Einstein Metrics,

Four-Manifolds, &

Gravitational Instantons

Claude LeBrun
Stony Brook University

AMS-AustMS-NZMS Joint Meeting,
Auckland, New Zealand/Aotearoa,
December 11, 2024.

Most recent results joint with

Most recent results joint with

Olivier Biquard
Sorbonne Université

Most recent results joint with

Olivier Biquard
Sorbonne Université

and

Paul Gauduchon
École Polytechnique

Most recent results joint with

Olivier Biquard
Sorbonne Université

and

Paul Gauduchon
École Polytechnique

Int. Math. Res. Not. IMRN

Most recent results joint with

Olivier Biquard
Sorbonne Université

and

Paul Gauduchon
École Polytechnique

Int. Math. Res. Not. IMRN
(2024) 13295-13311.

Definition. *A Riemannian metric* g *is said to be* Einstein *if it has constant Ricci curvature*

Definition. *A Riemannian metric* g *is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Definition. *A Riemannian metric* g *is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

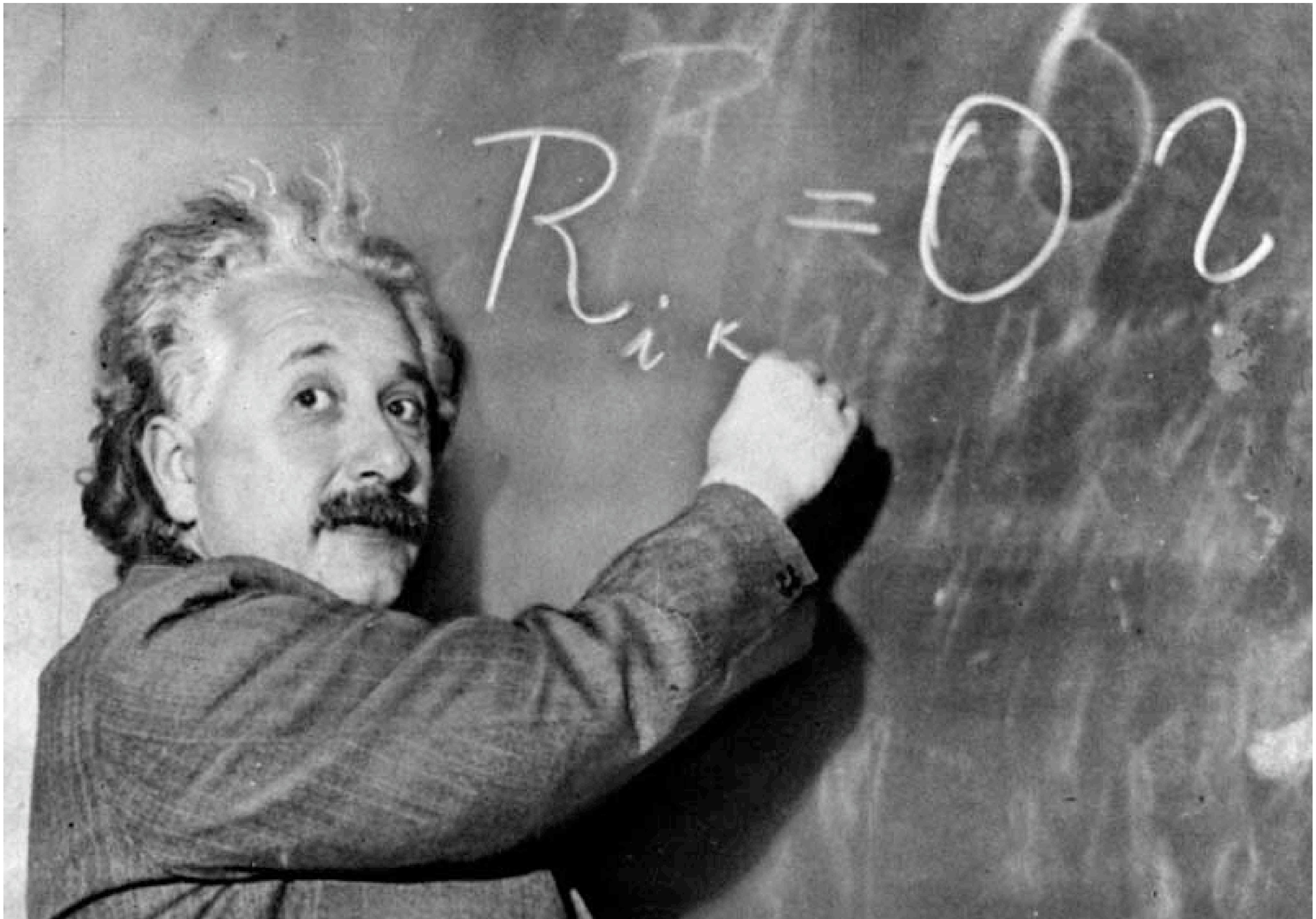
"...the greatest blunder of my life!"

— A. Einstein, to G. Gamow

Definition. *A Riemannian metric g is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.



Definition. *A Riemannian metric g is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

As punishment ...

Definition. *A Riemannian metric* g *is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Mathematicians call λ the Einstein constant.

Definition. *A Riemannian metric g is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Mathematicians call λ the Einstein constant.

Has same sign as the *scalar curvature*

$$s = r_j^{\ j} = \mathcal{R}^{ij}{}_{ij}.$$

Definition. *A Riemannian metric g is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

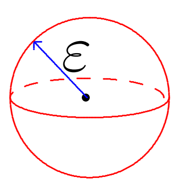
for some constant $\lambda \in \mathbb{R}$.

Mathematicians call λ the Einstein constant.

Has same sign as the *scalar curvature*

$$s = r_j^{\ j} = \mathcal{R}^{ij}{}_{ij}.$$

$$\frac{\mathrm{vol}_g(B_\varepsilon(p))}{c_n \varepsilon^n} = 1 - s\,\frac{\varepsilon^2}{6(n+2)} + O(\varepsilon^4)$$



Dimension ≤ 3:

Dimension ≤ 3:

Einstein's equations are "locally trivial:"

Dimension ≤ 3:

Einstein's equations are "locally trivial:"

Einstein metrics have constant sectional curvature.

Dimension ≤ 3:

Einstein's equations are "locally trivial:"

Einstein metrics have constant sectional curvature.

$\Longrightarrow$ Global rigidity results in these low dimensions.

Dimension ≥ 5:

Dimension ≥ 5:

On compact n-manifolds, $n \geq 5$, moduli space of Einstein metrics is often highly disconnected.

Dimension ≥ 5:

On compact n-manifolds, $n \geq 5$, moduli space of Einstein metrics is often highly disconnected.

For example, the moduli space of Einstein metrics on S^5 has infinitely many connected components.

Dimension ≥ 5:

On compact n-manifolds, $n \geq 5$, moduli space of Einstein metrics is often highly disconnected.

For example, the moduli space of Einstein metrics on S^5 has infinitely many connected components.

(Böhm,)

Dimension ≥ 5:

On compact n-manifolds, $n \geq 5$, moduli space of Einstein metrics is often highly disconnected.

For example, the moduli space of Einstein metrics on S^5 has infinitely many connected components.

(Böhm, Collins-Székelyhidi)

Dimension ≥ 5:

On compact n-manifolds, $n \geq 5$, moduli space of Einstein metrics is often highly disconnected.

For example, the moduli space of Einstein metrics on S^5 has infinitely many connected components.

Dimension ≥ 5:

On compact n-manifolds, $n \geq 5$, moduli space of Einstein metrics is often highly disconnected.

For example, the moduli space of Einstein metrics on S^5 has infinitely many connected components.

There are smooth compact 8-manifolds that admit pairs of Einstein metrics with opposite signs for λ.

Dimension ≥ 5:

On compact n-manifolds, $n \geq 5$, moduli space of Einstein metrics is often highly disconnected.

For example, the moduli space of Einstein metrics on S^5 has infinitely many connected components.

There are smooth compact 8-manifolds that admit pairs of Einstein metrics with opposite signs for λ.

(Catanese-LeBrun,)

Dimension ≥ 5:

On compact n-manifolds, $n \geq 5$, moduli space of Einstein metrics is often highly disconnected.

For example, the moduli space of Einstein metrics on S^5 has infinitely many connected components.

There are smooth compact 8-manifolds that admit pairs of Einstein metrics with opposite signs for λ.

(Catanese-LeBrun, Rasdeaconu-Şuvaina)

Dimension ≥ 5:

On compact n-manifolds, $n \geq 5$, moduli space of Einstein metrics is often highly disconnected.

Dimension ≥ 5:

On compact n-manifolds, $n \geq 5$, moduli space of Einstein metrics is often highly disconnected.

In high dimensions, the Einstein condition allows for a surprising degree of flexibility!

Four Dimensions is Exceptional

Four Dimensions is Exceptional

“Goldilocks Zone” for Einstein metrics:

Four Dimensions is Exceptional

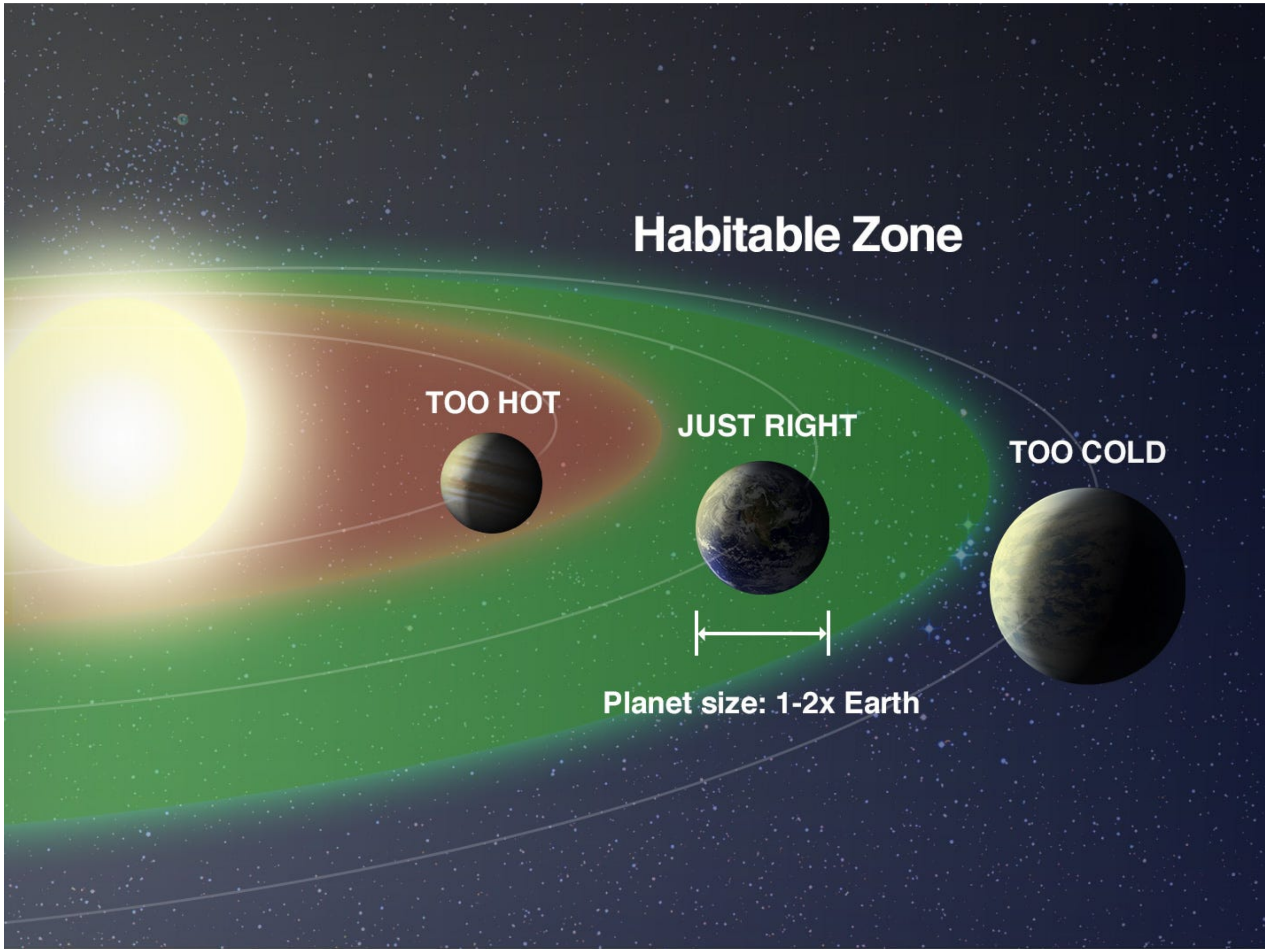
"Goldilocks Zone" for Einstein metrics:

Not too rigid. Not too flexible. Just right!

Four Dimensions is Exceptional

“Goldilocks Zone” for Einstein metrics:

Not too rigid. Not too flexible. Just right!



