Four-Manifolds,

*Einstein Metrics,* &

*Differential Topology*

Claude LeBrun
Stony Brook University

Ohio State University, 10/22/15
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Now choosing \(T_p M \xrightarrow{\cong} \mathbb{R}^n\) via some orthonormal basis gives us special coordinates on \(M\).
In these “geodesic normal” coordinates the metric volume measure is given by

\[ d\mu_g = \left[ 1 - \frac{1}{6} r^{jk} x_j x_k + O(|x|^3) \right] d\mu_{\text{Euclidean}}, \]
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“...the greatest blunder of my life!”

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$$\Delta x^j = 0 \implies r_{jk} = \frac{1}{2} \Delta g_{jk} + \text{lots}.$$
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- When $n \geq 6$, wide open. Maybe???
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Similar results for most simply connected spin 5-manifolds. (Boyer, Galicki, Kollár, et al.)
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(Terminology to be explained later!)
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**Theorem (L).** *There is only one Einstein metric on compact complex-hyperbolic 4-manifold $\mathcal{CH}_2/\Gamma$, up to scale and diffeos.*
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By contrast, high-dimensional Einstein metrics too common; have little to do with geometrization.
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$$\star^2 = 1.$$

$\Lambda^+$ self-dual 2-forms.
$\Lambda^-$ anti-self-dual 2-forms.
Riemann curvature of $g$
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splits into 4 irreducible pieces:
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\[
\begin{array}{c|c|c}
\Lambda^+ & \Lambda^{+-} & \Lambda^{-*} \\
\hline
W_+ + \frac{s}{12} & \hat{r} & \\
\hline
\hat{r} & W_- + \frac{s}{12} & \\
\end{array}
\]
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$W_+ = \text{self-dual Weyl curvature} \hspace{1cm} (\text{conformally invariant})$

$W_- = \text{anti-self-dual Weyl curvature}$
Thus \((M^4, g)\) Einstein \iff \(\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2\) commutes with

\(\star : \Lambda^2 \rightarrow \Lambda^2\):

\[
\mathcal{R} = \begin{pmatrix}
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\[ K(P) = K(P^\perp) \]
\((M, g)\) compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

\[
\chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + \ldots \right) d\mu
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$$\chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 \right) d\mu$$
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\chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\tilde{\nabla}|^2}{2} \right) d\mu
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\]

for Euler-characteristic \(\chi(M) = \sum_j (-1)^j b_j(M)\).
4-dimensional Hirzebruch signature formula

$$\tau(M) = \frac{1}{12\pi^2} \int_M \left( |W^+|^2 \right) d\mu$$
4-dimensional Hirzebruch signature formula

$$\tau(M) = \frac{1}{12\pi^2} \int_M \left( |W_+|^2 - |W_-|^2 \right) d\mu$$
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for signature \( \tau(M) = b_+(M) - b_-(M) \).
Here $\tau(M) = b_+(M) - b_-(M)$ defined in terms of intersection pairing
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$$H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) \rightarrow \mathbb{R}$$

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Diagonalize:

$$\begin{bmatrix}
+1 \\
\vdots \\
+1 \\
-1 \\
\vdots \\
-1
\end{bmatrix}.$$
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Diagonalize:

$$
\begin{bmatrix}
+1 \\
\vdots \\
+1
\end{bmatrix}
\begin{cases}
b_+(M) \\
-b_-(M)
\end{cases}
.$$
For \((M^4, g)\) compact oriented Riemannian,

Euler characteristic

\[
\chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{\|\mathring{r}\|^2}{2} \right) d\mu
\]

