

Mass in

Kähler Geometry

Claude LeBrun

Stony Brook University

Complex Geometry and Lie Groups
Nara Women's University, 2016/03/22

Joint work with

Joint work with

Hans-Joachim Hein
University of Maryland

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e-print: [arXiv:1507.08885](https://arxiv.org/abs/1507.08885) [math.DG]

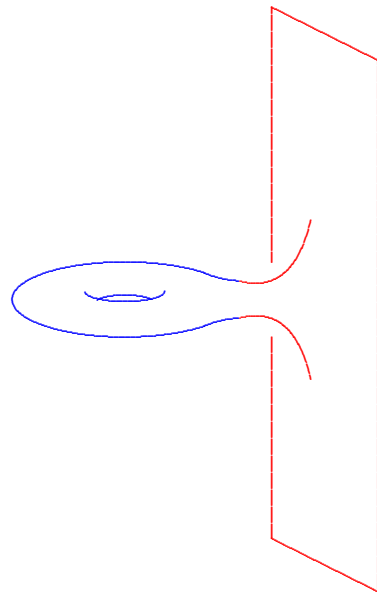
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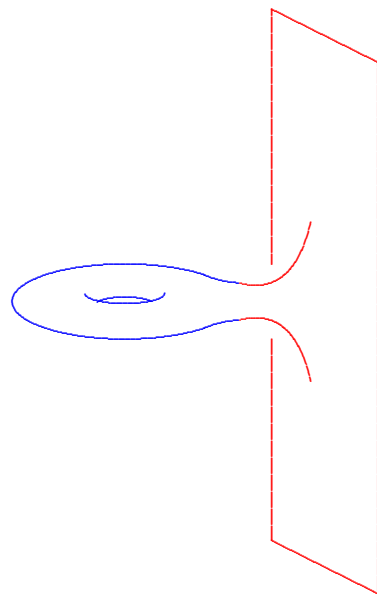
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To appear in *Comm. Math. Phys.*

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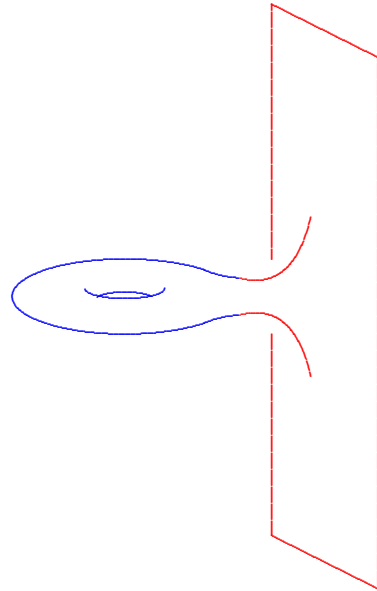


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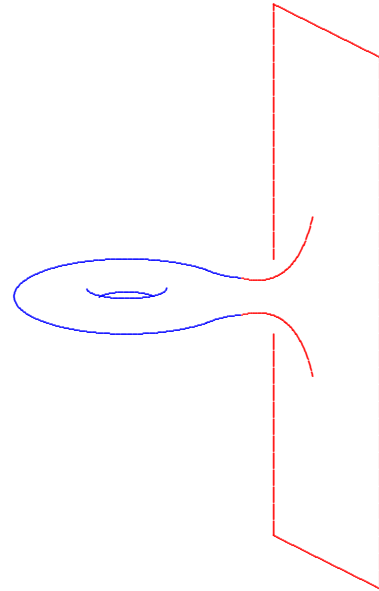
$$n \geq 3$$

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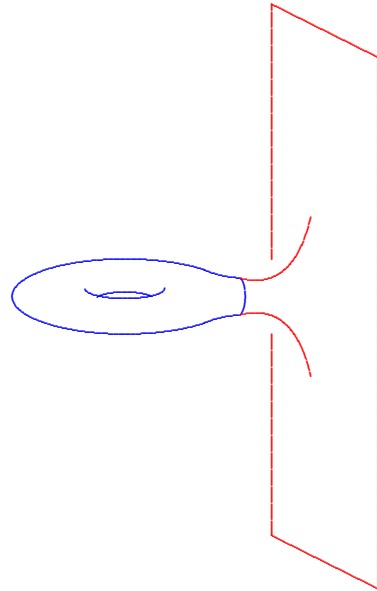
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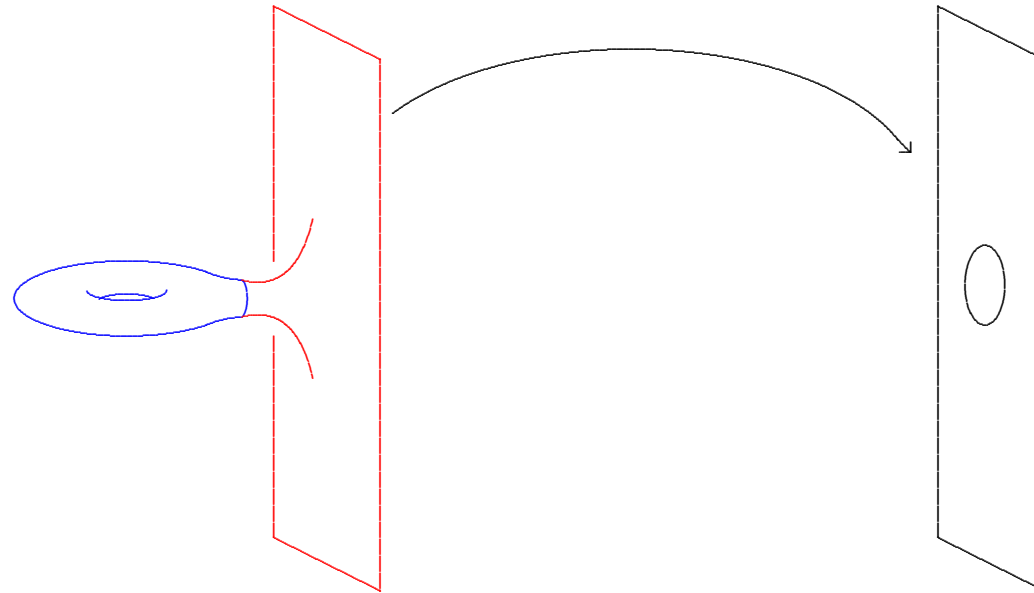