Dimension 4:

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus* T^4 *is flat.*

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus T^4 is flat.*

$\Longrightarrow$ Moduli space of Einstein metrics is connected.

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus T^4 is flat.*

$\Longrightarrow$ Moduli space of Einstein metrics is connected.

Theorem (Hitchin). *Any Einstein metric on $K3$ is Ricci-flat Kähler.*

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus T^4 is flat.*

$\Longrightarrow$ Moduli space of Einstein metrics is connected.

Theorem (Hitchin). *Any Einstein metric on $K3$ is Ricci-flat Kähler.*

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus T^4 is flat.*

$\Longrightarrow$ Moduli space of Einstein metrics is connected.

Theorem (Hitchin). *Any Einstein metric on $K3$ is Ricci-flat Kähler.*

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus T^4 is flat.*

$\Longrightarrow$ Moduli space of Einstein metrics is connected.

Theorem (Hitchin). *Any Einstein metric on $K3$ is Ricci-flat Kähler.*

$\Longrightarrow$ Moduli space of Einstein metrics is connected.

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus T^4 is flat.*

$\Longrightarrow$ Moduli space of Einstein metrics is connected.

Theorem (Hitchin). *Any Einstein metric on $K3$ is Ricci-flat Kähler.*

$\Longrightarrow$ Moduli space of Einstein metrics is connected.

(Kodaira, Yau, Siu, Kobayashi-Todorov)

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus* T^4 *is flat.*

$\Longrightarrow$ Moduli space of Einstein metrics is connected.

Theorem (Hitchin). *Any Einstein metric on* $K3$ *is Ricci-flat Kähler.*

$\Longrightarrow$ Moduli space of Einstein metrics is connected.
(Kodaira, Yau, Siu, Kobayashi-Todorov)

Theorem (Besson-Courtois-Gallot). *There is only one Einstein metric on compact hyperbolic 4-manifold* $\mathcal{H}^4/\Gamma$*, up to scale and diffeos.*

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus T^4 is flat.*

$\Longrightarrow$ Moduli space of Einstein metrics is connected.

Theorem (Hitchin). *Any Einstein metric on $K3$ is Ricci-flat Kähler.*

$\Longrightarrow$ Moduli space of Einstein metrics is connected.
(Kodaira, Yau, Siu, Kobayashi-Todorov)

Theorem (Besson-Courtois-Gallot). *There is only one Einstein metric on compact hyperbolic 4-manifold $\mathcal{H}^4/\Gamma$, up to scale and diffeos.*

Theorem (LeBrun). *There is only one Einstein metric on compact complex-hyperbolic 4-manifold $\mathbb{C}\mathcal{H}_2/\Gamma$, up to scale and diffeos.*

What's so special about dimension 4?

What's so special about dimension 4?

The Lie group $SO(4)$ is *not simple*

What's so special about dimension 4?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

What's so special about dimension 4?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g),

What's so special about dimension 4?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g), $\Longrightarrow$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

What's so special about dimension 4?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g), $\Longrightarrow$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^\pm$ are (± 1)-eigenspaces of

$$\star : \Lambda^2 \to \Lambda^2,$$

What's so special about dimension 4?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g), $\Longrightarrow$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^\pm$ are (± 1)-eigenspaces of

$$\star : \Lambda^2 \to \Lambda^2,$$

$$\star^2 = 1.$$

Λ^+ self-dual 2-forms.
Λ^- anti-self-dual 2-forms.

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^{+}	$W_+ + \frac{s}{12}$	$\mathring{r}$
Λ^{-}	$\mathring{r}$	$W_- + \frac{s}{12}$

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^+	$W_+ + \frac{s}{12}$	$\mathring{r}$
Λ^-	$\mathring{r}$	$W_- + \frac{s}{12}$

where

$$\begin{aligned} s &= \text{scalar curvature} \\ \mathring{r} &= \text{trace-free Ricci curvature} \\ W_+ &= \text{self-dual Weyl curvature} \\ W_- &= \text{anti-self-dual Weyl curvature} \end{aligned}$$

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^{+}	$W_+ + \frac{s}{12}$	$\mathring{r}$
Λ^{-}	$\mathring{r}$	$W_- + \frac{s}{12}$

where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W_+ = self-dual Weyl curvature *(conformally invariant)*

W_- = anti-self-dual Weyl curvature ″

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^+	$W_+ + \frac{s}{12}$	$\mathring{r}$
Λ^-	$\mathring{r}$	$W_- + \frac{s}{12}$

Which 4-manifolds admit Einstein metrics?

Symplectic 4-manifolds:

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω.

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω.

$$d\omega = 0, \qquad \lrcorner\omega : TM \stackrel{\cong}{\to} T^*M.$$

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω.

$$\omega = dx \wedge dy + dz \wedge dt$$

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω.

Seiberg-Witten theory provides powerful constraints on the differential topology of such manifolds.

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω.

Seiberg-Witten theory provides powerful constraints on the differential topology of such manifolds.

Some Suggestive Questions.

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω.

Seiberg-Witten theory provides powerful constraints on the differential topology of such manifolds.

Some Suggestive Questions. *If* (M^4, ω) *is a symplectic* 4-*manifold,*

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω.

Seiberg-Witten theory provides powerful constraints on the differential topology of such manifolds.

Some Suggestive Questions. *If* (M^4, ω) *is a symplectic* 4*-manifold, when does* M^4 *admit an Einstein metric* g

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω.

Seiberg-Witten theory provides powerful constraints on the differential topology of such manifolds.

Some Suggestive Questions. *If (M^4, ω) is a symplectic 4-manifold, when does M^4 admit an Einstein metric g (a priori unrelated to ω)?*

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω.

Seiberg-Witten theory provides powerful constraints on the differential topology of such manifolds.

Some Suggestive Questions. *If* (M^4, ω) *is a symplectic* 4*-manifold, when does* M^4 *admit an Einstein metric* g *(a priori unrelated to* ω*)? What if we also require* $\lambda > 0$*?*

Theorem (Chen-LeBrun-Weber '08)**.**

Theorem (CLW '08)**.** *Suppose that* M *is a smooth compact oriented* 4*-manifold which carries some symplectic form* ω*.*

Theorem (CLW '08). *Suppose that* M *is a smooth compact oriented* 4*-manifold which carries some symplectic form* ω. *Then* M *admits an (a priori unrelated) Einstein metric* g *with* $\lambda > 0$

Theorem (CLW '08). *Suppose that M is a smooth compact oriented 4-manifold which carries some symplectic form ω. Then M admits an (a priori unrelated) Einstein metric g with $\lambda > 0$*

$$\Longleftrightarrow \quad M \overset{diff}{\approx}$$

Theorem (CLW '08). *Suppose that M is a smooth compact oriented 4-manifold which carries some symplectic form ω. Then M admits an (a priori unrelated) Einstein metric g with $\lambda > 0$*

$$\iff \quad M \overset{diff}{\approx} \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}_2}, \\ \\ \end{cases}$$

Theorem (CLW '08). *Suppose that M is a smooth compact oriented 4-manifold which carries some symplectic form ω. Then M admits an (a priori unrelated) Einstein metric g with $\lambda > 0$*

$$\Longleftrightarrow \quad M \overset{diff}{\approx} \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}_2}, & 0 \leq k \leq 8, \\ \\ \end{cases}$$

Theorem (CLW '08). *Suppose that M is a smooth compact oriented 4-manifold which carries some symplectic form ω. Then M admits an (a priori unrelated) Einstein metric g with $\lambda > 0$*

$$\Longleftrightarrow \quad M \overset{diff}{\approx} \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, & 0 \leq k \leq 8, \\ \quad or \\ S^2 \times S^2 \end{cases}$$

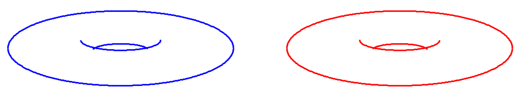
Conventions:

$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

Conventions:

$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

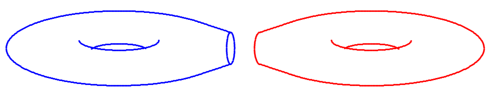
Connected sum #:



Conventions:

$\overline{\mathbb{CP}}_2$ = reverse oriented $\mathbb{CP}_2$.

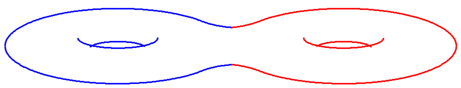
Connected sum #:



Conventions:

$\overline{\mathbb{CP}}_2$ = reverse oriented $\mathbb{CP}_2$.

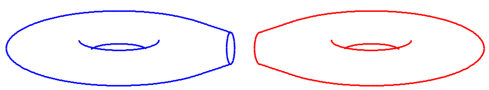
Connected sum #:



Conventions:

$\overline{\mathbb{CP}}_2$ = reverse oriented $\mathbb{CP}_2$.

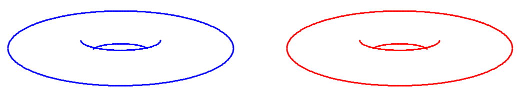
Connected sum $\#$:



Conventions:

$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

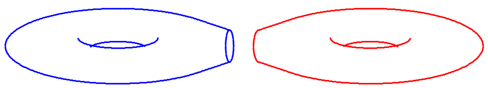
Connected sum #:



Conventions:

$\overline{\mathbb{CP}}_2$ = reverse oriented $\mathbb{CP}_2$.

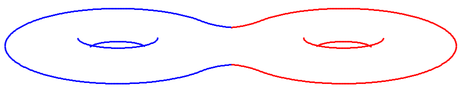
Connected sum $\#$:



Conventions:

$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

Connected sum #:



Theorem (CLW '08). *Suppose that M is a smooth compact oriented 4-manifold which carries some symplectic form ω. Then M admits an (unrelated) Einstein metric g with $\lambda > 0$*

$$\Longleftrightarrow \quad M \approx \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, & 0 \leq k \leq 8, \\ \quad or & \\ S^2 \times S^2 & \end{cases}$$

Theorem (CLW '08). *Suppose that M is a smooth compact oriented 4-manifold which carries some symplectic form ω. Then M admits an (unrelated) Einstein metric g with $\lambda > 0$*

$$\Longleftrightarrow \quad M \approx \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, & 0 \leq k \leq 8, \\ \quad or & \\ S^2 \times S^2 & \end{cases}$$

Diffeotypes: exactly the Del Pezzo surfaces.

Del Pezzo surfaces:

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points,
in general position,

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position,

Del Pezzo surfaces:

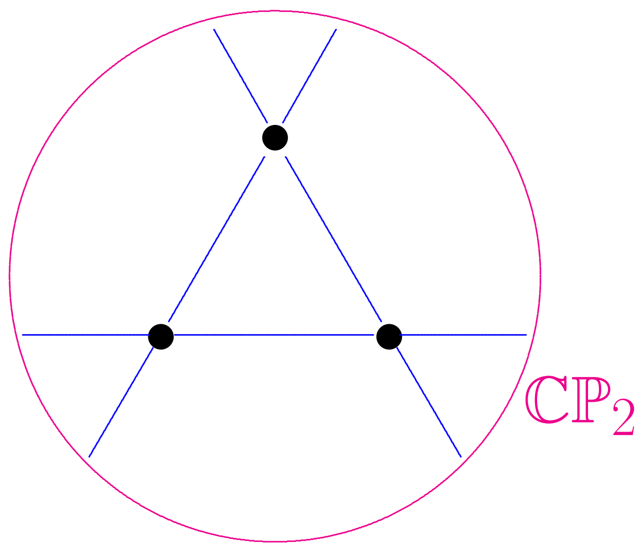
(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

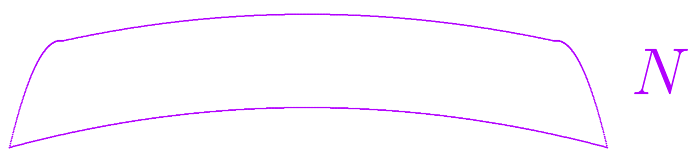
Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



Blowing up:

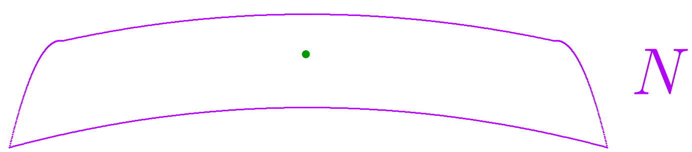
Blowing up:

If N is a complex surface,



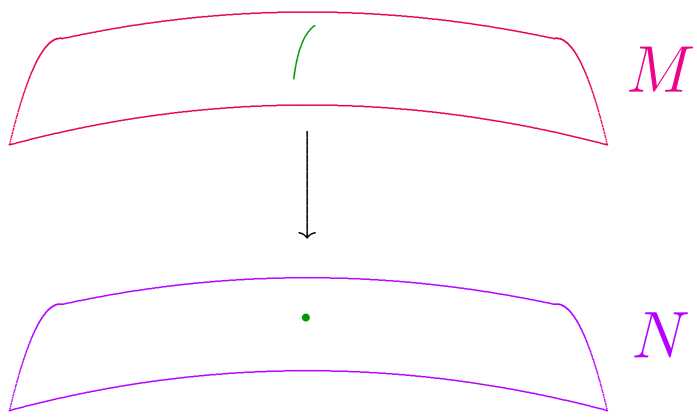
Blowing up:

