,

$$\begin{aligned}(1-\Pi_\omega)\ (\frac{d}{dt}\Pi_{\tilde{\omega}_t})\Big|_{t=0}\mathbf{s} &= (1-\Pi_\omega)\sum_j\left[\langle \frac{d}{dt}f^j_{\tilde{\omega}_t}\Big|_{t=0},\mathbf{s}\rangle_{L^2} f^j_{\omega_0} + \langle f^j_\omega,\mathbf{s}\rangle_{L^2}\ \frac{d}{dt}f^j_{\tilde{\omega}_t}\Big|_{t=0}\right]\\ &= \sum_j \langle f^j_\omega,\mathbf{s}\rangle_{L^2}(1-\Pi_\omega)\ \frac{d}{dt}f^j_{\tilde{\omega}_t}\Big|_{t=0}\,,\end{aligned} \tag{5.6}$$

because each $f^j_{\tilde{\omega}_0} = f^j_\omega$ is in the kernel of $(1-\Pi_\omega)$.

Lemma 1 *Suppose the Kähler metric g is extremal. Then*

$$(1-\Pi_\omega)\ (\frac{d}{dt}\Pi_{\tilde{\omega}_t})\Big|_{t=0}\mathbf{s} = (1-\Pi_\omega)[2i\mathbf{G}\overline{\partial}^*_g(\partial^\#_g\mathbf{s}\lrcorner\alpha) + (\partial\mathbf{s}\lrcorner\partial^\#_g\varphi)]\,.$$

Proof. If $\mathbf{s}$ were constant, the left-hand-side would vanish because $\Pi_{\tilde{\omega}_t}1 \equiv 1$ for all t. But in this case the right-hand-side would also vanish, since we would then have $\partial^\#_g\mathbf{s} = 0$.

We may thus henceforth assume that $\partial_g^{\#}\mathbf{s} \not\equiv 0$. Since the extremal condition implies that $\Im m\partial_g^{\#}\mathbf{s}$ is a Killing vector field, we may then choose our basis $\{\xi_j\}$ for $\mathbf{z}_0$ so that $J\xi_1 + i\xi_1 = \partial_g^{\#}\mathbf{s}$. Now recall that any choice of basis gives rise to a family of t-dependent potentials $p_j(\tilde{\omega}_t) = 2i\mathbf{G}\overline{\partial}_g^*((J\xi+i\xi)\lrcorner\tilde{\omega}_t)$, from which $f^0_{\tilde{\omega}_t}, \ldots, f^m_{\tilde{\omega}_t}$ are then manufactured by the Gram-Schmidt procedure. With the choice of basis made as indicated, $f^0_{\tilde{\omega}_t} = (\mathrm{vol}(M))^{-1/2}$, while

$$f^1_{\tilde{\omega}_t} = \frac{p_1(\tilde{\omega}_t)}{\|p_1(\tilde{\omega}_t)\|_{L^2}} \quad ,$$

where $p_1(\tilde{\omega}_t) = 2i\mathbf{G}\overline{\partial}_g^*(\partial_g^{\#}\mathbf{s}\lrcorner\tilde{\omega}_t)$. Also notice that $p_1(\tilde{\omega}_0) = \mathbf{s} - \mathbf{s}_0$, where $\mathbf{s}_0$ is the average value of $\mathbf{s}$.

Since $\mathbf{s}$ is perpendicular to each f^j_ω for $j > 1$, and since $\frac{d}{dt}f^0_{\tilde{\omega}_t} = 0$, the only surviving term in (5.6) corresponds to $j = 1$, and we have

$$\begin{aligned}(1-\Pi_\omega)\,(\frac{d}{dt}\Pi_{\tilde{\omega}_t})\Big|_{t=0}\mathbf{s} &= \langle\frac{\mathbf{s}-\mathbf{s}_0}{\|\mathbf{s}-\mathbf{s}_0\|}, \mathbf{s}\rangle(1-\Pi_\omega)\,\frac{d}{dt}f^1_{\tilde{\omega}_t}\Big|_{t=0}\\ &= \|\mathbf{s}-\mathbf{s}_0\|_{L^2}(1-\Pi_\omega)\,\frac{d}{dt}f^1_{\tilde{\omega}_t}\Big|_{t=0}.\end{aligned}$$