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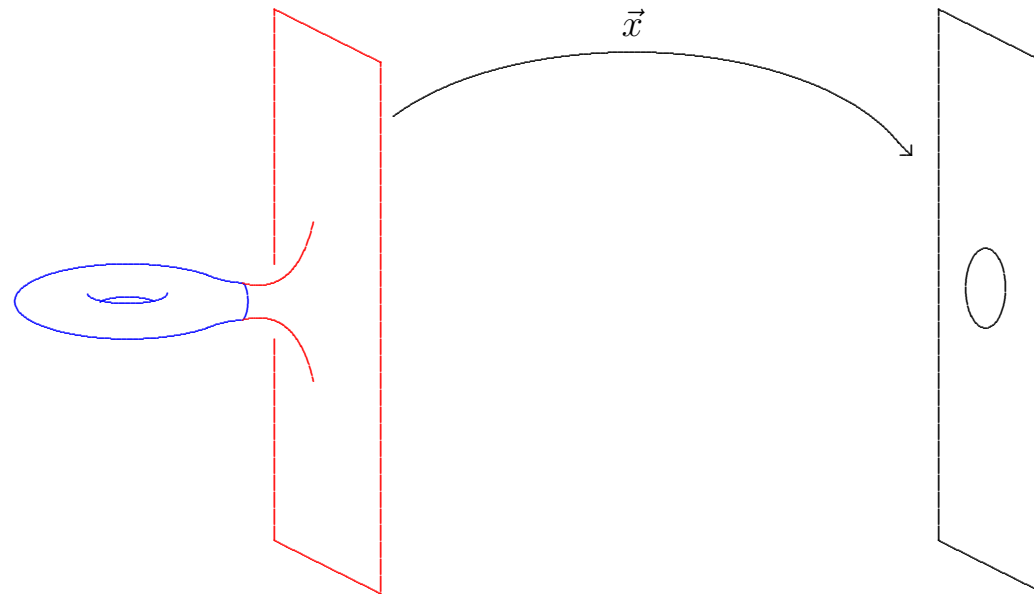
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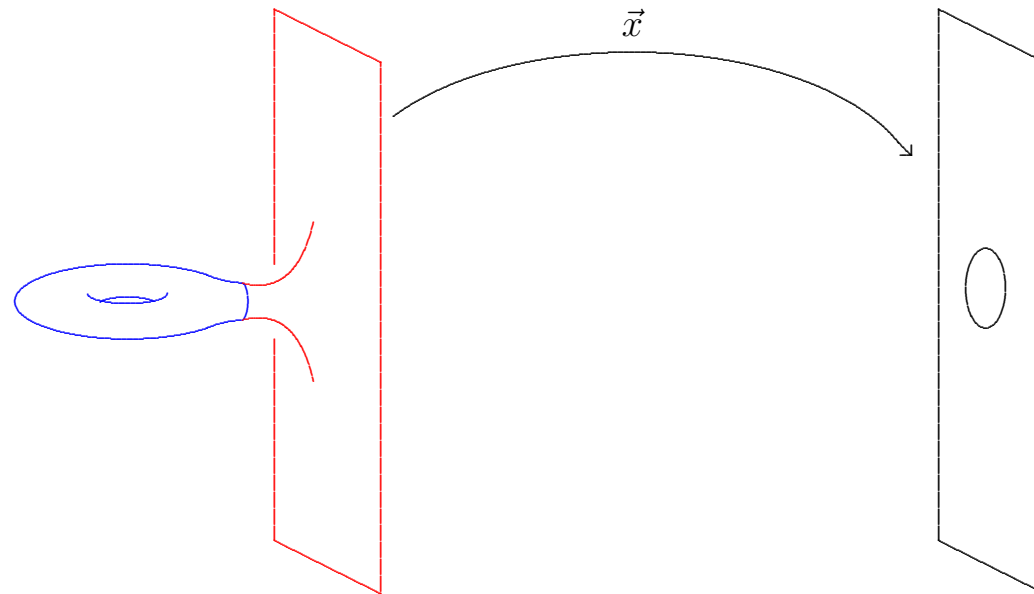