If N is a complex surface, may replace $p \in N$



Blowing up:

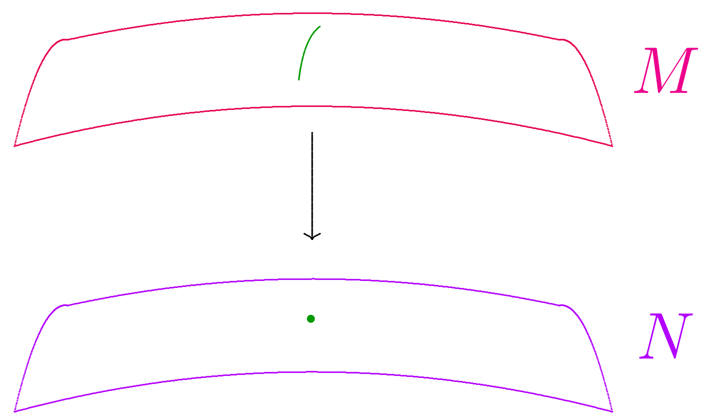
If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$



Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

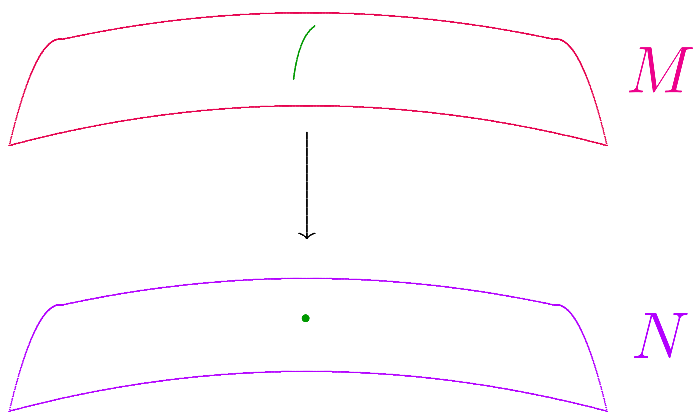


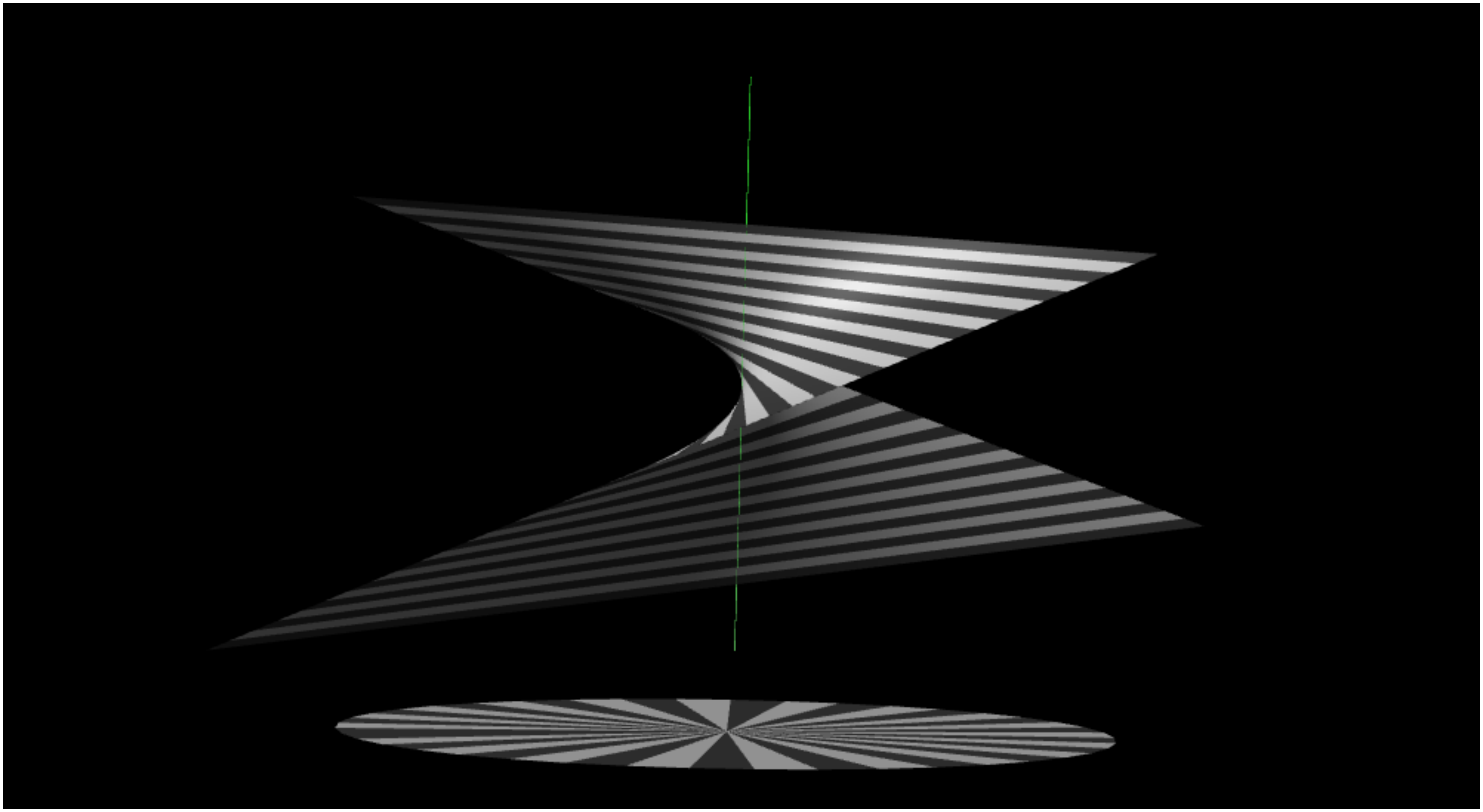
Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}_2}$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.



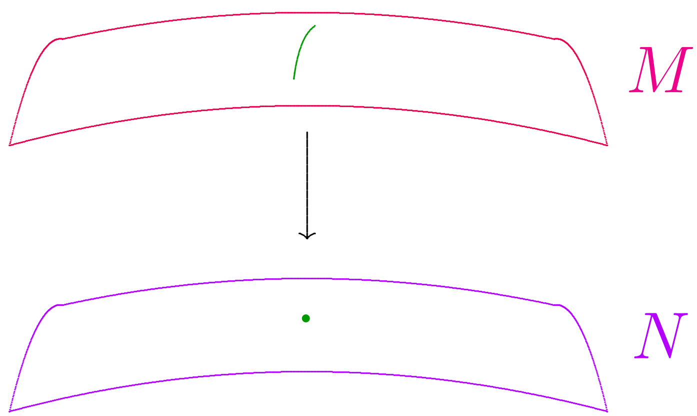


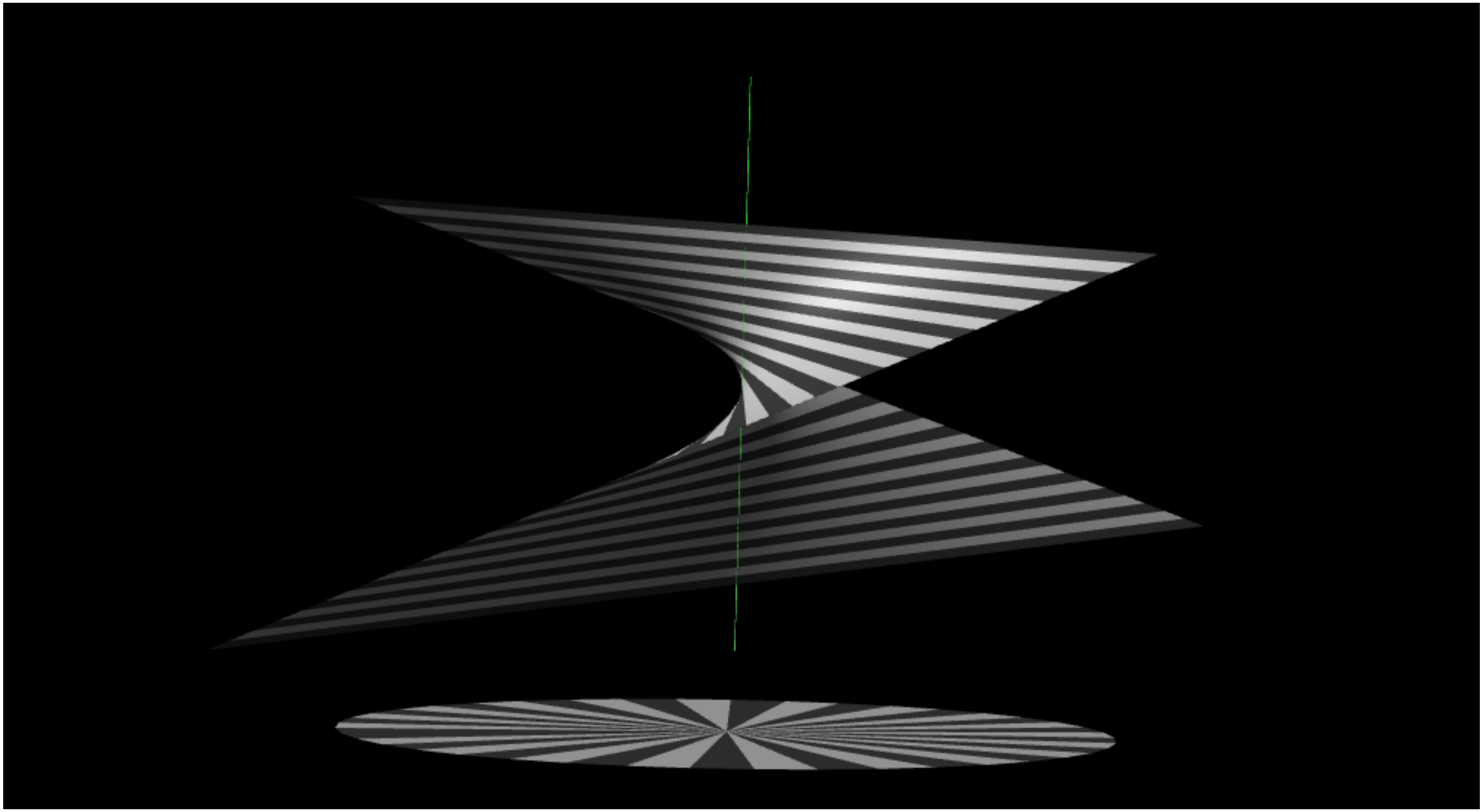
Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.