But

$$\frac{d}{dt}f^1_{\tilde{\omega}_t} \equiv \frac{1}{\|p_1(\tilde{\omega}_t)\|_{L^2}}\frac{d}{dt}p_1(\tilde{\omega}_t) \mod p_1(\tilde{\omega}_t)\,,$$

and $p_1(\tilde{\omega}_0) = \mathbf{s} - \mathbf{s}_0$ is in the kernel of $(1 - \Pi_\omega)$. It therefore follows that

$$(1-\Pi_\omega)\,(\frac{d}{dt}\Pi_{\tilde{\omega}_t})\Big|_{t=0}\mathbf{s} = (1-\Pi_\omega)\,\frac{d}{dt}p_1(\tilde{\omega}_t)\Big|_{t=0}.$$

To compute this last derivative, observe that

$$\begin{aligned}\tfrac{d}{dt}2i\mathbf{G}\overline{\partial}_g^*(\partial_g^{\#}\mathbf{s}\lrcorner\tilde{\omega}_t)\Big|_{t=0} &= 2i\mathbf{G}\overline{\partial}_g^*(\partial_g^{\#}\mathbf{s}\lrcorner\alpha) + 2i\mathbf{G}\overline{\partial}_g^*(\partial_g^{\#}\mathbf{s}\lrcorner i\partial\overline{\partial}\varphi)\\ &= 2i\mathbf{G}\overline{\partial}_g^*(\partial_g^{\#}\mathbf{s}\lrcorner\alpha) - 2i\mathbf{G}\overline{\partial}_g^*\overline{\partial}(\partial_g^{\#}\mathbf{s}\lrcorner i\partial\varphi)\\ &= 2i\mathbf{G}\overline{\partial}_g^*(\partial_g^{\#}\mathbf{s}\lrcorner\alpha) + (\partial_g^{\#}\mathbf{s}\lrcorner\partial\varphi) + \text{constant},\end{aligned}$$

since $\partial_g^{\#}\mathbf{s}$ is a holomorphic vector field and $2\mathbf{G}\overline{\partial}_g^*\overline{\partial}$ is the identity on the orthogonal complement of the constants. But

$$\partial^{\#}\mathbf{s}\lrcorner\partial\varphi = \overline{\overline{\partial}\bar{\mathbf{s}}\lrcorner\partial^{\#}\bar{\varphi}} = (\mathrm{grad}\,\mathbf{s} - iJ\,\mathrm{grad}\,\mathbf{s})\lrcorner d\varphi$$

is real, since $J\,\mathrm{grad}\,\mathbf{s}$ is a Killing field and φ is real and G-invariant. Our last expression thus equals $2i\mathbf{G}\overline{\partial}_g^*(\partial_g^\#\mathbf{s}\lrcorner\alpha) + (\partial\mathbf{s}\lrcorner\partial_g^\#\varphi) +$ constant, and the result therefore follows because the constant is killed by $(1-\Pi_\omega)$. ∎

Proposition 7 *For $k > n$, equation* (5.4) *defines a C^1 map whose Fréchet derivative at the origin is given by*

$$D\mathbf{S}_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1-\Pi_\omega \end{pmatrix} \begin{bmatrix} 1 & 0 \\ -2(\rho,\ \cdot\ ) - 2i\mathbf{G}\overline{\partial}_g^*(\partial_g^\#\mathbf{s}\lrcorner\ \cdot\ ) & -2(\overline{\partial}\partial^\#)^*\overline{\partial}\partial^\# \end{bmatrix}. \tag{5.7}$$

Proof. That the map is C^1 follows from the fact that the scalar curvature operator $\mathcal{S}_G$ is C^1 and the fact that the projection $\Pi_{\omega+\alpha+i\partial\overline{\partial}\varphi}$ depends smoothly on (α,φ).