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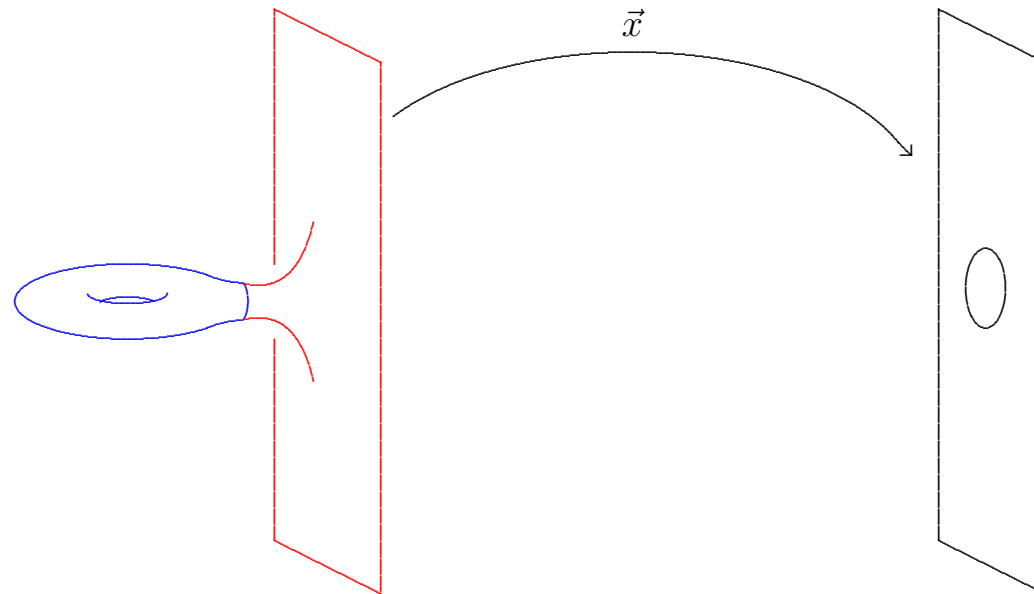
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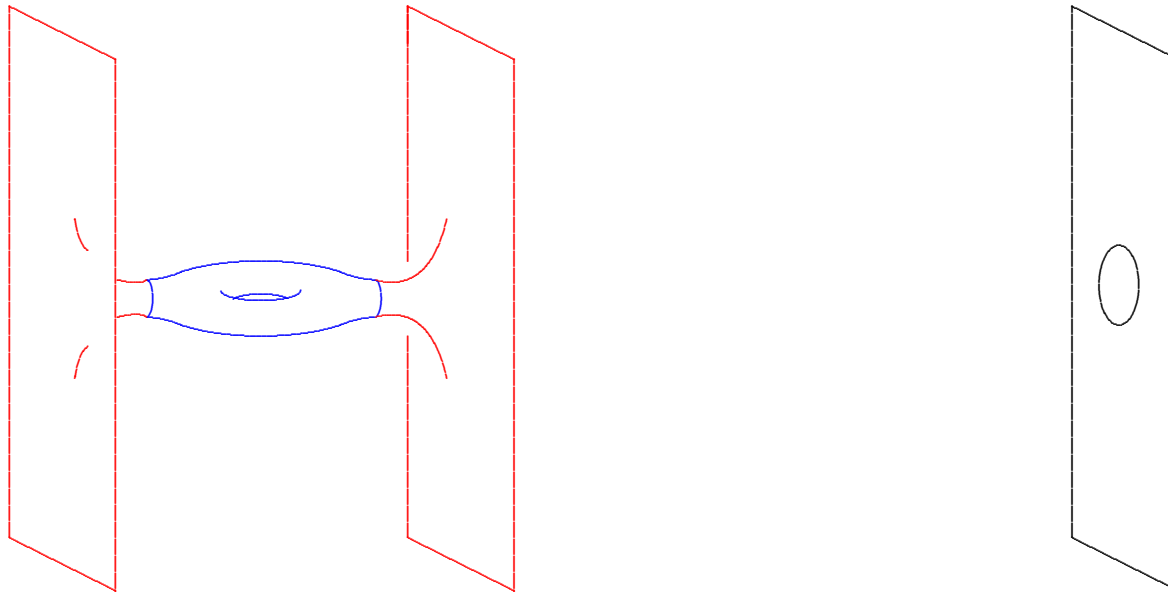
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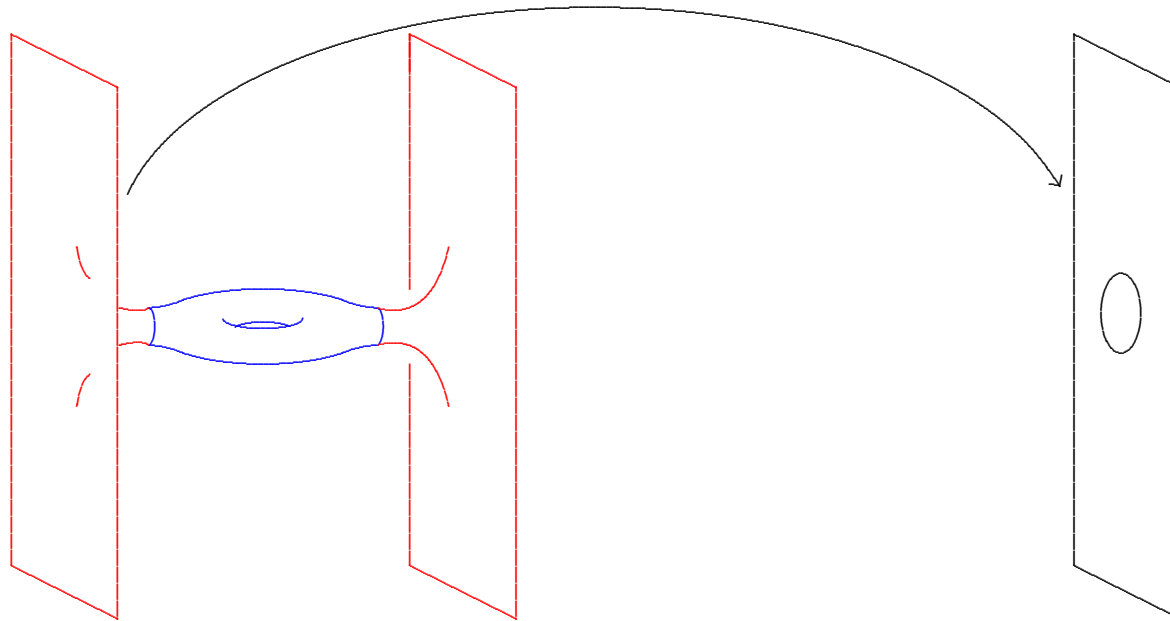
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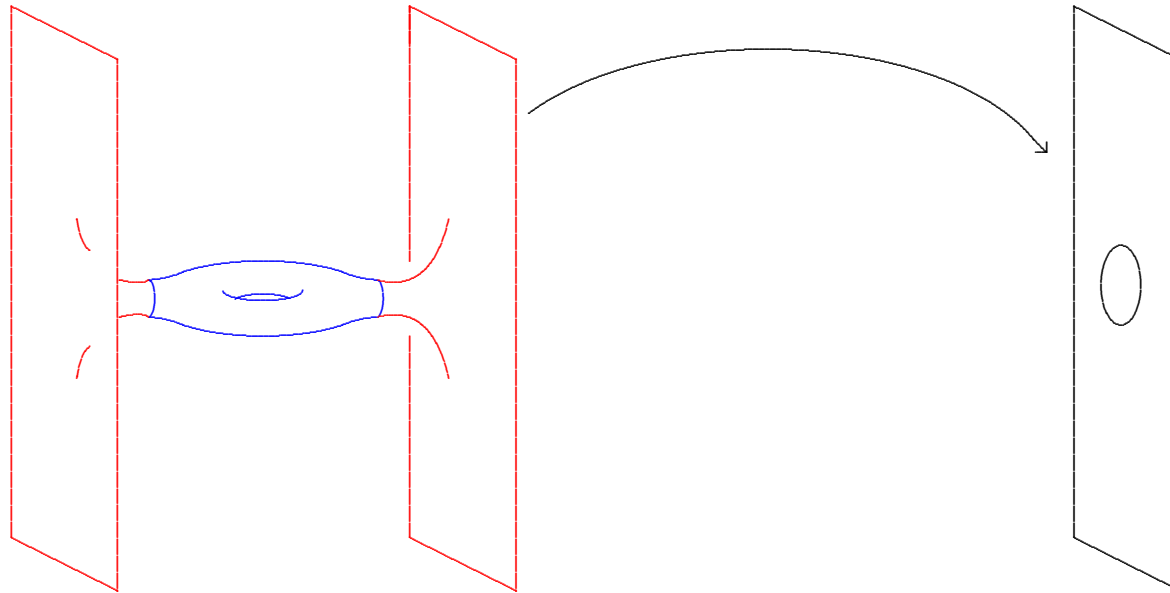