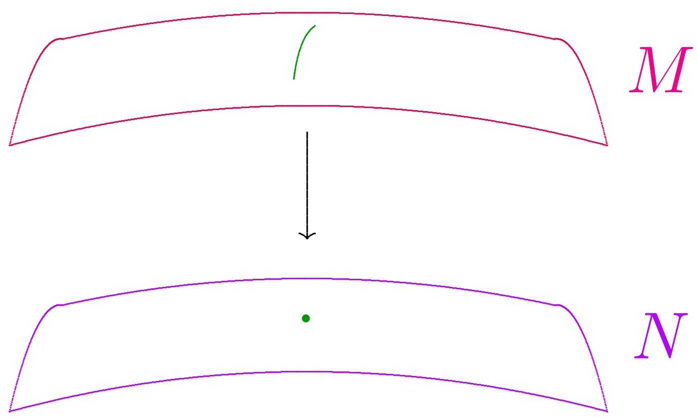


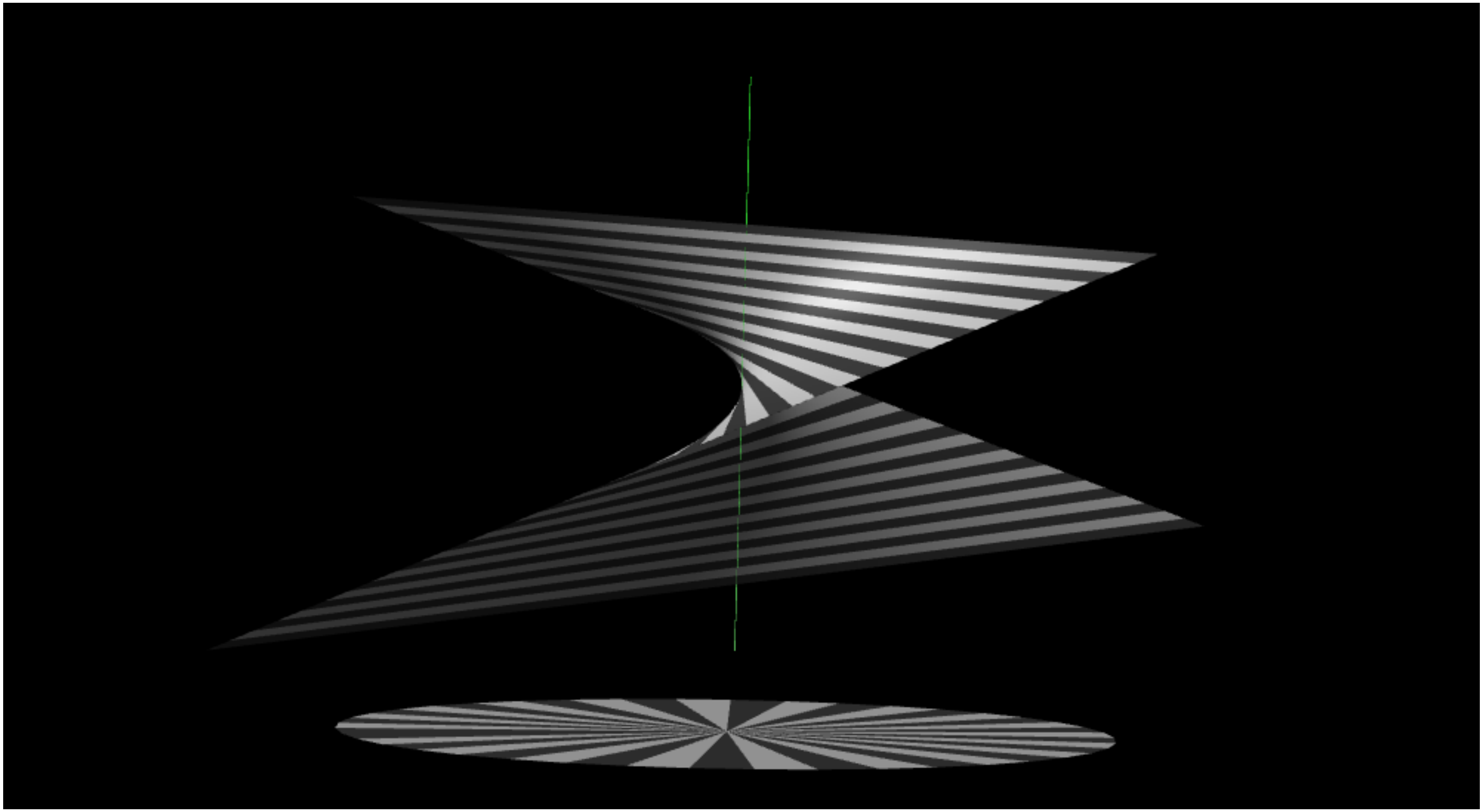
Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.



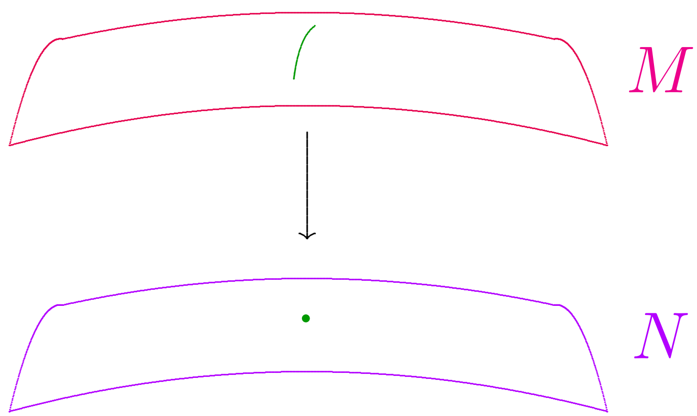


Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}_2}$$

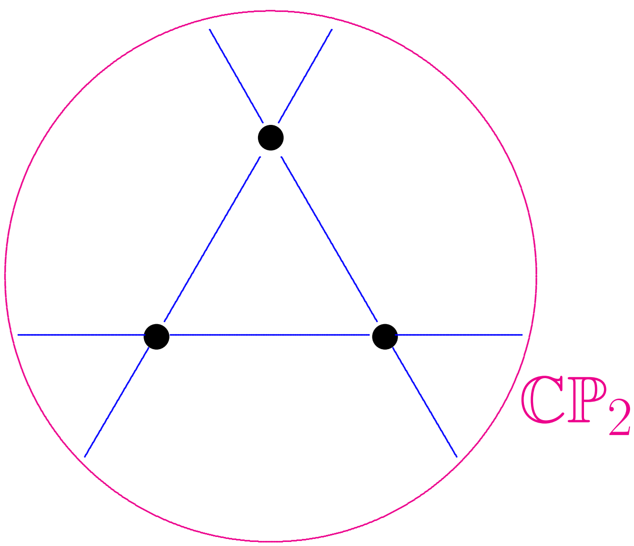
in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.



Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

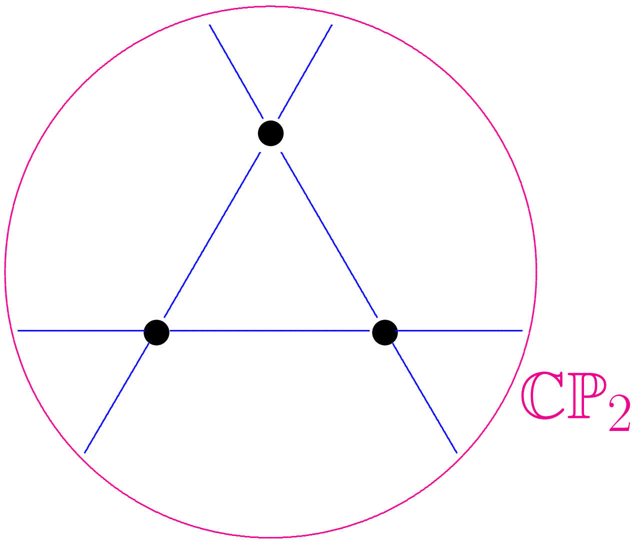
Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

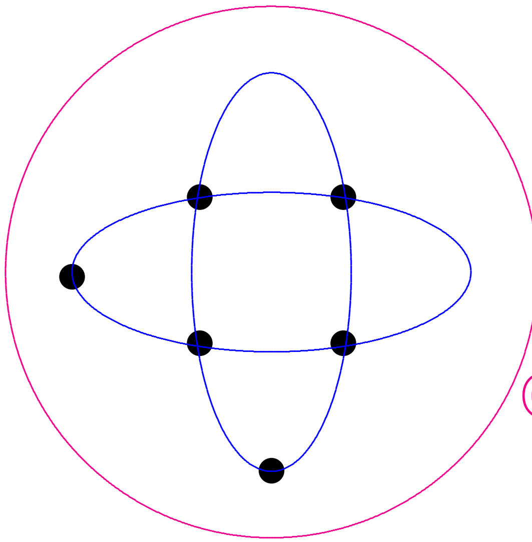


No 3 on a line,

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

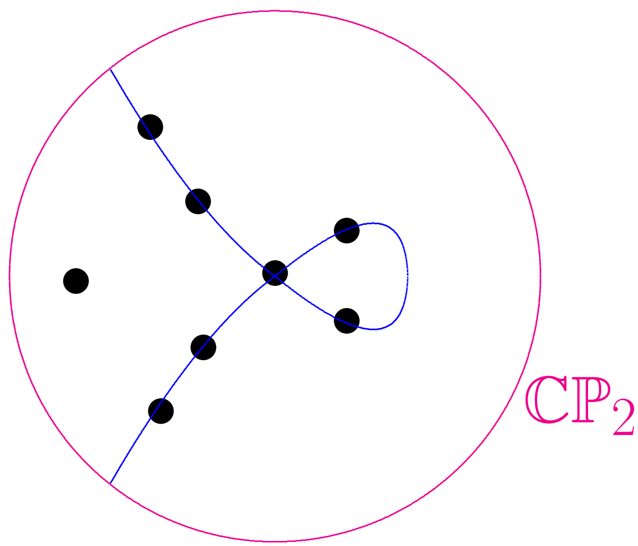


No 3 on a line, no 6 on conic,

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



No 3 on a line, no 6 on conic, no 8 on nodal cubic.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible Hermitian,*

$$g = g(J\cdot, J\cdot)$$

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible Hermitian, Einstein metric,*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible Hermitian, Einstein metric, and this metric is unique*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible Hermitian, Einstein metric, and this metric is unique up to complex automorphisms and constant rescalings.*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

L '97: Such Einstein metrics are necessarily conformal to extremal Kähler metrics, in the sense of Calabi.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

L '97: Such Einstein metrics are necessarily conformal to extremal Kähler metrics, in the sense of Calabi. Most of them are actually Kähler-Einstein.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

Existence: Page

1978

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński,

1983

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński, Siu,

1986

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński, Siu, Tian-Yau,

1987

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo* (M^4, J) *admits a* J*-compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński, Siu, Tian-Yau, Tian,

1990

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński, Siu, Tian-Yau, Tian, Odaka-Spotti-Sun,

2016

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński, Siu, Tian-Yau, Tian, Odaka-Spotti-Sun, Chen-L-Weber.

2008

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

Uniqueness: Bando-Mabuchi '87

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible Hermitian, Einstein metric, and this metric is geometrically unique.*

Uniqueness: Bando-Mabuchi '87, L '12.

One fundamental open problem:

One fundamental open problem:

Understand all Einstein metrics on del Pezzos.

One fundamental open problem:

Understand all Einstein metrics on del Pezzos.

Is Einstein moduli space connected?

One fundamental open problem:

Understand all Einstein metrics on del Pezzos.

Is Einstein moduli space connected?

Progress to date:

One fundamental open problem:

Understand all Einstein metrics on del Pezzos.

Is Einstein moduli space connected?

Progress to date:

Completely understand one connected component.

One fundamental open problem:

Understand all Einstein metrics on del Pezzos.

Is Einstein moduli space connected?

Progress to date:

Completely understand one connected component.

Peng Wu proposed one beautiful characterization of these metrics,

One fundamental open problem:

Understand all Einstein metrics on del Pezzos.

Is Einstein moduli space connected?

Progress to date:

Completely understand one connected component.

Peng Wu proposed one beautiful characterization of these metrics, in terms of an open condition on

$$W_+ : \Lambda^+ \to \Lambda^+.$$

One fundamental open problem:

Understand all Einstein metrics on del Pezzos.

Is Einstein moduli space connected?

Progress to date:

Completely understand one connected component.

Peng Wu proposed one beautiful characterization of these metrics, in terms of an open condition on

$$W_+ : \Lambda^+ \to \Lambda^+.$$

Wu's criterion:

$$\det(W_+) > 0.$$

Theorem (Wu/L '21).

Theorem (Wu/L '21). *Let* (M, g) *be a compact oriented Einstein* 4*-manifold,*

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein* 4*-manifold,*

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M.

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M. *Then* M *is diffeomorphic to a del Pezzo surface,*

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M*. Then* M *is diffeomorphic to a del Pezzo surface, and* g *is one of the conformally Kähler, Einstein metrics we've discussed.*

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M. *Then* M *is diffeomorphic to a del Pezzo surface, and* g *is one of the conformally Kähler, Einstein metrics we've discussed.*

Simplified version of my argument:

Theorem (Wu/L '21). *Let (M, g) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M. Then M is diffeomorphic to a del Pezzo surface, and g is one of the conformally Kähler, Einstein metrics we've discussed.

Simplified version of my argument:

$$\det(W^+) > 0 \qquad \Longrightarrow \qquad W^+ \sim \begin{bmatrix} + & & \\ & - & \\ & & - \end{bmatrix}$$

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M*. Then* M *is diffeomorphic to a del Pezzo surface, and* g *is one of the conformally Kähler, Einstein metrics we've discussed.*

Simplified version of my argument:

$$\det(W^+) > 0 \quad \Longrightarrow \quad W^+ \sim \begin{bmatrix} + & & \\ & - & \\ & & - \end{bmatrix}$$

Top eigenspace $L \subset \Lambda^+$ of W_+ is a line bundle.

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M*. Then* M *is diffeomorphic to a del Pezzo surface, and* g *is one of the conformally Kähler, Einstein metrics we've discussed.*

Simplified version of my argument:
Set $h = \alpha^{2/3} g$, where α top eigenvalue of W_{+g},

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M*. Then* M *is diffeomorphic to a del Pezzo surface, and* g *is one of the conformally Kähler, Einstein metrics we've discussed.*

Simplified version of my argument:
Set $h = \alpha^{2/3} g$, where α top eigenvalue of W_+g, and choose top eigenform $\omega \in \Lambda^+$ with $|\omega|_h \equiv 1$.

Theorem (Wu/L '21). *Let (M, g) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M. Then M is diffeomorphic to a del Pezzo surface, and g is one of the conformally Kähler, Einstein metrics we've discussed.