The first row of our formula for the derivative is obvious, while the second row is a consequence of the Leibnitz and chain rules. Indeed, we know that the derivative at $(0,0)$ in the direction of (α,φ) of the second component of (5.4) is given by (5.5); and, in view of the Lemma 1 and (4.2), this is just

$$(1-\Pi_\omega)\left(-2(\rho,\ \cdot\ ) - 2i\mathbf{G}\overline{\partial}_g^*(\partial_g^\#\mathbf{s}\lrcorner\ \cdot\ ) \qquad \mathcal{L} - \partial\mathbf{s}\lrcorner\partial^\#\ \cdot\ \right)\begin{pmatrix}\alpha \\ \varphi\end{pmatrix}.$$

The result now follows from (4.3). ∎

Proposition 8 *Let (M,J,g) be a compact extremal Kähler manifold. Then the map* $\mathbf{S}$ *defined in* (5.4) *becomes a diffeomorphism when restricted to a sufficiently small neighborhood of the origin.*

Proof. By the Banach-Space Inverse Function Theorem [1] [10], it suffices to show that $D\mathbf{S}_{(0,0)}$ is an isomorphism. Since $(\overline{\partial}\partial^\#)^*\overline{\partial}\partial^\#$ is elliptic of fourth order, $D\mathbf{S}_{(0,0)}$ is Fredholm, and we therefore need merely to show that its kernel and cokernel are trivial.

Let us first examine the kernel. Suppose that $D\mathbf{S}_{(0,0)}(\alpha,\varphi) = 0$. Then (5.7) tells us that $\alpha = 0$ and

$$(1-\Pi_\omega)(\overline{\partial}\partial^\#)^*\overline{\partial}\partial^\#\varphi = 0\,.$$

By (5.2), the latter is equivalent to saying that

$$(\overline{\partial}\partial^\#)^*\overline{\partial}\partial^\#\varphi = \sum_{j=0}^m c_j f^j_\omega$$

for suitable constants c_j. Therefore,

$$\begin{aligned}\langle(\overline{\partial}\partial^\#)^*\overline{\partial}\partial^\#\varphi, f^j_\omega\rangle_{L^2} &= \langle\varphi,(\overline{\partial}\partial^\#)^*\overline{\partial}\partial^\# f^j_\omega\rangle_{L^2}\\ &= 0\end{aligned}$$

because the f^j_ω are, by construction, a basis for the kernel of $(\overline{\partial}\partial^\#)^*\overline{\partial}\partial^\#$. The coefficients c_j are thus all zero, and $(\overline{\partial}\partial^\#)^*\overline{\partial}\partial^\#\varphi = 0$. But $\varphi \in I_{k+4}$, and the latter is by definition the orthogonal complement of $\ker(\overline{\partial}\partial^\#)^*\overline{\partial}\partial^\# \subset L^2_{k+4,G}$. Hence $\varphi = 0$, and $\ker D\mathbf{S}_{(0,0)} = \{(0,0)\}$.

Now for the cokernel. If $(\beta,\psi) \in \mathcal{H}^{1,1}(M)\times I_k \subset \mathcal{H}^{1,1}(M)\times L^2_{k,G}(M)$ is L^2-orthogonal to the image of (5.7), then

$$\langle\alpha,\beta\rangle_{L^2} - \langle(1-\Pi_\omega)[2(\rho,\alpha)+2i\mathbf{G}\overline{\partial}^*_g(\partial^\#_g\mathbf{s}\lrcorner\alpha+2(\overline{\partial}\partial^\#)^*\overline{\partial}\partial^\#\psi],\psi\rangle_{L^2} = 0 \quad (5.8)$$

for all (α,ψ) in $\mathcal{H}^{1,1}(M)\times I_{k+4}$. In particular, setting $\alpha = 0$, we see that

$$\begin{aligned}\langle\psi,(\overline{\partial}\partial^\#)^*\overline{\partial}\partial^\#\psi\rangle_{L^2} &= \langle\psi,(\overline{\partial}\partial^\#)^*\overline{\partial}\partial^\#(1-\Pi_\omega)\psi\rangle_{L^2}\\ &= \langle(1-\Pi_\omega)(\overline{\partial}\partial^\#)^*\overline{\partial}\partial^\#\psi,\psi\rangle_{L^2} = 0\end{aligned}$$