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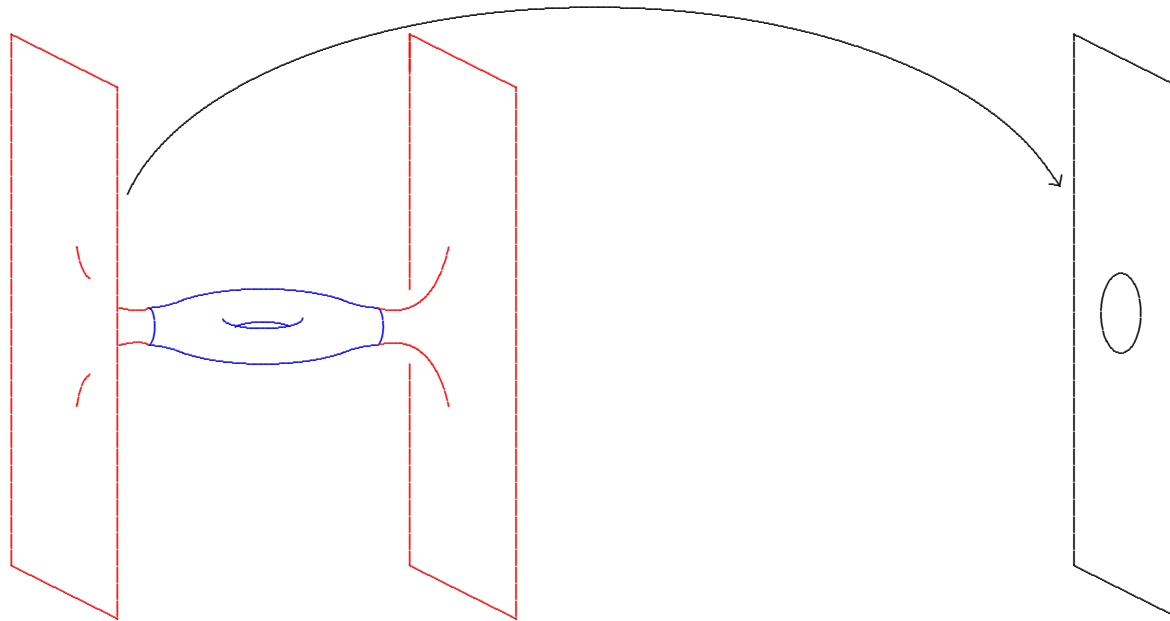
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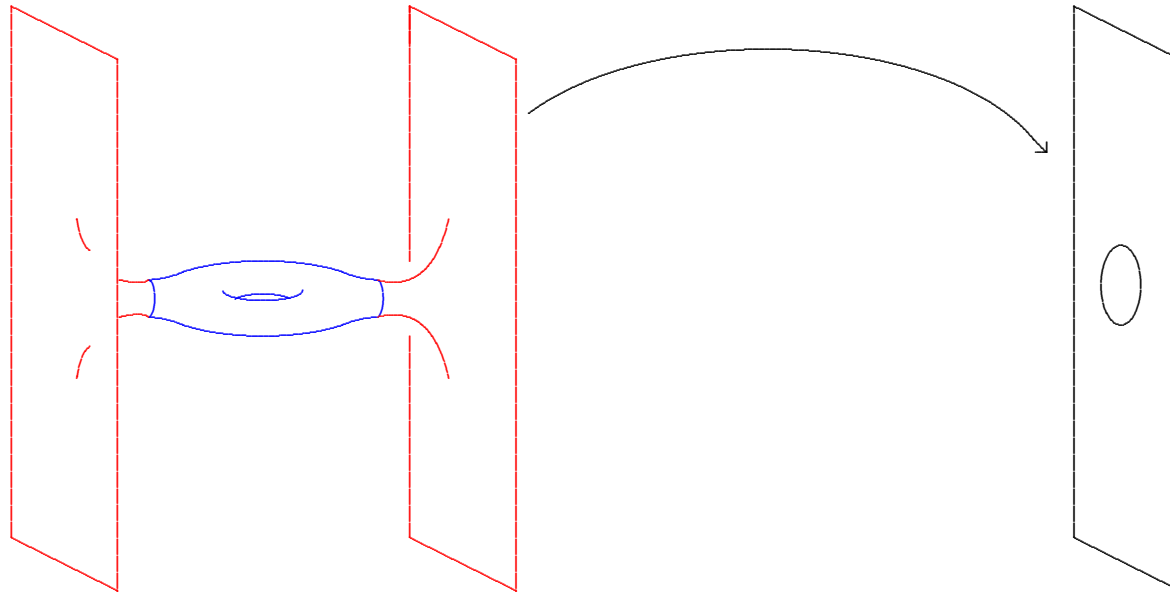
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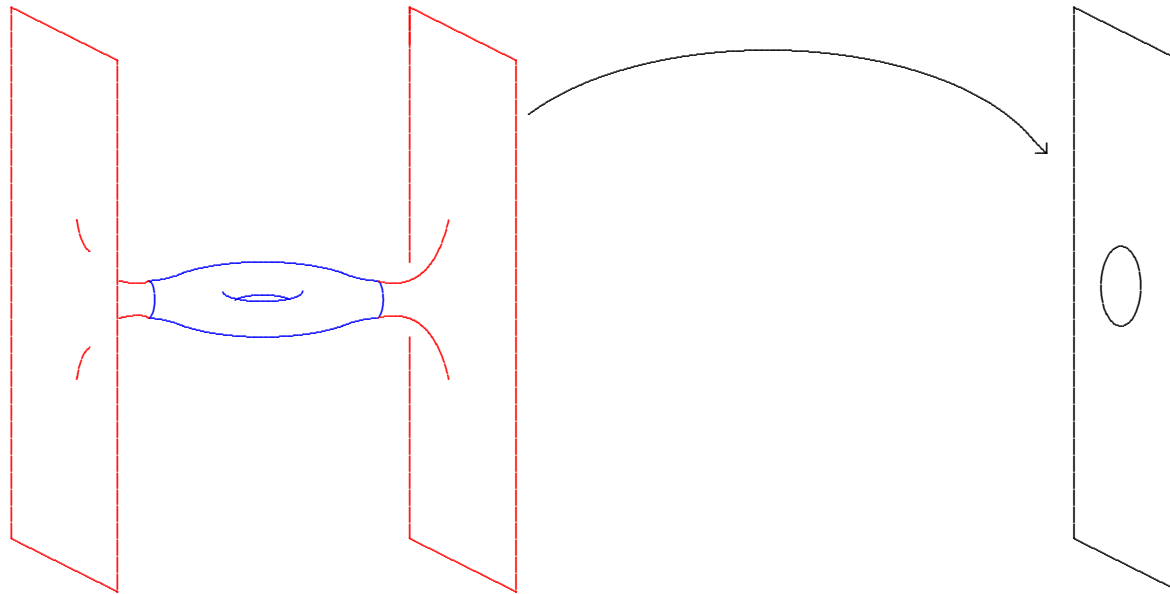
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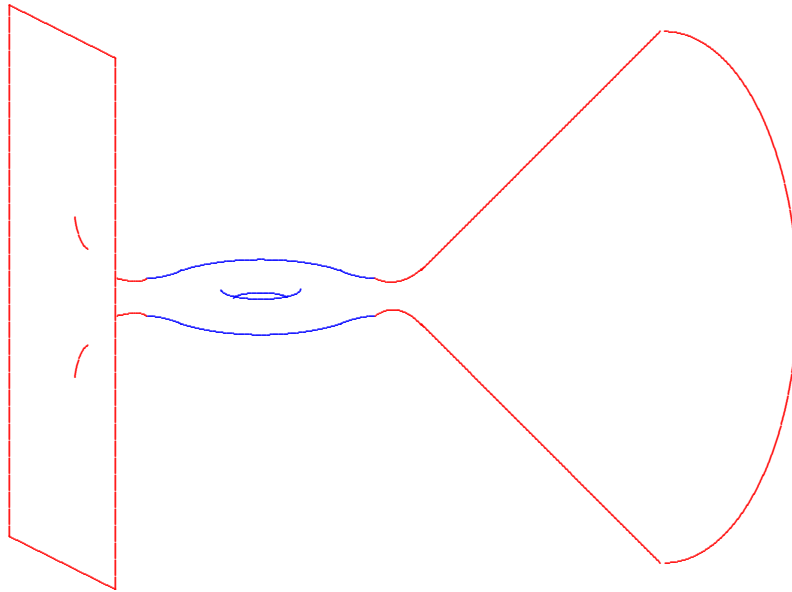
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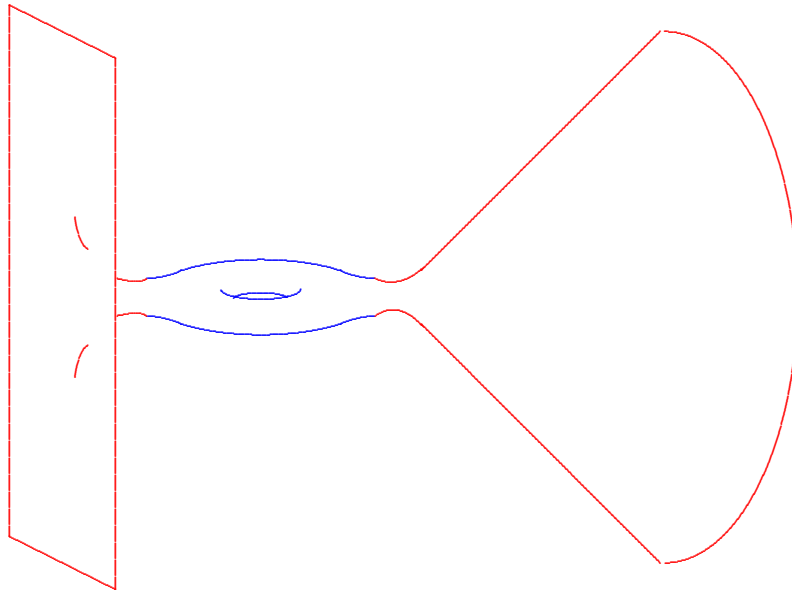