Simplified version of my argument:
Set $h = \alpha^{2/3} g$, where α top eigenvalue of W_{+g},
and choose top eigenform $\omega \in \Lambda^+$ with $|\omega|_h \equiv 1$.
A Weitzenböck argument

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M*. Then* M *is diffeomorphic to a del Pezzo surface, and* g *is one of the conformally Kähler, Einstein metrics we've discussed.*

Simplified version of my argument:
Set $h = \alpha^{2/3} g$, where α top eigenvalue of W_{+g}, and choose top eigenform $\omega \in \Lambda^+$ with $|\omega|_h \equiv 1$. A Weitzenböck argument then shows that

$$0 \geq |\nabla\omega|^2 + 3\langle\omega, (d + d^*)^2\omega\rangle$$

Theorem (Wu/L '21). *Let (M, g) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M. Then M is diffeomorphic to a del Pezzo surface, and g is one of the conformally Kähler, Einstein metrics we've discussed.

Simplified version of my argument:
Set $h = \alpha^{2/3} g$, where α top eigenvalue of W_{+g}, and choose top eigenform $\omega \in \Lambda^+$ with $|\omega|_h \equiv 1$. A Weitzenböck argument then shows that

$$0 \geq |\nabla \omega|^2 + 3\langle \omega, (d + d^*)^2 \omega \rangle$$

at every point, with respect to h.

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M. *Then* M *is diffeomorphic to a del Pezzo surface, and* g *is one of the conformally Kähler, Einstein metrics we've discussed.*

Simplified version of my argument:
Set $h = \alpha^{2/3} g$, where α top eigenvalue of W_{+g}, and choose top eigenform $\omega \in \Lambda^+$ with $|\omega|_h \equiv 1$. A Weitzenböck argument then shows that

$$0 \geq |\nabla\omega|^2 + 3\langle\omega, (d + d^*)^2\omega\rangle$$

at every point, with respect to h. Now integrate!

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M. *Then* M *is diffeomorphic to a del Pezzo surface, and* g *is one of the conformally Kähler, Einstein metrics we've discussed.*

Theorem (Wu/L '21). *Let (M, g) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M. Then M is diffeomorphic to a del Pezzo surface, and g is one of the conformally Kähler, Einstein metrics we've discussed.

Corollary. *Every simply-connected compact oriented Einstein (M^4, h) with $\det(W_+) > 0$ is diffeomorphic to a del Pezzo surface.*

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M*. Then* M *is diffeomorphic to a del Pezzo surface, and* g *is one of the conformally Kähler, Einstein metrics we've discussed.*

Corollary. *Every simply-connected compact oriented Einstein* (M^4, h) *with* $\det(W_+) > 0$ *is diffeomorphic to a del Pezzo surface. Conversely, every del Pezzo* M^4 *carries Einstein* h *with* $\det(W_+) > 0$*, and these sweep out exactly one connected component of moduli space* $\mathscr{E}(M)$*.*

We've focused on compact Einstein manifolds.

We've focused on compact Einstein manifolds.

But complete, non-compact solutions are often key to proving theorems about compact ones.

We've focused on compact Einstein manifolds.

But complete, non-compact solutions are often key to proving theorems about compact ones.

We'll now see that some of these same ideas lead to interesting results in the non-compact setting.

Definition. *A gravitational instanton is a*

Definition. *A gravitational instanton is a complete,*

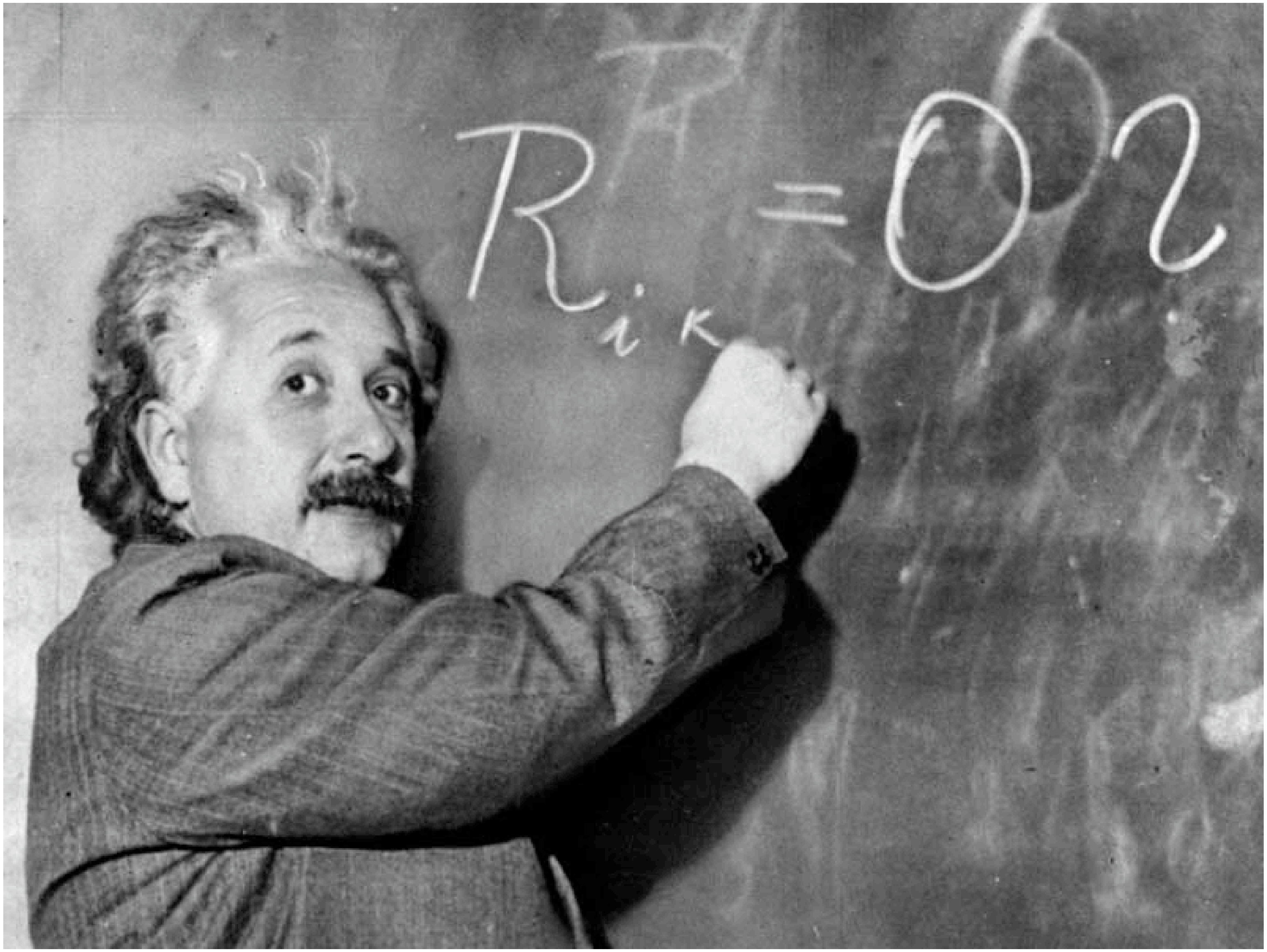
Definition. *A gravitational instanton is a complete, non-compact,*

Definition. *A gravitational instanton is a complete, non-compact, non-flat,*

Definition. *A gravitational instanton is a complete, non-compact, non-flat, Ricci-flat*

Definition. *A gravitational instanton is a complete, non-compact, non-flat, Ricci-flat Riemannian* 4*-manifold.*

Definition. *A gravitational instanton is a complete, non-compact, non-flat, Ricci-flat Riemannian* 4*-manifold.*



Definition. *A gravitational instanton is a complete, non-compact, non-flat, Ricci-flat Riemannian* 4*-manifold.*

Definition. *A gravitational instanton is a complete, non-compact, non-flat, Ricci-flat Riemannian* 4*-manifold.*

Terminology due to Gibbons & Hawking, late '70s

Definition. *A gravitational instanton is a complete, non-compact, non-flat, Ricci-flat Riemannian* 4*-manifold.*

Terminology due to Gibbons & Hawking, late '70s

Key examples:

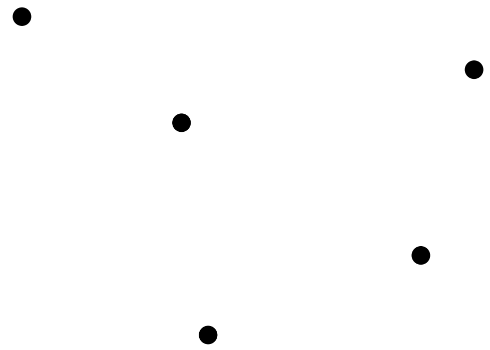
Key examples:

Discovered by Gibbons & Hawking, 1979.

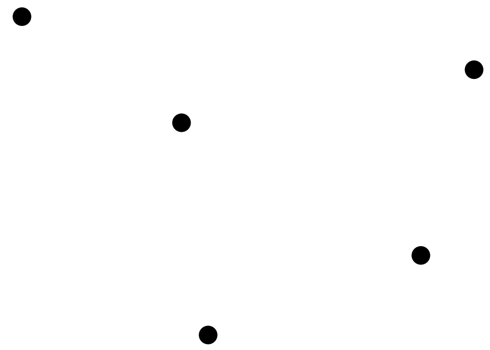
Key examples:

Discovered by Gibbons & Hawking, 1979.

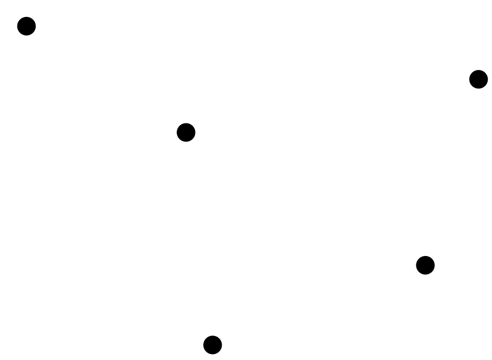
Data: ℓ points in $\mathbb{R}^3$ and a constant $\kappa^2 \geq 0$.



Data: ℓ points in $\mathbb{R}^3$ and a constant $\kappa^2 \geq 0$.

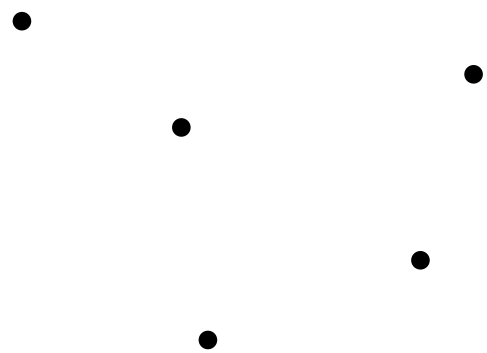


Data: ℓ points in $\mathbb{R}^3$ and κ^2



Data: ℓ points in $\mathbb{R}^3$ and $\kappa^2 \Longrightarrow V$ with $\Delta V = 0$

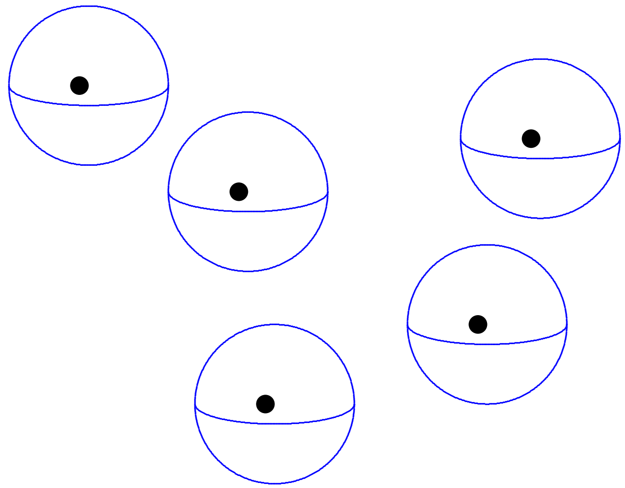
$$V = \kappa^2 + \sum_{j=1}^{\ell} \frac{1}{2\varrho_j}$$



Data: ℓ points in $\mathbb{R}^3$ and $\kappa^2 \Longrightarrow V$ with $\Delta V = 0$

$$V = \kappa^2 + \sum_{j=1}^{\ell} \frac{1}{2\varrho_j}$$

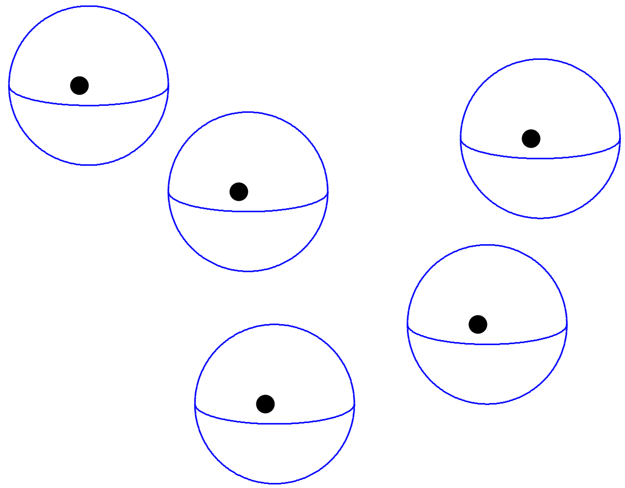
$F = \star dV$ closed 2-form,



Data: ℓ points in $\mathbb{R}^3$ and $\kappa^2 \Longrightarrow V$ with $\Delta V = 0$

$$V = \kappa^2 + \sum_{j=1}^{\ell} \frac{1}{2\varrho_j}$$

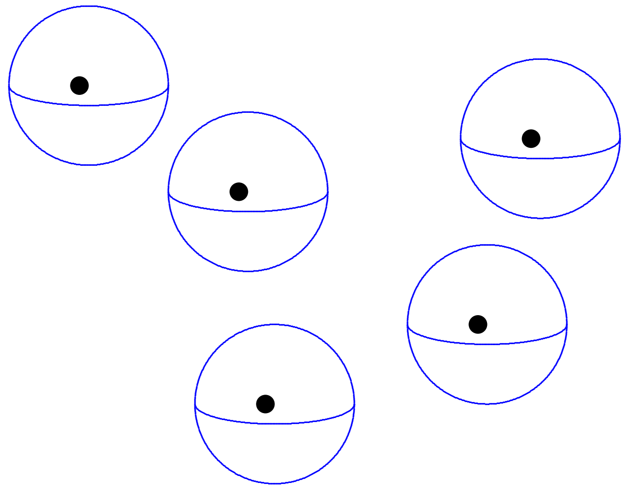
$F = \star dV$ closed 2-form, $[\frac{1}{2\pi}F] \in H^2(\mathbb{R}^3 - \{p_j\}, \mathbb{Z})$.