Signature

\[
\tau(M) = \frac{1}{12\pi^2} \int_M \left( |W_+|^2 - |W_-|^2 \right) d\mu
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Theorem (Freedman/Donaldson). Two smooth compact simply connected oriented 4-manifolds are orientedly homeomorphic if and only if
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Warning: “Exotic differentiable structures!”
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Typically, one homeotype \( \leftrightarrow \infty \) many diffeotypes.
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Corollary. Any smooth compact simply connected non-spin 4-manifold $M$ is homeomorphic to
Corollary. Any smooth compact simply connected non-spin 4-manifold $M$ is homeomorphic to a connect sum

$$j \mathbb{CP}^2 \# k \overline{\mathbb{CP}^2} = \mathbb{CP}^2 \# \cdots \# \mathbb{CP}^2 \# \overline{\mathbb{CP}^2} \# \cdots \# \overline{\mathbb{CP}^2}$$
Corollary. Any smooth compact simply connected non-spin 4-manifold $M$ is homeomorphic to a connect sum

$$j\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2} = \underbrace{\mathbb{C}P^2 \# \cdots \# \mathbb{C}P^2}_j \# \underbrace{\overline{\mathbb{C}P^2} \# \cdots \# \overline{\mathbb{C}P^2}}_k$$

where $j = b_+(M)$ and $k = b_-(M)$. 
Convention:

$\overline{\mathbb{CP}}_2 = \text{reverse oriented } \mathbb{CP}_2.$
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---

Connected sum \#:
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Connected sum \#: 

![Diagram of connected sum]

127
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![Connected sum diagram](image-url)
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Connected sum \#: 

\[
\begin{array}{c}
\text{Diagram}
\end{array}
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What about spin case?
**Corollary.** *Any smooth compact simply connected non-spin 4-manifold $M$ is homeomorphic to a connect sum $j\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$.\n
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Need new building block!*
Corollary. Any smooth compact simply connected non-spin 4-manifold $M$ is homeomorphic to a connect sum $j\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$.

What about spin case?

Need new building block!

$K3$ manifold...
$K3$ = Kummer-Kähler-Kodaira manifold.
$K3 = \text{Kummer-Kähler-Kodaira manifold}.$

Simply connected complex surface with $c_1 = 0.$
\( K3 = \text{Kummer-Kähler-Kodaira manifold.} \)

Simply connected complex surface with \( c_1 = 0 \).

Only one deformation type.
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Simply connected complex surface with $c_1 = 0$.

Only one diffeomorphism type.
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Simply connected complex surface with $c_1 = 0$.

Only one diffeomorphism type.

Spin, $\chi = 24$, $\tau = -16$. 
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Kummer construction:
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Kummer construction:

Begin with $T^4/\mathbb{Z}_2$: 
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

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Begin with $T^4 / \mathbb{Z}_2$: 

![Diagram of Kummer construction](image_url)
\[ K3 = \text{Kummer-Kähler-Kodaira manifold}. \]

Kummer construction:

Begin with \( T^4/\mathbb{Z}_2 \):

\[ \text{Diagram of } T^4/\mathbb{Z}_2 \]
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Kummer construction:

Begin with $T^4/\mathbb{Z}_2$: 

![Diagram of $T^4/\mathbb{Z}_2$]

1. Consider $T^4/\mathbb{Z}_2$ as a quotient of the 4-torus $T^4$ by the action of the 2-torus $\mathbb{Z}_2$.
2. The quotient space $T^4/\mathbb{Z}_2$ is obtained by identifying antipodal points on $T^4$.
3. The resulting space is a K3 surface, which is a complex surface with trivial canonical bundle and vanishing first Betti number.

The Kummer construction involves taking a certain quotient of a 4-torus and ensuring that the resulting space satisfies the properties of a K3 surface.
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Kummer construction:

Begin with $T^4/\mathbb{Z}_2$: 

![Diagram of $T^4/\mathbb{Z}_2$]

$T^2$
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

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![Diagram of T^4/Z_2]
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\[
\begin{array}{c}
\includegraphics[width=0.4\textwidth]{kummerconstruction.png}
\end{array}
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Begin with $\mathbb{T}^4 / \mathbb{Z}_2$: 

![Diagram of T^4/\mathbb{Z}_2 construction]
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Kummer construction:

Begin with $T^4/\mathbb{Z}_2$: 

Replace $\mathbb{R}^4/\mathbb{Z}_2$ neighborhood of each singular point with copy of $T^*S^2$. 
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Result is a $K3$ surface.
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