for all $\psi \in I_{k+4}$. Hence the component of $(\overline{\partial}\partial^\#)^*\overline{\partial}\partial^\#\psi$ perpendicular to $\ker(\overline{\partial}\partial^\#)^*\overline{\partial}\partial^\#$ must be zero, and

$$(\overline{\partial}\partial^\#)^*\overline{\partial}\partial^\#\psi = \sum_{j=0}^m c_j f^j_\omega$$

for some coefficients c_j. Our previous argument then shows that the coefficients c_j are all zero. Hence $\psi \in I_k \cap \ker(\overline{\partial}\partial^\#)^*\overline{\partial}\partial^\# = \{0\}$. Equation (5.8) now tells us that $\langle\alpha,\beta\rangle_{L^2} = 0$ for all $\alpha \in \mathcal{H}^{1,1}(M)$, so that $\beta = 0$, too. Thus $\mathrm{coker} D\mathbf{S}_{(0,0)} = \{(0,0)\}$, and $D\mathbf{S}_{(0,0)}$ is an isomorphism. ∎

This now implies our main result:

Theorem A *Let (M, J) be a compact complex manifold, and let $\mathbf{E} \subset H^{1,1}(M, \mathbf{R})$ be the set of Kähler classes of extremal Kähler metrics on M. Then $\mathbf{E}$ is open.*

Proof. By Proposition 7, there is a neighborhood $\hat{\mathcal{W}} \subset \mathcal{W}$ of the origin in $\mathcal{H}^{1,1} \times I_{k+4}$ such that $\mathbf{S}|_{\hat{\mathcal{W}}}$ is a diffeomorphism onto an open neighborhood of the origin in $\mathcal{H}^{1,1} \times I_k$. Define $V_0 \subset \mathcal{H}^{1,1}$ by $[V_0 \times \{0\}] := \mathbf{S}(\hat{\mathcal{W}}) \cap [\mathcal{H}^{1,1} \times \{0\}]$, and let $\phi : V_0 \to I_{k+4}$ be defined by the diagram

$$\begin{array}{ccc} V_0 & \stackrel{\phi}{\longrightarrow} & I_{k+4} \\ \downarrow & & \uparrow \\ V_0 \times \{0\} & \stackrel{(\mathbf{S}|_{\hat{\mathcal{W}}})^{-1}}{\longrightarrow} & \mathcal{H}^{1,1} \times I_{k+4} \end{array} .$$

For each harmonic form $\alpha \in V_0$, we then have

$$(\alpha, 0) = \mathbf{S}(\alpha, \phi(\alpha)) = (\alpha, (1 - \Pi_\omega)(1 - \Pi_{\tilde{\omega}})\mathbf{s}(\alpha, \phi(\alpha))) ,$$

where $\tilde{\omega} = \omega + \alpha + i\partial\overline{\partial}\phi(\alpha)$. By (5.3), we therefore have $(1 - \Pi_{\tilde{\omega}})\mathbf{s}(\alpha, \phi(\alpha)) = 0$, which is to say that the scalar curvature $\tilde{\mathbf{s}}$ of the metric $\tilde{g}$ corresponding to $\tilde{\omega}$ is in the span of $\{f^j_{\tilde{\omega}}\}_{j=0}^m$. Hence $\bar{\partial}\partial^{\#}_{\tilde{g}}\tilde{\mathbf{s}} = 0$, and $\tilde{g}$ is an extremal Kähler metric. Identifying $H^{1,1}(M, \mathbf{R})$ with $\mathcal{H}^{1,1}$ and setting $V := [\omega] + V_0$ then completes the proof. ∎

The solution space of the extremal Kähler metric equation thus behaves stably with respect to deformations of the Kähler class. However, examples of Burns and de Bartolomeis [2] show that there is no analogous stability with respect to deformations of the complex structure.

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