Data: ℓ points in $\mathbb{R}^3$ and $\kappa^2 \Longrightarrow V$ with $\Delta V = 0$

$$V = \kappa^2 + \sum_{j=1}^{\ell} \frac{1}{2\varrho_j}$$

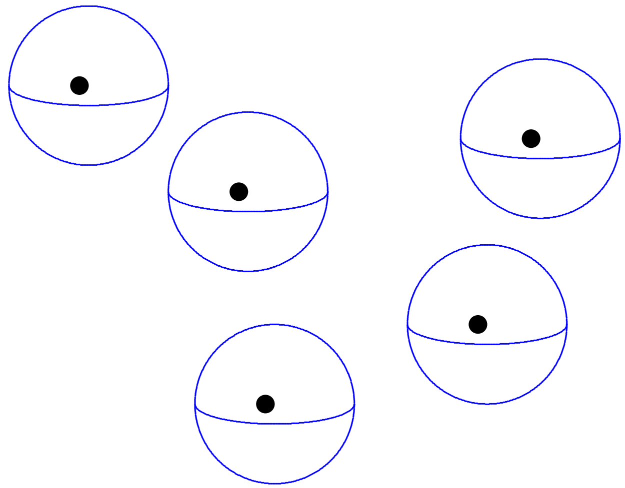
$F = \star dV$ curvature θ on $P \to \mathbb{R}^3 - \{\text{pts}\}$.



Data: ℓ points in $\mathbb{R}^3$. $\Longrightarrow V$ with $\Delta V = 0$

$$g = Vh + V^{-1}\theta^2$$

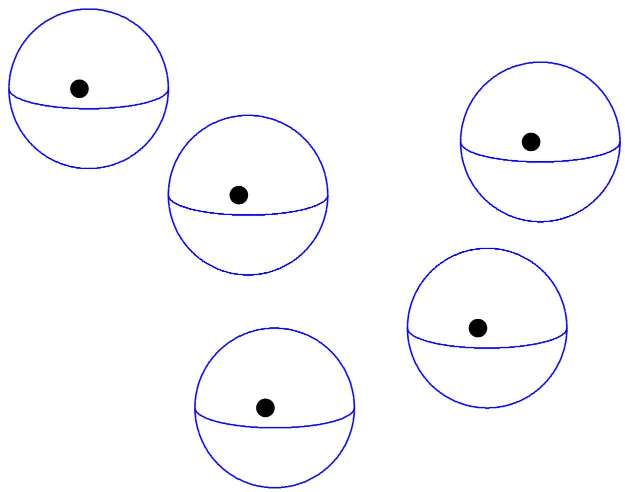
$F = \star dV$ curvature θ on $P \to \mathbb{R}^3 - \{\text{pts}\}$.



Data: ℓ points in $\mathbb{R}^3$. $\Longrightarrow$ V with $\Delta V = 0$

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

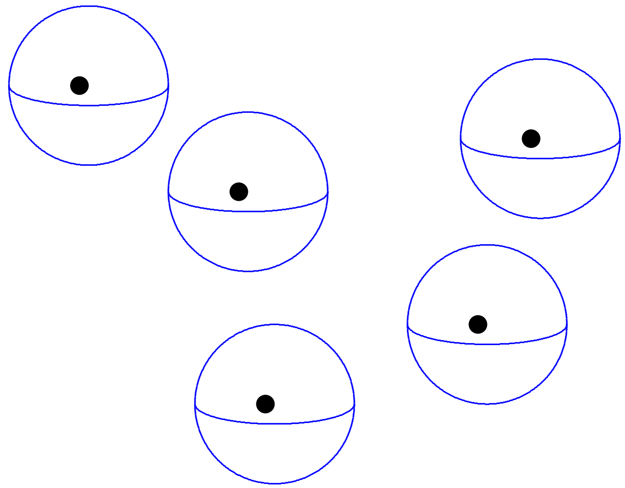
$F = \star dV$ curvature θ on $P \to \mathbb{R}^3 - \{\text{pts}\}$.



Data: ℓ points in $\mathbb{R}^3$. $\Longrightarrow$ V with $\Delta V = 0$

$$g = Vh + V^{-1}\theta^2$$

$F = \star dV$ curvature θ on $P \to \mathbb{R}^3 - \{\text{pts}\}$.



Data: ℓ points in $\mathbb{R}^3$. $\Longrightarrow$ V with $\Delta V = 0$

$$g = Vh + V^{-1}\theta^2$$

on P. Then take M^4 = Riemannian completion.

Gibbons-Hawking gravitational instantons:

Gibbons-Hawking gravitational instantons:

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

Gibbons-Hawking gravitational instantons:

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

$$V = \kappa^2 + \sum_{j=1}^{\ell} \frac{1}{2\varrho_j}$$

Gibbons-Hawking gravitational instantons:

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

$$V = \kappa^2 + \sum_{j=1}^{\ell} \frac{1}{2\varrho_j}$$

$$d\theta = \star dV$$

Gibbons-Hawking gravitational instantons:

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

$$V = \kappa^2 + \sum_{j=1}^{\ell} \frac{1}{2\varrho_j}$$

$$d\theta = \star dV$$

Kähler with respect to three complex structures

Gibbons-Hawking gravitational instantons:

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

$$V = \kappa^2 + \sum_{j=1}^{\ell} \frac{1}{2\varrho_j}$$

$$d\theta = \star dV$$

Kähler with respect to three complex structures

$$dx \mapsto V^{-1}\theta, \qquad dy \mapsto dz$$

Gibbons-Hawking gravitational instantons:

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

$$V = \kappa^2 + \sum_{j=1}^{\ell} \frac{1}{2\varrho_j}$$

$$d\theta = \star dV$$

Kähler with respect to three complex structures

$$dx \mapsto V^{-1}\theta, \qquad dy \mapsto dz$$

$$dy \mapsto V^{-1}\theta, \qquad dz \mapsto dx$$

Gibbons-Hawking gravitational instantons:

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

$$V = \kappa^2 + \sum_{j=1}^{\ell} \frac{1}{2\varrho_j}$$

$$d\theta = \star dV$$

Kähler with respect to three complex structures

$$dx \mapsto V^{-1}\theta, \qquad dy \mapsto dz$$

$$dy \mapsto V^{-1}\theta, \qquad dz \mapsto dx$$

$$dz \mapsto V^{-1}\theta, \qquad dx \mapsto dy$$

Gibbons-Hawking gravitational instantons:

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

$$V = \kappa^2 + \sum_{j=1}^{\ell} \frac{1}{2\varrho_j}$$

$$d\theta = \star dV$$

Kähler with respect to three complex structures

Gibbons-Hawking gravitational instantons:

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

$$V = \kappa^2 + \sum_{j=1}^{\ell} \frac{1}{2\varrho_j}$$

$$d\theta = \star dV$$

Kähler with respect to three complex structures

Hence holonomy $\subset \mathbf{Sp}(1) = \mathbf{SU}(2)$.

Gibbons-Hawking gravitational instantons:

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

$$V = \kappa^2 + \sum_{j=1}^{\ell} \frac{1}{2\varrho_j}$$

$$d\theta = \star dV$$

Kähler with respect to three complex structures

Hence holonomy $\subset \mathbf{Sp}(1) = \mathbf{SU}(2)$.

Hence Ricci-flat!

Gibbons-Hawking gravitational instantons:

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

$$V = \kappa^2 + \sum_{j=1}^{\ell} \frac{1}{2\varrho_j}$$

$$d\theta = \star dV$$

Kähler with respect to three complex structures

Hence holonomy $\subset \mathbf{Sp}(1) = \mathbf{SU}(2)$.

Hence Ricci-flat!

Calabi later called such metrics "hyper-Kähler."

Gibbons-Hawking gravitational instantons:

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

$$V = \kappa^2 + \sum_{j=1}^{\ell} \frac{1}{2\varrho_j}$$

$$d\theta = \star dV$$

Kähler with respect to three complex structures

Hence holonomy $\subset \mathbf{Sp}(1) = \mathbf{SU}(2)$.

Hence Ricci-flat!

Calabi later called such metrics "hyper-Kähler."

$M \to \mathbb{R}^3$ hyper-Kähler moment map of S^1 action.

Gibbons-Hawking gravitational instantons:

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

$$V = \kappa^2 + \sum_{j=1}^{\ell} \frac{1}{2\varrho_j}$$

$$d\theta = \star dV$$

Kähler with respect to three complex structures

Hence holonomy $\subset \mathbf{Sp}(1) = \mathbf{SU}(2)$.

Hence Ricci-flat!

Calabi later called such metrics "hyper-Kähler."

Gibbons and Hawking were unaware of all this!

Gibbons-Hawking gravitational instantons:

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

$$V = \kappa^2 + \sum_{j=1}^{\ell} \frac{1}{2\varrho_j}$$

$$d\theta = \star dV$$

Gibbons-Hawking gravitational instantons:

These spaces have just one end, $\approx(\mathbb{R}^4 - \{0\})/\mathbb{Z}_\ell$

Gibbons-Hawking gravitational instantons:

These spaces have just one end, $\approx(\mathbb{R}^4 - \{0\})/\mathbb{Z}_\ell$

When $\kappa = 0$, they are ALE:

Gibbons-Hawking gravitational instantons:

These spaces have just one end, $\approx(\mathbb{R}^4 - \{0\})/\mathbb{Z}_\ell$

When $\kappa = 0$, they are ALE:

Gibbons-Hawking gravitational instantons:

These spaces have just one end, $\approx(\mathbb{R}^4 - \{0\})/\mathbb{Z}_\ell$

When $\kappa = 0$, they are ALE:

"Asymptotically locally Euclidean"

Gibbons-Hawking gravitational instantons:

These spaces have just one end, $\approx (\mathbb{R}^4 - \{0\})/\mathbb{Z}_\ell$

When $\kappa = 0$, they are ALE:

"Asymptotically locally Euclidean"

$$g_{jk} = \delta_{jk} + O(|x|^{-4})$$

Gibbons-Hawking gravitational instantons:

These spaces have just one end, $\approx(\mathbb{R}^4 - \{0\})/\mathbb{Z}_\ell$

When $\kappa = 0$, they are ALE:

"Asymptotically locally Euclidean"

$$g_{jk} = \delta_{jk} + O(|x|^{-4})$$

In particular, volume of large ball is

Gibbons-Hawking gravitational instantons:

These spaces have just one end, $\approx(\mathbb{R}^4 - \{0\})/\mathbb{Z}_\ell$

When $\kappa = 0$, they are ALE:

"Asymptotically locally Euclidean"

$$g_{jk} = \delta_{jk} + O(|x|^{-4})$$

In particular, volume of large ball is

$$\mathrm{Vol}(B_\rho) \sim \mathrm{const} \cdot \rho^4$$

Gibbons-Hawking gravitational instantons:

These spaces have just one end, $\approx(\mathbb{R}^4 - \{0\})/\mathbb{Z}_\ell$

When $\kappa = 0$, they are ALE:

"Asymptotically locally Euclidean"

$$g_{jk} = \delta_{jk} + O(|x|^{-4})$$

In particular, volume of large ball is

$$\mathrm{Vol}(B_\rho) \sim \frac{\pi^2/2}{\ell} \cdot \rho^4$$

Gibbons-Hawking gravitational instantons:

These spaces have just one end, $\approx(\mathbb{R}^4 - \{0\})/\mathbb{Z}_\ell$

When $\kappa = 0$, they are ALE:

"Asymptotically locally Euclidean"

$$g_{jk} = \delta_{jk} + O(|x|^{-4})$$

In particular, volume of large ball is

$$\mathrm{Vol}(B_\rho) \sim \frac{\pi^2/2}{\ell} \cdot \rho^4$$

Notice that $\ell = 1$ case is just flat $\mathbb{R}^4$!