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$T^4 = \text{Picard torus of curve of genus 2.}$
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Remove singularities by deforming equation.
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Generic quartic is then a $K3$ surface.
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Generic quartic is then a $K3$ surface. Example:

$$0 = x^4 + y^4 + z^4 + w^4$$
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Kummer construction:

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Generic quartic is then a \( K3 \) surface. Example:

\[
0 = (x^2 + y^2 + z^2 - w^2)^2 - 8[(1 - z^2)^2 - 2x^2][(1 + z^2)^2 - 2y^2]
\]
Theorem (Freedman/Donaldson). Two smooth compact simply connected oriented 4-manifolds are orientedly homeomorphic if and only if

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Conjecture (11/8 Conjecture). Any smooth compact simply connected spin 4-manifold $M$ is (un-orientedly) homeomorphic to either $S^4$ or a connected sum $jK3 \# k(S^2 \times S^2)$. 
**Corollary.** Any smooth compact simply connected non-spin 4-manifold $M$ is homeomorphic to a connect sum $j\mathbb{CP}_2\# k\overline{\mathbb{CP}_2}$.

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Equivalent to asserting that such manifolds satisfy

$$b_2 \geq \frac{11}{8} |\tau|.$$
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Certainly true of all examples in this lecture!
Question. Which smooth compact 4-manifolds $M^4$ admit Einstein metrics?
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Complex geometry is rich source of examples.
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On suitable 4-manifolds, Seiberg-Witten theory allows one to mimic Kähler geometry when treating non-Kähler metrics.

Narrower Question. If $(M^4, J)$ is a compact complex surface, when does $M^4$ admit an Einstein metric $g$ (unrelated to $J$)?
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On suitable 4-manifolds, Seiberg-Witten theory allows one to mimic Kähler geometry when treating non-Kähler metrics.

**Narrower Question.** If $(M^4, J)$ is a compact complex surface, when does $M^4$ admit an Einstein metric $g$ (unrelated to $J$)?

**Even Narrower Question.** If $(M^4, J)$ is a compact complex surface, when does $M^4$ admit an Einstein metric $g$ (unrelated to $J$) with Einstein constant $\lambda \geq 0$?
Theorem (L ‘09). Suppose that $M$ is a smooth compact oriented 4-manifold which admits an integrable complex structure $J$. 
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Theorem (L ’09). Suppose that $M$ is a smooth compact oriented 4-manifold which admits an integrable complex structure $J$. Then $M$ also admits an Einstein metric $g$ with $\lambda \geq 0$ if and only if

$$M \cong \begin{cases} \mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}, \\ \mathbb{S}^2 \times \mathbb{S}^2, \\ K^3, \\ K^3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, \\ T^4/\mathbb{Z}_3, \\ T^4/\mathbb{Z}_4, \\ T^4/ (\mathbb{Z}_2 \oplus \mathbb{Z}_2), \\ T^4/ (\mathbb{Z}_3 \oplus \mathbb{Z}_3), \\ or \\ T^4/ (\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{cases}$$
Theorem (L ’09). Suppose that $M$ is a smooth compact oriented $4$-manifold which admits an integrable complex structure $J$. Then $M$ also admits an Einstein metric $g$ with $\lambda \geq 0$ if and only if

$$M \overset{\text{diff}}{\cong} \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}_2}, & 0 \leq k \leq 8, \\ \mathbb{T}^4, & \mathbb{T}^4 / \mathbb{Z}_2, \mathbb{T}^4 / \mathbb{Z}_3, \mathbb{T}^4 / \mathbb{Z}_4, \mathbb{T}^4 / (\mathbb{Z}_2 \oplus \mathbb{Z}_2), \mathbb{T}^4 / (\mathbb{Z}_3 \oplus \mathbb{Z}_3), \mathbb{T}^4 / (\mathbb{Z}_2 \oplus \mathbb{Z}_4) & \end{cases}$$
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K3, \\
K3/\mathbb{Z}_2,
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Del Pezzo surfaces, 
K3 surface, Enriques surface, 
Abelian surface, Hyper-elliptic surfaces.
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Del Pezzo surfaces,
K3 surface, Enriques surface,
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Similarly when $M$ symplectic instead of complex.
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Theorem (L ’09). Suppose that $M$ is a smooth compact oriented 4-manifold which admits an integrable complex structure $J$. Then $M$ also admits an Einstein metric $g$ with $\lambda \geq 0$ if and only if

$$M \overset{\text{diff}}{\approx} \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}_2}, & 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{cases}$$

