Einstein Metrics

and

Global Conformal Geometry

I

Claude LeBrun
SUNY Stony Brook
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Now choosing \(T_p M \cong \mathbb{R}^n\) via some orthonormal basis gives us special coordinates on \(M\).
In these “geodesic normal” coordinates the metric volume measure is given by

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\[ d\mu_g = \left[ 1 - \frac{1}{6} r^{jk} x^j x^k + O(\left| x \right|^3) \right] d\mu_{\text{Euclidean}}, \]
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Why?

\[ g_{jk} = \delta_{jk} - \frac{1}{3} \mathcal{R}_{j\ell km} x^\ell x^m + O(|x|^3) \]

in these coordinates.
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(Use Jacobi’s equation for geodesic deviation.)
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\[ d\mu_g = \sqrt{\det[g_{jk}]} \, dx^1 \wedge \cdots \wedge dx^n \]
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The Ricci curvature
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The \textit{Ricci curvature} is by definition the function on the unit tangent bundle.
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\[ v \mapsto r(v, v). \]
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature.
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$$r = \lambda g$$

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Generalizes constant sectional curvature condition, but weaker.
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same number of equations as unknowns.
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- $\mathcal{R}_{j klm}$: $\frac{n^2(n^2-1)}{12}$ components.
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Elliptic non-linear PDE after gauge fixing.

$$\Delta x^j = 0 \implies r_{jk} = \frac{1}{2} \Delta g_{jk} + \text{lots}.$$
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**Proposition.** If \( n \geq 3 \), a Riemannian \( n \)-manifold \((M^n, g)\) is Einstein iff the trace-free part of its Ricci tensor vanishes:

\[ \hat{r} := r - \frac{s}{n} g = 0. \]
Here $s$ denotes the *scalar curvature*

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Meaning?
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Meaning? Metric distance balls

$$B_{\varepsilon}(p) = \{ q \in M \mid \exists \text{ path from } p \text{ to } q \text{ of length } < \varepsilon \}$$
Here $s$ denotes the *scalar curvature*

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**Meaning?** Metric distance balls

$$B_\varepsilon(p) = \{q \in M \mid \exists \text{ path from } p \text{ to } q \text{ of length } < \varepsilon\}$$

have metric volume
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$$\text{vol}_g(B_\varepsilon(p))$$
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$$\frac{\text{vol}_g(B_\varepsilon(p))}{c_n \varepsilon^n}.$$
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$$ \frac{\text{vol}_g(B_\varepsilon(p))}{c_n \varepsilon^n} = 1 - s \frac{\varepsilon^2}{6(n+2)} + $$
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\[ \frac{\text{vol}_g(B_\varepsilon(p))}{c_n\varepsilon^n} = 1 - s \frac{\varepsilon^2}{6(n+2)} + O(\varepsilon^4) \]
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$$\frac{\text{vol}_g(B_{\varepsilon}(p))}{c_n \varepsilon^n} = 1 - s \frac{\varepsilon^2}{6(n+2)} + O(\varepsilon^4)$$

where $c_n = \pi^{n/2}/(n/2)!$
Question (Yamabe). Does every smooth compact 1-connected $n$-manifold admit an Einstein metric?
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What we know:
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**What we know:**

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On a 3-manifold,

$$\frac{s}{2} - r(v,v) = K(v^\perp)$$

for any unit vector $v$, so Einstein $\Rightarrow$ constant sectional curvature $\lambda/2$. 
**Question** (Yamabe). *Does every smooth compact 1-connected $n$-manifold admit an Einstein metric?*

What we know:

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- When $n = 3$: $\Longleftrightarrow$ Poincaré conjecture. Hamilton, Perelman, ... Yes!
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- When $n = 5$: Yes?? (Boyer-Galicki-Kollár)
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- When $n \geq 6$, wide open. Maybe???
Variational Approach

If $M$ smooth compact $n$-manifold, $n \geq 3$,

$\mathcal{G}_M = \{ \text{smooth metrics } g \text{ on } M \}$
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then Einstein metrics = critical points of

*Einstein-Hilbert action* functional
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$$\mathcal{S} : \mathcal{G}_M \longrightarrow \mathbb{R}$$

$$g \longmapsto \int_M s_g d\mu_g$$
Variational Approach

If $M$ smooth compact $n$-manifold, $n \geq 3$,

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then Einstein metrics = critical points of normalized \textit{Einstein-Hilbert action} functional

$$S : \mathcal{G}_M \longrightarrow \mathbb{R}$$

$$g \longmapsto V^{(2-n)/n} \int_M s_g d\mu_g$$
Variational Approach

If $M$ smooth compact $n$-manifold, $n \geq 3$,

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$$\mathcal{S} : \mathcal{G}_M \longrightarrow \mathbb{R}$$