Gibbons-Hawking gravitational instantons:

These spaces have just one end, $\approx (\mathbb{R}^4 - \{0\})/\mathbb{Z}_\ell$

When $\kappa = 0$, they are ALE:

"Asymptotically locally Euclidean"

$$g_{jk} = \delta_{jk} + O(|x|^{-4})$$

In particular, volume of large ball is

$$\mathrm{Vol}(B_\rho) \sim \frac{\pi^2/2}{\ell} \cdot \rho^4$$

Notice that $\ell = 1$ case is just flat $\mathbb{R}^4$!

The $\ell = 2$ case is Eguchi-Hanson $\approx T^*S^2$.

Gibbons-Hawking gravitational instantons:

These spaces have just one end, $\approx(\mathbb{R}^4 - \{0\})/\mathbb{Z}_\ell$

But when $\kappa \neq 0$, they are instead ALF:

Gibbons-Hawking gravitational instantons:

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

$$V = \kappa^2 + \sum_{j=1}^{\ell} \frac{1}{2\varrho_j}$$

Gibbons-Hawking gravitational instantons:

These spaces have just one end, $\approx (\mathbb{R}^4 - \{0\})/\mathbb{Z}_\ell$

But when $\kappa \neq 0$, they are instead ALF:

Gibbons-Hawking gravitational instantons:

These spaces have just one end, $\approx(\mathbb{R}^4 - \{0\})/\mathbb{Z}_\ell$

But when $\kappa \neq 0$, they are instead ALF:

"Asymptotically locally flat"

Gibbons-Hawking gravitational instantons:

These spaces have just one end, $\approx(\mathbb{R}^4 - \{0\})/\mathbb{Z}_\ell$

But when $\kappa \neq 0$, they are instead ALF:

"Asymptotically locally flat"

Curvature still falls off at infinity,

Gibbons-Hawking gravitational instantons:

These spaces have just one end, $\approx (\mathbb{R}^4 - \{0\})/\mathbb{Z}_\ell$

But when $\kappa \neq 0$, they are instead ALF:

"Asymptotically locally flat"

Curvature still falls off at infinity,

$$|\mathcal{R}| \sim \text{const} \cdot \rho^{-3}$$

Gibbons-Hawking gravitational instantons:

These spaces have just one end, $\approx (\mathbb{R}^4 - \{0\})/\mathbb{Z}_\ell$

But when $\kappa \neq 0$, they are instead ALF:

"Asymptotically locally flat"

Curvature still falls off at infinity,

Gibbons-Hawking gravitational instantons:

These spaces have just one end, $\approx(\mathbb{R}^4 - \{0\})/\mathbb{Z}_\ell$

But when $\kappa \neq 0$, they are instead ALF:

"Asymptotically locally flat"

Curvature still falls off at infinity,

but volume growth is only cubic:

$$\mathrm{Vol}(B_\rho) \sim \mathrm{const} \cdot \rho^3$$

Gibbons-Hawking gravitational instantons:

These spaces have just one end, $\approx (\mathbb{R}^4 - \{0\})/\mathbb{Z}_\ell$

But when $\kappa \neq 0$, they are instead ALF:

"Asymptotically locally flat"

Curvature still falls off at infinity,

but volume growth is only cubic:

$$\mathrm{Vol}(B_\rho) \sim \mathrm{const} \cdot \rho^3$$

This last property distinguishes the ALF spaces from other classes of gravitational instantons:

ALG, ALH, ALG*, ALH*, . . .

Example.

Example. Taub-NUT metric:

Example. Taub-NUT metric:

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

$$V = 1 + \frac{1}{2\varrho}$$

Example. Taub-NUT metric:

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

$$V = 1 + \frac{1}{2\varrho}$$

Can also write as

$$g = \frac{r+1}{4r}dr^2 + r(1+r)[\sigma_1{}^2 + \sigma_2{}^2] + \frac{r}{r+1}\sigma_3{}^2$$

Example. Taub-NUT metric:

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

$$V = 1 + \frac{1}{2\varrho}$$

Can also write as

$$g = \frac{r+1}{4r}dr^2 + r(1+r)[\sigma_1{}^2 + \sigma_2{}^2] + \frac{r}{r+1}\sigma_3{}^2$$

for left-invariant coframe $\{\sigma_j\}$ on $S^3 = \mathbf{SU}(2)$.

Example. Taub-NUT metric:

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

$$V = 1 + \frac{1}{2\varrho}$$

Can also write as

$$g = \frac{r+1}{4r}dr^2 + r(1+r)[\sigma_1{}^2 + \sigma_2{}^2] + \frac{r}{r+1}\sigma_3{}^2$$

for left-invariant coframe $\{\sigma_j\}$ on $S^3 = \mathbf{SU}(2)$.

Taub-NUT becomes Hermitian metric on $\mathbb{C}^2$.

Example. Taub-NUT metric:

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

$$V = 1 + \frac{1}{2\varrho}$$

Can also write as

$$g = \frac{r+1}{4r}dr^2 + r(1+r)[\sigma_1{}^2 + \sigma_2{}^2] + \frac{r}{r+1}\sigma_3{}^2$$

for left-invariant coframe $\{\sigma_j\}$ on $S^3 = \mathbf{SU}(2)$.

Taub-NUT becomes Hermitian metric on $\mathbb{C}^2$.

$$dr \mapsto \frac{2r}{1+r}\sigma_3, \quad \sigma_1 \mapsto \sigma_2$$

Example. Taub-NUT metric:

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

$$V = 1 + \frac{1}{2\varrho}$$

Can also write as

$$g = \frac{r+1}{4r}dr^2 + r(1+r)[{\sigma_1}^2 + {\sigma_2}^2] + \frac{r}{r+1}{\sigma_3}^2$$

for left-invariant coframe $\{\sigma_j\}$ on $S^3 = \mathbf{SU}(2)$.

Taub-NUT becomes Hermitian metric on $\mathbb{C}^2$.

Example. Taub-NUT metric:

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

$$V = 1 + \frac{1}{2\varrho}$$

Can also write as

$$g = \frac{r+1}{4r}dr^2 + r(1+r)[\sigma_1{}^2 + \sigma_2{}^2] + \frac{r}{r+1}\sigma_3{}^2$$

for left-invariant coframe $\{\sigma_j\}$ on $S^3 = \mathbf{SU}(2)$.

Taub-NUT becomes Hermitian metric on $\mathbb{C}^2$.

This J determines opposite orientation from the hyper-Kähler complex structures.

Example. Taub-NUT metric:

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

$$V = 1 + \frac{1}{2\varrho}$$

Can also write as

$$g = \frac{r+1}{4r}dr^2 + r(1+r)[\sigma_1{}^2 + \sigma_2{}^2] + \frac{r}{r+1}\sigma_3{}^2$$

for left-invariant coframe $\{\sigma_j\}$ on $S^3 = \mathbf{SU}(2)$.

Taub-NUT becomes Hermitian metric on $\mathbb{C}^2$.

Non-Kähler, but conformally Kähler!

Hawking also explored non-hyper-Kähler examples. . .

Example.

Example. Riemannian Schwarzschild metric:

Example. Riemannian Schwarzschild metric:

$$g = \left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2 + \varrho^2 g_{S^2}$$

Example. Riemannian Schwarzschild metric:

$$g = \left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2 + \varrho^2 g_{S^2}$$

Conformal to

$$h = \frac{1}{\varrho^2}\left[\left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2\right] + g_{S^2}$$

Example. Riemannian Schwarzschild metric:

$$g = \left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2 + \varrho^2 g_{S^2}$$

Conformal to

$$h = \frac{1}{\varrho^2} \left[\left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2 \right] + g_{S^2}$$

Kähler!

Example. Riemannian Schwarzschild metric:

$$g = \left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2 + \varrho^2 g_{S^2}$$

Conformal to

$$h = \frac{1}{\varrho^2}\left[\left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2\right] + g_{S^2}$$

Kähler! In fact, extremal Kähler!

Example. Riemannian Schwarzschild metric:

$$g = \left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2 + \varrho^2 g_{S^2}$$

Conformal to

$$h = \frac{1}{\varrho^2}\left[\left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2\right] + g_{S^2}$$

Kähler! In fact, extremal Kähler!

$$\bar{\partial}\nabla^{1,0}s = 0$$

Example. Riemannian Schwarzschild metric:

$$g = \left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2 + \varrho^2 g_{S^2}$$

Conformal to

$$h = \frac{1}{\varrho^2}\left[\left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2\right] + g_{S^2}$$

Hawking: set $t = 4m\theta$ and $\varrho = 2m + \frac{r^2}{8m}$.
This makes g into a Ricci-flat metric on $\mathbb{R}^2 \times S^2$.

Example. Riemannian Schwarzschild metric:

$$g = \left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2 + \varrho^2 g_{S^2}$$

Conformal to

$$h = \frac{1}{\varrho^2}\left[\left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2\right] + g_{S^2}$$

Hawking: set $t = 4m\theta$ and $\varrho = 2m + \frac{r^2}{8m}$.
This makes g into a Ricci-flat metric on $\mathbb{R}^2 \times S^2$.

$$g = dr^2 + r^2 d\theta^2 + 4m^2 g_{S^2} + O(r^2)$$

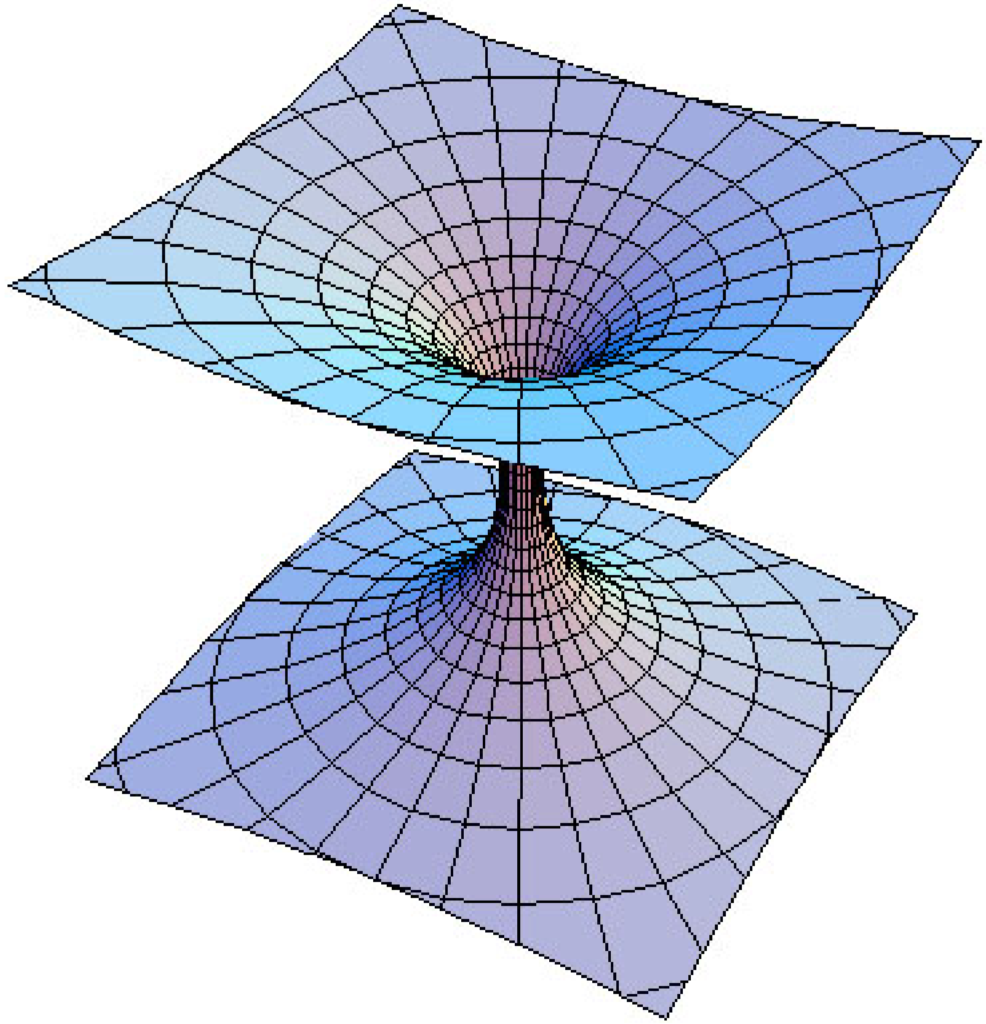
Example. Riemannian Schwarzschild metric:

$$g = \left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2 + \varrho^2 g_{S^2}$$

Conformal to

$$h = \frac{1}{\varrho^2}\left[\left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2\right] + g_{S^2}$$

Hawking: set $t = 4m\theta$ and $\varrho = 2m + \frac{r^2}{8m}$.
This makes g into a Ricci-flat metric on $\mathbb{R}^2 \times S^2$.
Makes h into extremal Kähler metric on $\mathbb{C} \times \mathbb{CP}_1$.