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No others: Hitchin-Thorpe, Seiberg-Witten, …
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Kähler metrics:
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$$g = \sum_{j,k=1}^{m} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k} \left[ dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j \right]$$
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Modern definition:

$(M^{2m}, g)$ has holonomy $\subset U(m)$.
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Ricci-flat Kähler:

\((\widetilde{M}^{2m}, g)\) has holonomy \( \subset \text{SU}(m) \).
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“Calabi-Yau metrics.”
Corollary. $\exists \lambda = 0$ \textit{Einstein metrics on }$K3$. 
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Indeed, $\exists$ sequences of these $\longrightarrow$ flat orbifold $T^4/\mathbb{Z}_2$. 
Hitchin-Thorpe Inequality:

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W + |^2 - \frac{|\hat{r}|^2}{2} \right) d\mu_g$$
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Einstein \(\Rightarrow\) 

\[
\frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 \right) d\mu_g
\]

**Theorem** (Hitchin-Thorpe Inequality). *If smooth compact oriented \(M^4\) admits Einstein \(g\), then

\[(2\chi + 3\tau)(M) \geq 0,\]

with equality only if \((M, g)\) finitely covered by flat \(T^4\) or Calabi-Yau \(K3\).
Theorem (L ’09). Suppose that $M$ is a smooth compact oriented 4-manifold which admits an integrable complex structure $J$. Then $M$ also admits an Einstein metric $g$ with $\lambda \geq 0$ if and only if

\[ M^\text{diff} \approx \begin{cases} 
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S^2 \times S^2, \\
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Moduli space \( \mathcal{E}(M) \)
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Moduli space \( \mathcal{E}(M) = \{ \text{Einstein } g \}/(\text{Diffeos} \times \mathbb{R}^+) \)
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Every Einstein metric is Ricci-flat Kähler.

Moduli space \( \mathcal{E}(M) \) completely understood.
But we understand some cases better than others!

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Moduli space \( \mathcal{E}(M) \) connected!
\[ \mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8, \quad S^2 \times S^2, \]

\[ K3, \]
\[ K3/\mathbb{Z}_2, \]
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Below the line:

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\[ \mathbb{CP}_2 \#^k \overline{\mathbb{CP}_2}, \quad 0 \leq k \leq 8, \]
\[ S^2 \times S^2, \]
\[ K3, \]
\[ K3/\mathbb{Z}_2, \]
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Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space \( \mathcal{E}(M) \) connected!
Above the line:

Know an Einstein metric on each manifold.

\[ \mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}, \quad 0 \leq k \leq 8, \]
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\[ K3/\mathbb{Z}_2, \]
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Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space \(\mathcal{E}(M)\) connected!
Above the line:

Moduli space $\mathcal{E}(M) \neq \emptyset$.

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Moduli space $\mathcal{E}(M)$ connected!
Above the line:

Moduli space $\mathcal{E}(M) \neq \emptyset$. But is it connected?

\[
\begin{align*}
\mathbb{CP}^2 & \# k\overline{\mathbb{CP}^2}, \quad 0 \leq k \leq 8, \\
S^2 \times S^2, \\
K3, \\
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\end{align*}
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Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathcal{E}(M)$ connected!
In the remaining cases,
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\[ g = uh \]
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for some Kähler metric \( h \) and a positive function \( u \).
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Derdziński ’83: breakthrough paper on this subject.
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\[ W_+ (\omega, \omega) > 0 \]
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As Riemannian metrics, they satisfy a curious curvature condition. Namely, if \( \omega \) is a non-trivial self-dual harmonic 2-form, they satisfy

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everywhere on \( M \). This scalar condition is a conformally invariant analog of the more familiar condition \( s > 0 \).
Theorem (L’14).