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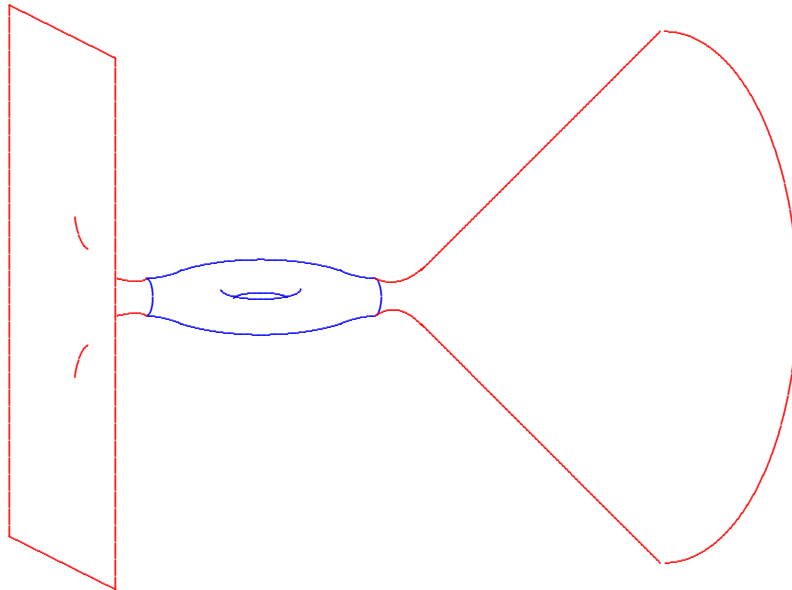
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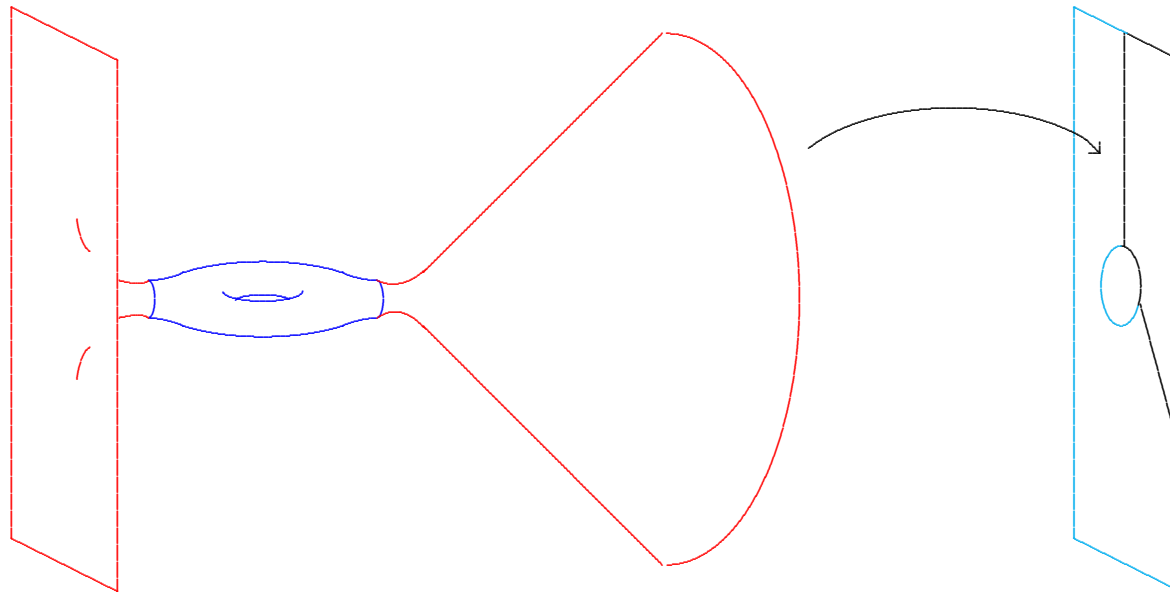
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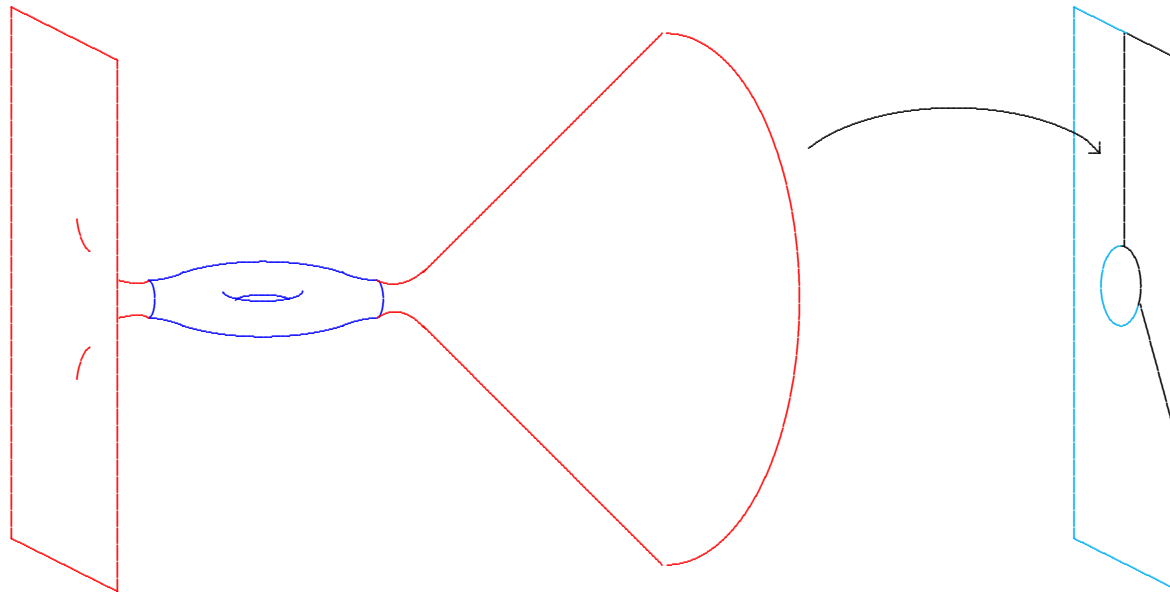
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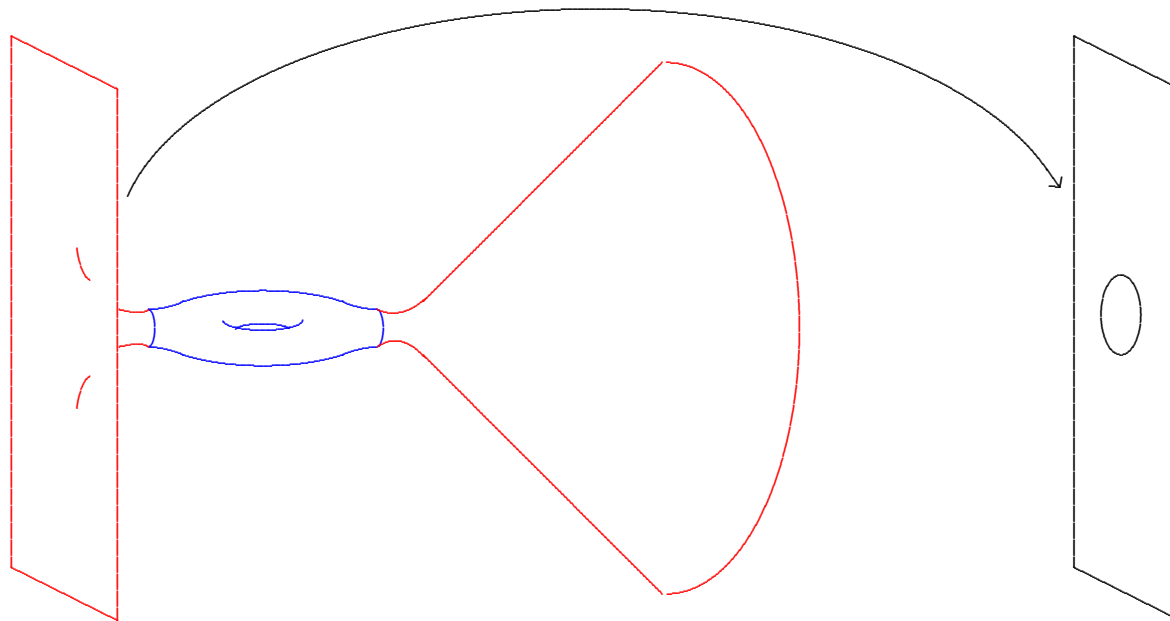