$\mathbb{R} \times S^2 \subset \mathbb{R}^2 \times S^2$

Definition. *A gravitational instanton is a complete, non-compact, non-flat, Ricci-flat Riemannian* 4*-manifold.*

Definition. *A gravitational instanton is a complete, non-compact, non-flat, Ricci-flat Riemannian* 4*-manifold.*

Many excellent mathematical papers cleverly narrow the definition for technical convenience, by assuming at the outset that the metric is hyper-Kähler.

Definition. *A gravitational instanton is a complete, non-compact, non-flat, Ricci-flat Riemannian* 4*-manifold.*

Many excellent mathematical papers cleverly narrow the definition for technical convenience, by assuming at the outset that the metric is hyper-Kähler.

Hitchin, Kronheimer, Cherkis-Hitchin, Minerbe, Hein, Chen-Chen, Hein-Sun-Viaclovsky-Zhang. . .

Definition. *A gravitational instanton is a complete, non-compact, non-flat, Ricci-flat Riemannian* 4*-manifold.*

Many excellent mathematical papers cleverly narrow the definition for technical convenience, by assuming at the outset that the metric is hyper-Kähler.

But my collaborators

Definition. *A gravitational instanton is a complete, non-compact, non-flat, Ricci-flat Riemannian* 4*-manifold.*

Many excellent mathematical papers cleverly narrow the definition for technical convenience, by assuming at the outset that the metric is hyper-Kähler.

But my collaborators Biquard and Gauduchon

Definition. *A gravitational instanton is a complete, non-compact, non-flat, Ricci-flat Riemannian* 4*-manifold.*

Many excellent mathematical papers cleverly narrow the definition for technical convenience, by assuming at the outset that the metric is hyper-Kähler.

But my collaborators Biquard and Gauduchon have fortunately done us all the favor of reminding us

Definition. *A gravitational instanton is a complete, non-compact, non-flat, Ricci-flat Riemannian 4-manifold.*

Many excellent mathematical papers cleverly narrow the definition for technical convenience, by assuming at the outset that the metric is hyper-Kähler.

But my collaborators Biquard and Gauduchon have fortunately done us all the favor of reminding us that the hyper-Kähler gravitons are only one small part of the story!

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat,*

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat*

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is*

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is toric,*

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is toric,*

$\mathbb{T}^2$ acts effectively and isometrically

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is toric, ALF,*

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is toric, ALF,*

$$\Longrightarrow \quad \mathrm{Vol}(B_\rho) \sim \mathrm{const} \cdot \rho^3$$

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is toric, ALF, and Hermitian*

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is toric, ALF, and Hermitian with respect to some integrable complex structure* J.

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is toric, ALF, and Hermitian with respect to some integrable complex structure* J*.*

$$g(J\cdot, J\cdot) = g$$

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is toric, ALF, and Hermitian with respect to some integrable complex structure* J.

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric*, ALF, and Hermitian with respect to some integrable complex structure* J. *Also assume that* (M, g, J) *is not Kähler.*

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric*, ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

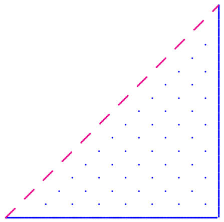
Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is toric, ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT metric;*

Diffeomorphic to $\mathbb{R}^4$

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric*, ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT* metric*;*



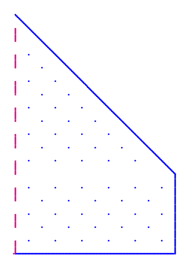
Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric, *ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT* metric;
- *the Taub-bolt* metric;

Diffeomorphic to $\mathbb{CP}_2 - \{pt\}$

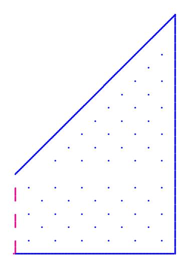
Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric*, ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT* metric*;*
- *the Taub-bolt* metric*;*



Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric*, ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT* metric*;*
- *the Taub-bolt* metric*;*



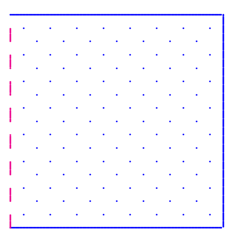
Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is toric, ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT metric;*
- *the Taub-bolt metric;*
- *a metric of the Kerr family;*

Diffeomorphic to $S^2 \times \mathbb{R}^2$

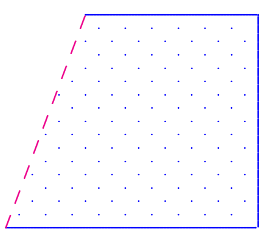
Theorem (Biquard-Gauduchon '23)**.** *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric*, ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT metric;*
- *the Taub-bolt metric;*
- *a metric of the Kerr family;*



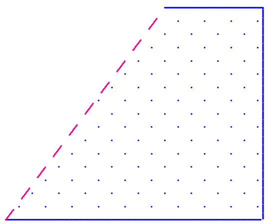
Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric, *ALF, and Hermitian with respect to some integrable complex structure* J. *Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT* metric;
- *the Taub-bolt metric;*
- *a metric of the Kerr family;*



Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is toric, ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT metric;*
- *the Taub-bolt metric;*
- *a metric of the Kerr family;*



Theorem (Biquard-Gauduchon '23)**.** *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is toric, ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT metric;*
- *the Taub-bolt metric;*
- *a metric of the Kerr family; or*
- *a metric in the Chen-Teo family.*

Y. Chen & E. Teo, 2011

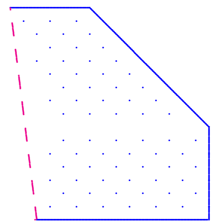
Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is toric, ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT metric;*
- *the Taub-bolt metric;*
- *a metric of the Kerr family; or*
- *a metric in the Chen-Teo family.*

Diffeomorphic to $\mathbb{CP}_2\#(S^2 \times \mathbb{R}^2)$

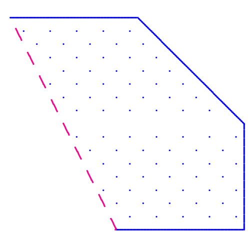
Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric*, ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT* metric*;*
- *the Taub-bolt* metric*;*
- *a metric of the Kerr family; or*
- *a metric in the Chen-Teo family.*



Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric*, ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT metric;*
- *the Taub-bolt metric;*
- *a metric of the Kerr family; or*
- *a metric in the Chen-Teo family.*



Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric, *ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT* metric*;*
- *the Taub-bolt* metric*;*
- *a metric of the Kerr family; or*
- *a metric in the Chen-Teo family.*

Each of the metrics g in question is conformal to a complete extremal Kähler metric with $s > 0$.

Each of the metrics g in question is conformal to a complete extremal Kähler metric with $s > 0$.

This implies that they always satisfy Peng Wu's criterion

$$\det(W^+) > 0,$$

Each of the metrics g in question is conformal to a complete extremal Kähler metric with $s > 0$.

This implies that they always satisfy Peng Wu's criterion

$$\det(W^+) > 0,$$

allowing one to generalize methods first explored in the compact case.

Theorem A (BGL '24).

Theorem A (BGL '24). *Let* (M, g_0) *be*

Theorem A (BGL '24)**.** *Let* (M, g_0) *be any of the ALF toric Hermitian gravitational instantons*

Theorem A (BGL '24)**.** *Let (M, g_0) be any of the ALF toric Hermitian gravitational instantons appearing in the Biquard-Gauduchon classification.*

Theorem A (BGL '24). *Let* (M, g_0) *be any of the ALF toric Hermitian gravitational instantons appearing in the Biquard-Gauduchon classification. Then any other Ricci-flat Riemannian metric* g *on* M

Theorem A (BGL '24)**.** *Let* (M, g_0) *be any of the ALF toric Hermitian gravitational instantons appearing in the Biquard-Gauduchon classification. Then any other Ricci-flat Riemannian metric* g *on* M *which is sufficiently* C^3_1*-close to* g

Theorem A (BGL '24). *Let (M, g_0) be any of the ALF toric Hermitian gravitational instantons appearing in the Biquard-Gauduchon classification. Then any other Ricci-flat Riemannian metric g on M which is sufficiently C^3_1-close to g*

$$\|g - g_0\|_{C^3_1} < \varepsilon$$

Theorem A (BGL '24). *Let* (M, g_0) *be any of the ALF toric Hermitian gravitational instantons appearing in the Biquard-Gauduchon classification. Then any other Ricci-flat Riemannian metric* g *on* M *which is sufficiently* C^3_1-*close to* g

$$\|g - g_0\|_{C^3_1} < \varepsilon$$

$$\|\mho\|_{C^3_1} := \sup_M \sum_{j=0}^{3} (1 + \mathsf{dist})^{j+1} |\nabla^j \mho|_{g_0}$$

Theorem A (BGL '24). *Let (M, g_0) be any of the ALF toric Hermitian gravitational instantons appearing in the Biquard-Gauduchon classification. Then any other Ricci-flat Riemannian metric g on M which is sufficiently C^3_1-close to g*

$$\|g - g_0\|_{C^3_1} < \varepsilon$$

$$\|\mho\|_{C^3_1} := \sup_M \sum_{j=0}^{3} (1 + \mathsf{dist})^{j+1} |\nabla^j \mho|_{g_0}$$

$$|\mho|_{g_0} = O(\varrho^{-1}), \quad |\nabla \mho|_{g_0} = O(\varrho^{-2}), \quad \ldots$$

Theorem A (BGL '24)**.** *Let* (M, g_0) *be any of the ALF toric Hermitian gravitational instantons appearing in the Biquard-Gauduchon classification. Then any other Ricci-flat Riemannian metric* g *on* M *which is sufficiently* C^3_1*-close to* g

Theorem A (BGL '24). *Let* (M, g_0) *be any of the ALF toric Hermitian gravitational instantons appearing in the Biquard-Gauduchon classification. Then any other Ricci-flat Riemannian metric* g *on* M *which is sufficiently* C^3_1*-close to* g *is conformal to some strictly extremal Kähler metric* h*,*

Theorem A (BGL '24). *Let (M, g_0) be any of the ALF toric Hermitian gravitational instantons appearing in the Biquard-Gauduchon classification. Then any other Ricci-flat Riemannian metric g on M which is sufficiently C^3_1-close to g is conformal to some strictly extremal Kähler metric h, and so is, in particular, Hermitian.*

Theorem A (BGL '24). *Let* (M, g_0) *be any of the ALF toric Hermitian gravitational instantons appearing in the Biquard-Gauduchon classification. Then any other Ricci-flat Riemannian metric* g *on* M *which is sufficiently* C^3_1*-close to* g *is conformal to some strictly extremal Kähler metric* h*, and so is, in particular, Hermitian. Moreover, every such* g *carries at least one Killing field.*

Theorem A (BGL '24). *Let (M, g_0) be any of the ALF toric Hermitian gravitational instantons appearing in the Biquard-Gauduchon classification. Then any other Ricci-flat Riemannian metric g on M which is sufficiently C^3_1-close to g is conformal to some strictly extremal Kähler metric h, and so is, in particular, Hermitian. Moreover, every such g carries at least one Killing field.*

Set $h = \alpha^{2/3} g$, where α top eigenvalue of W_{+g}, and choose top eigenform $\omega \in \Lambda^+$ with $|\omega|_h \equiv 1$. Then

$$0 \geq |\nabla \omega|^2 + 3\langle \omega, (d + d^*)^2 \omega \rangle$$

at every point, with respect to h.

Theorem B (BGL '24)**.** *Let* (M, g_0) *be any toric Hermitian ALF gravitational instanton.*

Theorem B (BGL '24)**.** *Let* (M, g_0) *be any toric Hermitian ALF gravitational instanton. Then any Ricci-flat metric* g *on* M *which is sufficiently* C^3_1 *close to* g_0

Theorem B (BGL '24). *Let* (M, g_0) *be any toric Hermitian ALF gravitational instanton. Then any Ricci-flat metric* g *on* M *which is sufficiently* C^3_1 *close to* g_0 *must be another one of the gravitational instantons classified by Biquard-Gauduchon.*

Theorem B (BGL '24). *Let* (M, g_0) *be any toric Hermitian ALF gravitational instanton. Then any Ricci-flat metric* g *on* M *which is sufficiently* C^3_1 *close to* g_0 *must be another one of the gravitational instantons classified by Biquard-Gauduchon.*

This optimal result combines **Theorem A** with a result of Mingyang Li, `arXiv:2310.13197`.

Ngā mihi i tō manaaki!

Thanks for your hospitality!

It's a real pleasure to be here!

It's a real pleasure to be here!