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where $V = \text{Vol}(M, g)$ inserted to make scale-invariant.
Basic difficulty:
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\[ S(g) = V^{(2-n)/n} \int_M s_g d\mu_g \]

not bounded above or below.
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Yamabe:
Consider any conformal class

\[ \gamma = [g_0] = \{ f g_0 \mid u : M \to \mathbb{R}^+ \}, \]
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Consider any conformal class

\[ \gamma = [g_0] = \{ fg_0 \mid u : M \to \mathbb{R}^+ \} , \]

Then restriction \( S|\gamma \) is bounded below.
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Set $p = \frac{2n}{n-2}$. 
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Conformal rescaling:

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Conformal rescaling:

$\hat{g} = u^{p-2}g$ then has $d\mu = u^p d\mu$

and its scalar curvature satisfies

$$\hat{s}u^{p-1} = [(p + 2)\Delta + s]u$$
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Set \( p = \frac{2n}{n-2} \).

Conformal rescaling:
\( \hat{g} = u^{p-2} g \) then has \( \hat{d}\mu = u^p d\mu \)
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where \( \Delta = -\nabla \cdot \nabla \).
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\[ \hat{s} u^{p-1} = [(p + 2) \Delta + s] u \]

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S(\hat{g}) = \frac{\int_M \left( s u^2 + (p + 2) |\nabla u|^2 \right) d\mu}{\left[ \int_M u^p d\mu \right]^{2/p}}
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Difficulty: \( L_1^2 \hookrightarrow L^p \) bounded, but not compact.
Yamabe (1950s)
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Trudinger (1960s)
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Aubin (1970s)
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Has \( s = \text{ constant} \).
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∃ metric $g \in \gamma$ which minimizes $S|_{\gamma}$.
Has $s = \text{constant}$.
Unique up to scale when $s \leq 0$. 
\[ Y_\gamma = \inf_{g \in \gamma} \frac{\int_M s_g \, d\mu_g}{(\int_M d\mu_g)^{\frac{n-2}{n}}}; \]
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If \( g \) has \( s \) of fixed sign, agrees with sign of \( Y_{[g]} \).
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\[ Y_\gamma \leq S(S^n, g_{\text{round}}) \]
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Aubin:
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Schoen:
\[ = \text{ only for round sphere.} \]
Yamabe’s Dream
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Yamabe’s Dream

Too good to be true!
Yamabe’s Dream

$g \in \gamma$

Too good to be true! But ...
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$$\mathcal{Y}(M) = \sup_{\gamma} Y_{\gamma}$$
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H. Yamabe, O. Kobayashi, R. Schoen.
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H. Yamabe, O. Kobayashi, R. Schoen.

$Y(M) > 0 \iff M$ admits $g$ with $s > 0$. 
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H. Yamabe, O. Kobayashi, R. Schoen.

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\mathcal{Y}(M) > 0 \iff M \text{ admits } g \text{ with } s > 0.
$$

**Problem.** Compute actual value of $\mathcal{Y}(M)$ for concrete, interesting manifolds.
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**Definition.** An Einstein metric $g$ on a smooth compact manifold $M$ will be called a *supreme* Einstein metric if

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**Example**  The round metric on $S^n$ is a supreme Einstein metric.
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**Problem.** Which manifolds admit supreme Einstein metrics?

**Problem.** Think of your favorite examples of Einstein metrics. Are any of them supreme?
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$S^3/\Gamma$ open, except when $\Gamma = \mathbb{Z}_2$. 
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In particular, complex-hyperbolic metric on $\mathbb{CH}_2/\Gamma$ is supreme Einstein.
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Open question for hyperbolic 4-manifolds $\mathcal{H}^4/\Gamma$!
Theorem (Petean). Let $M^n$ be a simply connected $n$-manifold, $n \geq 5$. Then $\mathcal{Y}(M) \geq 0$. 
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Inspiration:

**Theorem (Gromov/Lawson).** Let $M^n$ be a simply connected $n$-manifold, $n \geq 5$. If $M$ is not spin, then $M$ carries a metric $g$ with $s > 0$. That is,

$$w_2(TM) \neq 0 \implies \mathcal{Y}(M) > 0.$$
Theorem. Let $M$ be a compact simply connected $n$-manifold, $n \geq 3$. If $n \neq 4$, $\mathcal{V}(M) \geq 0$. 
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Theorem. There exist infinitely many compact simply connected 4-manifolds with $\mathcal{Y}(M) < 0$. 
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There are topological 4-manifolds which admit an Einstein metric for one smooth structure, but not for others.

This is intimately tied to the fact that $\mathcal{Y}(M)$ depends strongly on the smooth structure in dimension four.