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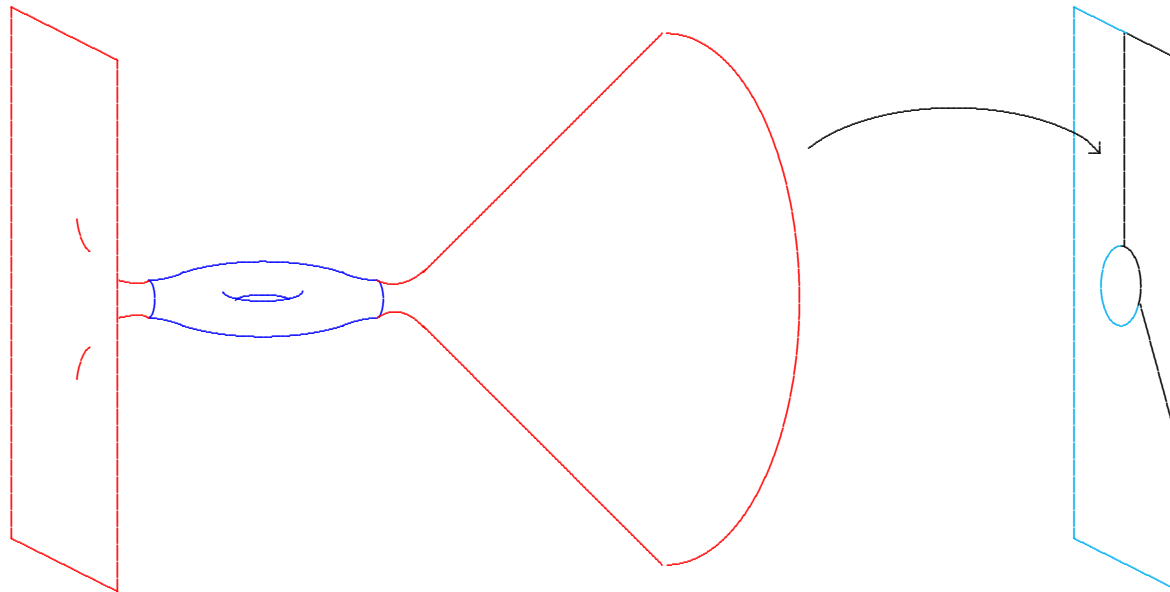
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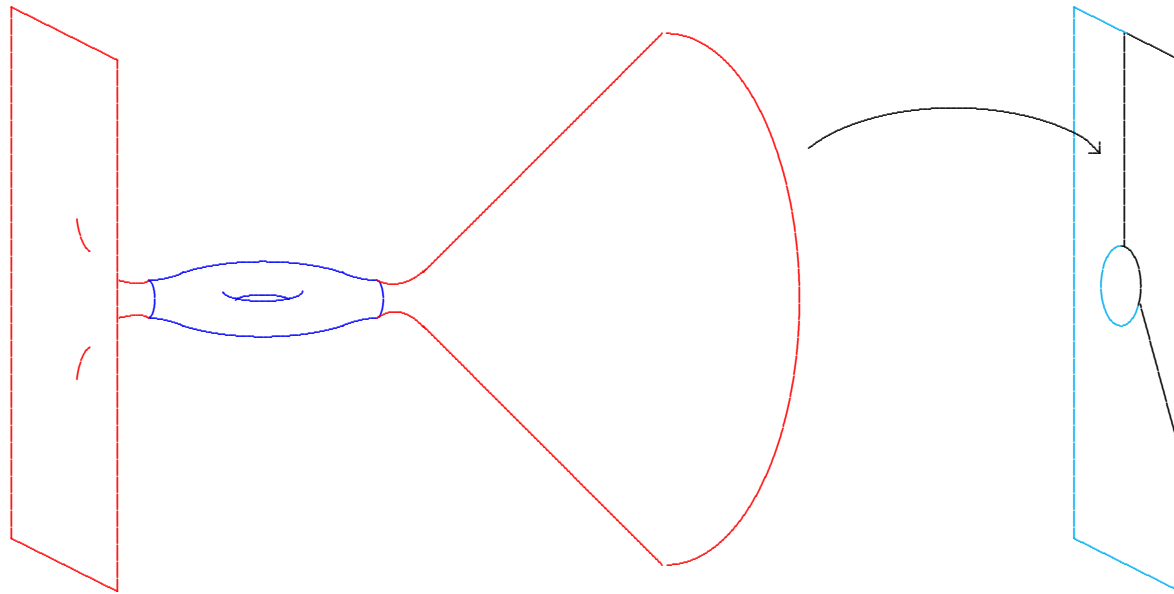
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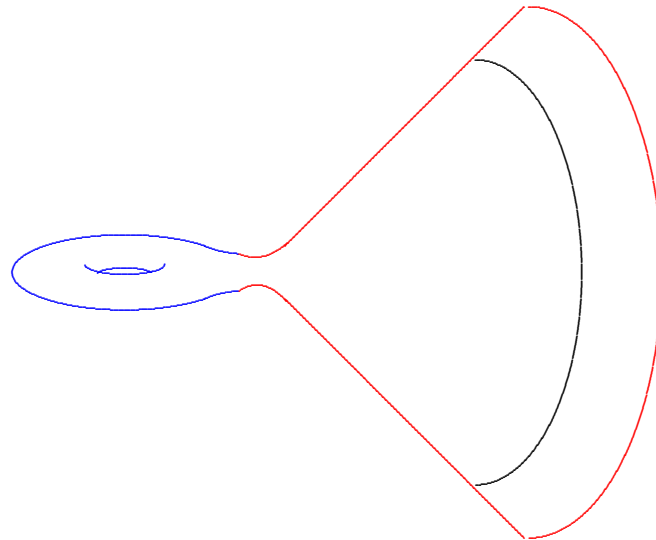
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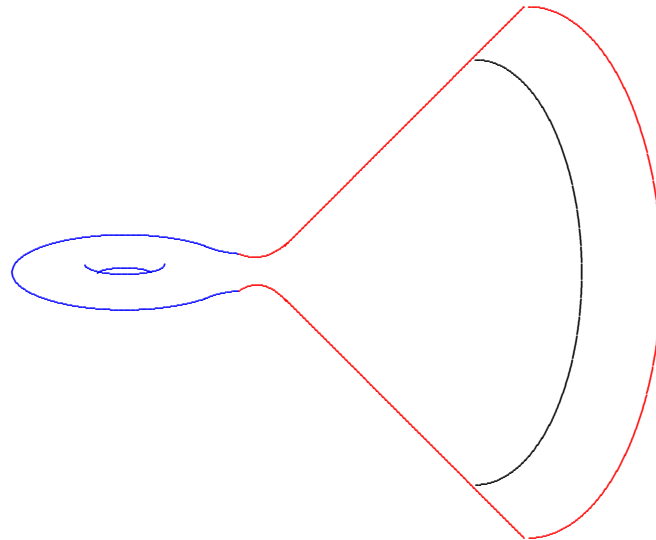