Let \((M, g)\) be a smooth compact Einstein 4-manifold with \(b^+ = 1\). If \(h\) satisfies everywhere on \(M\), then \(h\) is conformally Kähler and has Einstein constant \(\lambda > 0\). Moreover, \(M\) is diffeomorphic to a Del Pezzo surface. Conversely, every Del Pezzo surface admits Einstein metrics with these properties.
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In fact, all known Einstein metrics on Del Pezzo surfaces have these properties.
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Blow-up of \(\mathbb{CP}^2\) at \(k\) distinct points, in general position,
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Blow-up of \(\mathbb{CP}_2\) at \(k\) distinct points, \(0 \leq k \leq 8\), in general position,
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Blow-up of $\mathbb{CP}^2$ at $k$ distinct points, $0 \leq k \leq 8$,  
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![Diagram of blowing up](image)
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Blow-up of \(\mathbb{C}P_2\) at \(k\) distinct points, \(0 \leq k \leq 8\), in general position, or \(\mathbb{C}P_1 \times \mathbb{C}P_1\).

No 3 on a line,
Del Pezzo surfaces:

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Blow-up of $\mathbb{CP}^2$ at $k$ distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}^1 \times \mathbb{CP}^1$.

No 3 on a line, no 6 on conic,
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Shorthand: “\(c_1 > 0\).”

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No 3 on a line, no 6 on conic, no 8 on nodal cubic.
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**Theorem.** *Each Del Pezzo* \((M^4, J)\) *admits a compatible conformally Kähler Einstein metric, and this metric is unique up to automorphisms and rescaling.*
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**Theorem.** Each Del Pezzo $(M^4, J)$ admits a compatible conformally Kähler Einstein metric, and this metric is unique up to automorphisms and rescaling.

Existence: Tian, Odaka-Spotti-Sun, Chen-L-Weber…

Uniqueness: Bando-Mabuchi, L ’12…
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For each topological type:
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For each topological type:

Moduli space of such \((M^4, J)\) is connected.
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Just a point if \(b_2(M) \leq 5\).
For $M^4$ a Del Pezzo surface, set
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$$\mathcal{E}(M) = \{\text{Einstein } g \text{ on } M\}/(\text{Diffeos } \times \mathbb{R}^+)$$
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**Theorem** (L ’14).
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**Theorem (L ’14). $E^+_\omega(M)$ is connected.**
For $M^4$ a Del Pezzo surface, set

$$\mathcal{E}(M) = \{\text{Einstein } g \text{ on } M\}/(\text{Diffeos} \times \mathbb{R}^+)$$

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**Theorem** (L ’14). $\mathcal{E}_\omega^+(M)$ is connected. Moreover, if $b_2(M) \leq 5$, 

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**Theorem (L ’14).** $\mathcal{E}^+_\omega(M)$ is connected. Moreover, if $b_2(M) \leq 5$, then $\mathcal{E}^+_\omega(M) = \{\text{point}\}$.

**Corollary.**
For $M^4$ a Del Pezzo surface, set

$\mathcal{E}(M) = \{\text{Einstein } g \text{ on } M\}/(\text{Diffeos} \times \mathbb{R}^+)$

$\mathcal{E}_\omega^+(M) = \{\text{Einstein } g \text{ with } W^+(\omega, \omega) > 0\}/\sim$

**Theorem (L '14).** $\mathcal{E}_\omega^+(M)$ is connected. Moreover, if $b_2(M) \leq 5$, then $\mathcal{E}_\omega^+(M) = \{\text{point}\}$.

**Corollary.** $\mathcal{E}_\omega^+(M)$ is exactly one connected component of $\mathcal{E}(M)$.
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In this setting, Seiberg-Witten theory plays the starring role.

But that would be the subject of an an entirely different colloquium!