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Seems to depend on choice of coordinates!

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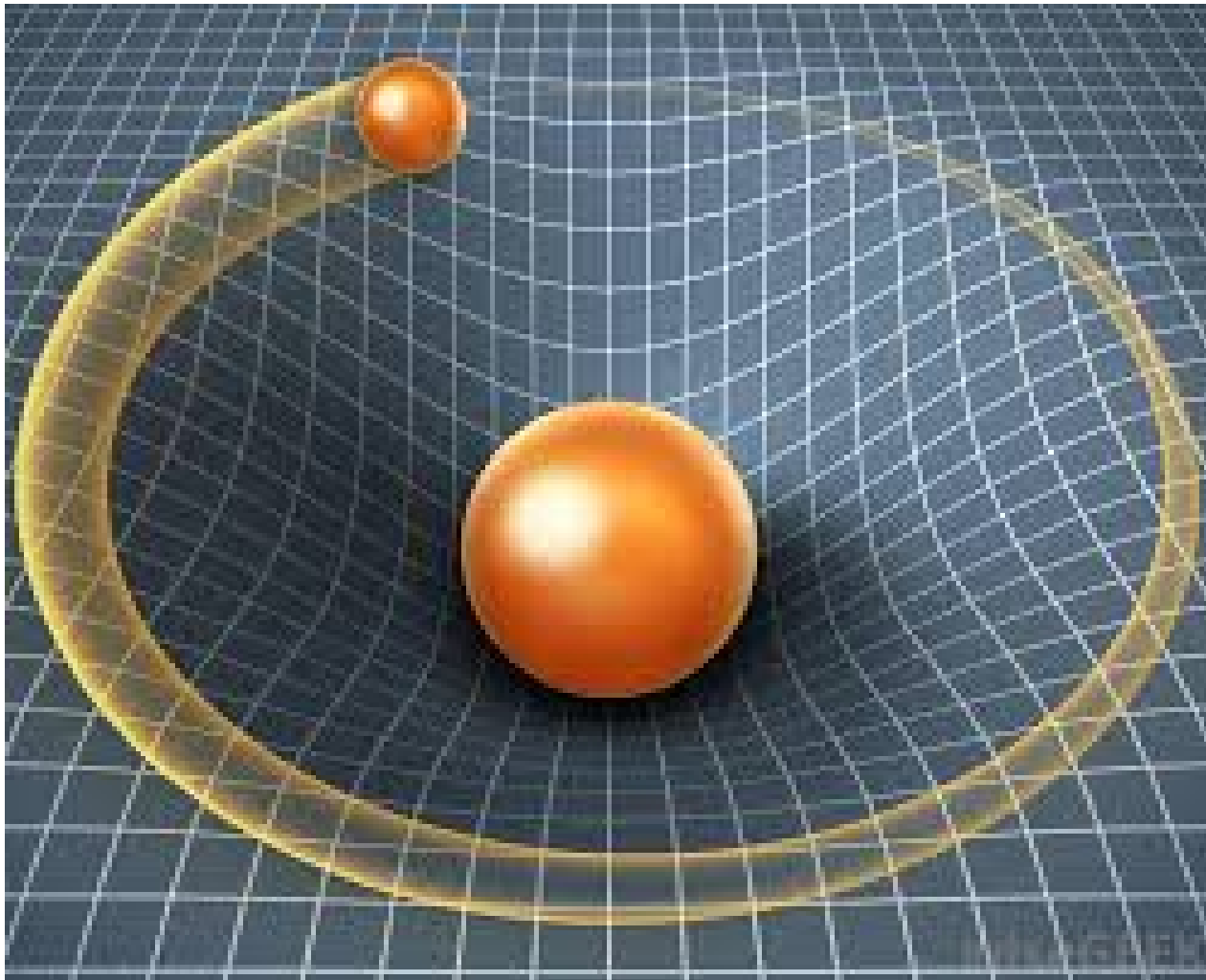
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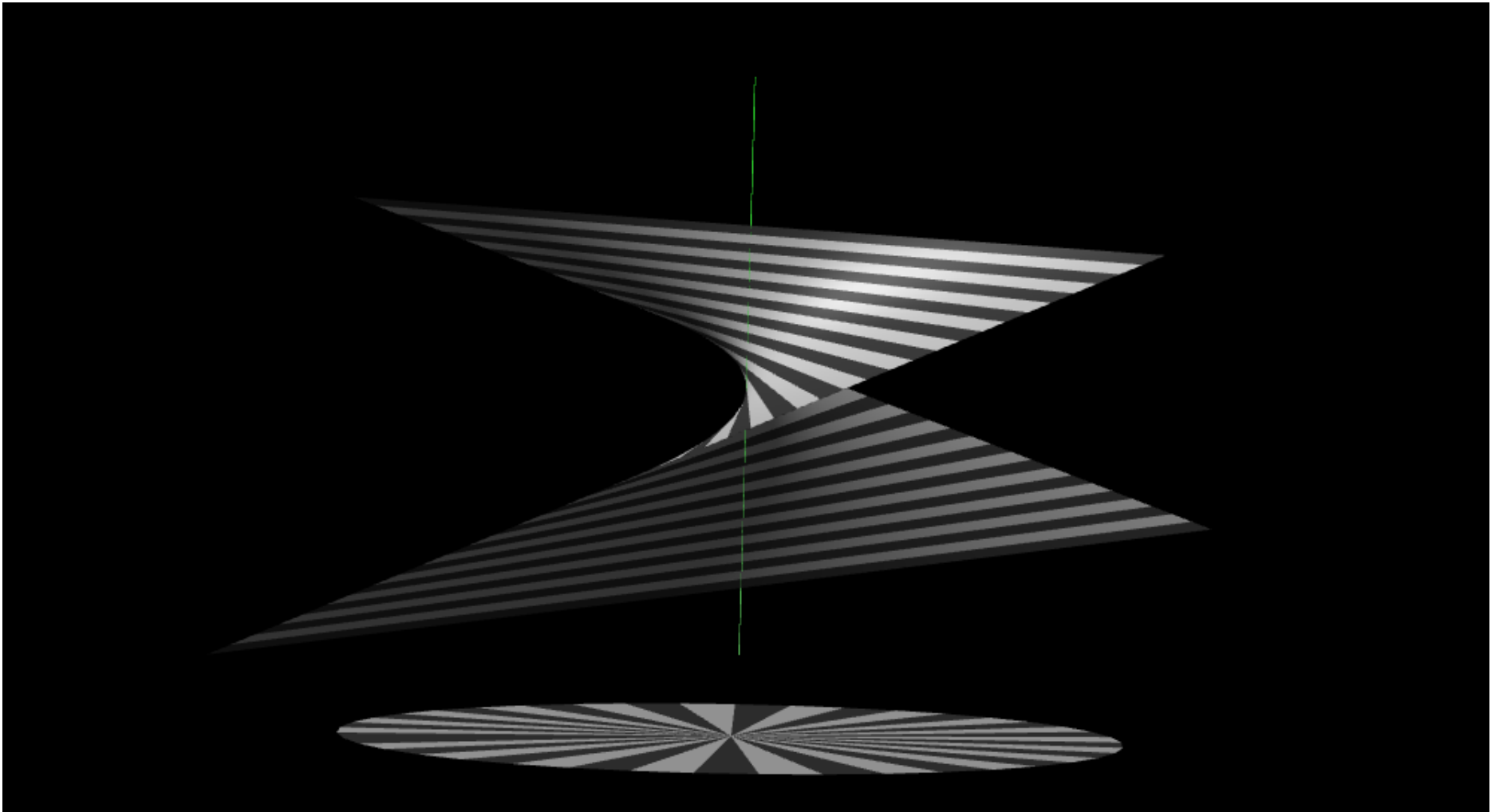
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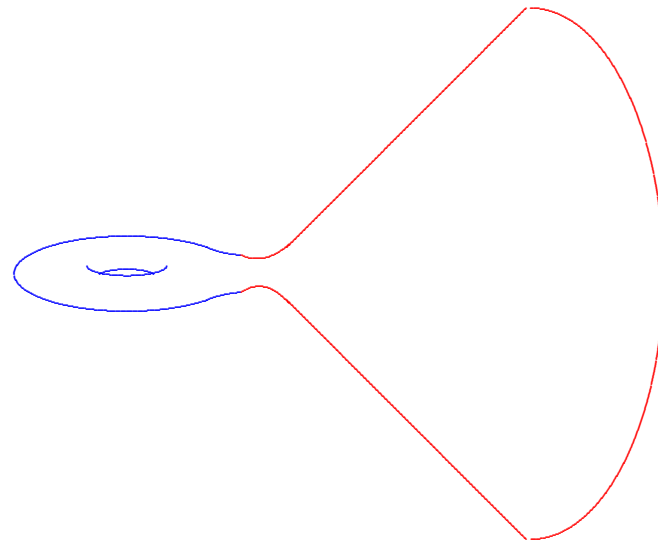
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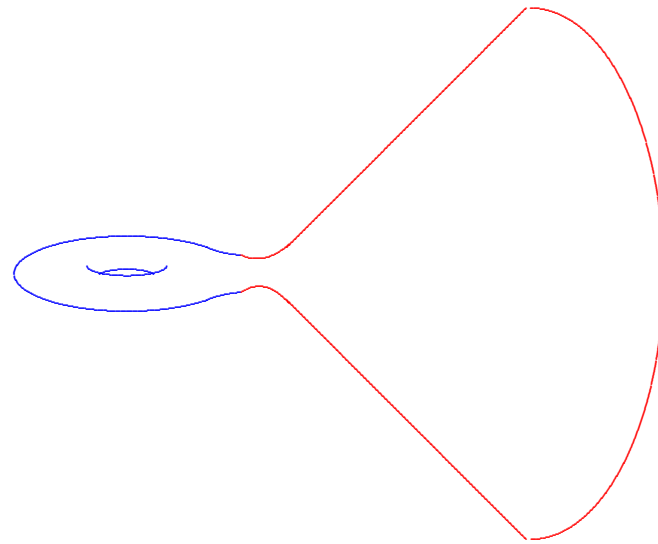
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Mass of an **ALE Kähler** manifold is unambiguous.

Does not depend on the choice of an end!

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Theorem A.

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In fact, we will see that there is an explicit formula for the mass in terms of these data!

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(Discovered independently by Rollin, Singer, & Şuvaina, using different methods.)

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For a compact Kähler manifold (M^{2m}, g, J) ,

$$\int_M s_g d\mu_g = \frac{4\pi}{(m-1)!} \langle c_1, [\omega]^{m-1} \rangle$$

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So the mass is a “boundary correction” to the topological formula for the total scalar curvature.

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Corollary. Any ALE scalar-flat Kähler manifold (M, g, J) of complex dimension m has mass given by

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Corollary. Any ALE scalar-flat Kähler manifold (M, g, J) of complex dimension m has mass given by

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So **Theorem A** is an immediate consequence!

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$$g^{jk} (g_{j\ell,k} - g_{jk,\ell}) \nu^\ell \alpha_E = -\star d \log \left(\sqrt{\det g} \right) + O(\varrho^{-3-\varepsilon}).$$

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However, since $s = 0$,

$$d(\theta \wedge \omega) = \rho \wedge \omega = \frac{s}{4} \omega^2 = 0.$$

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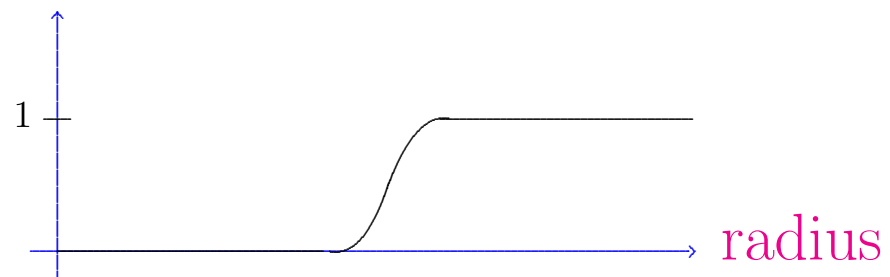
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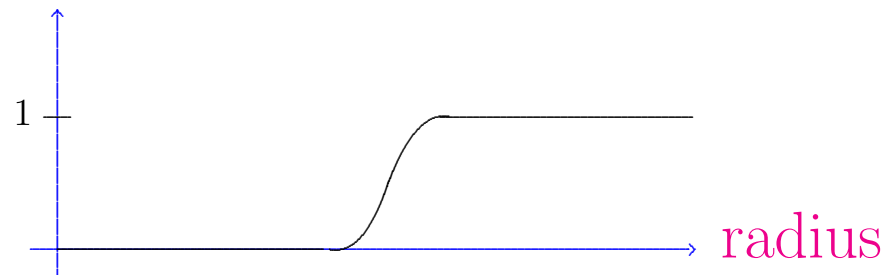
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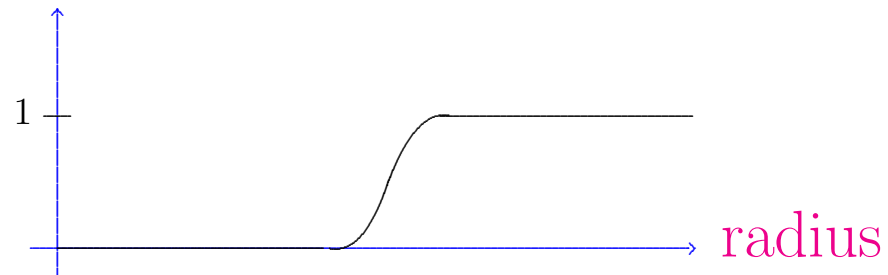
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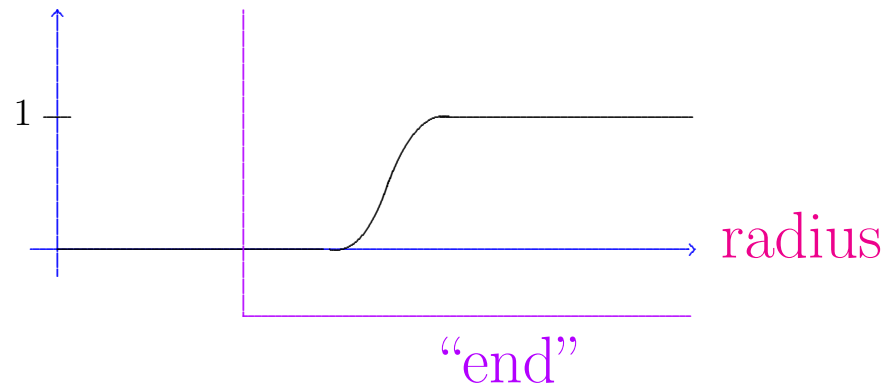
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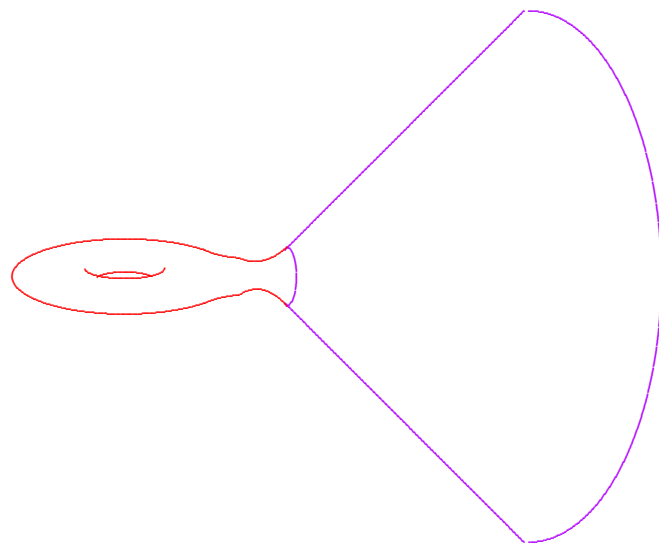
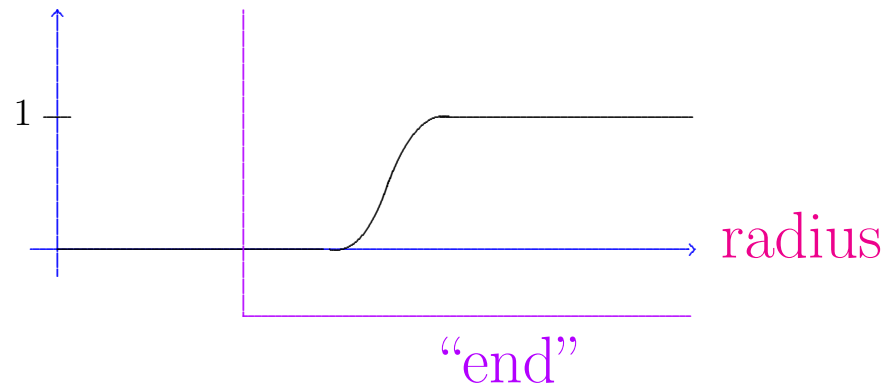
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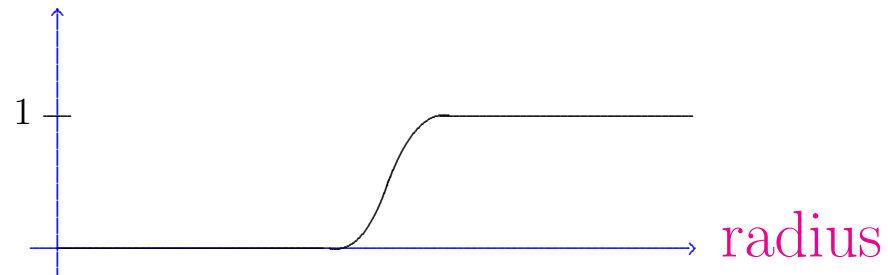
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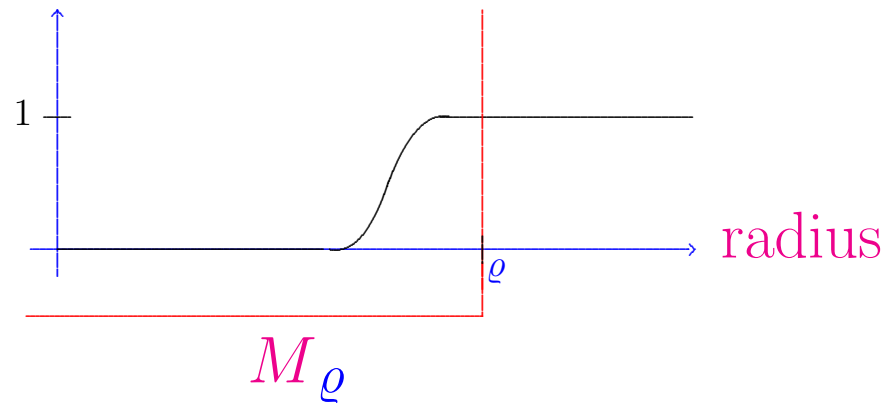
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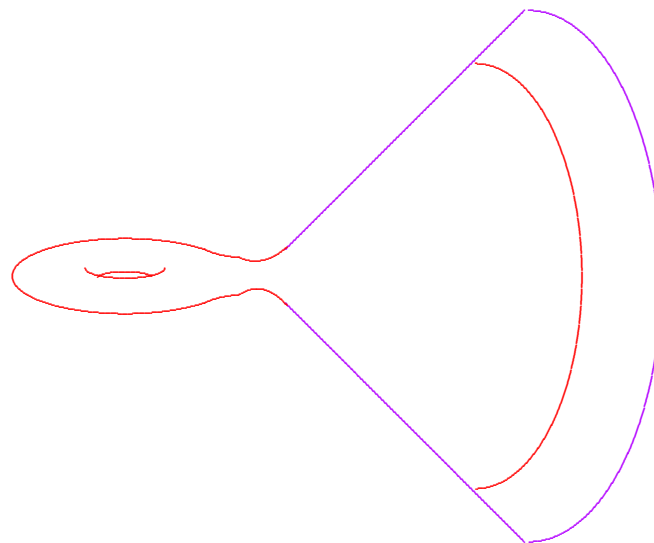
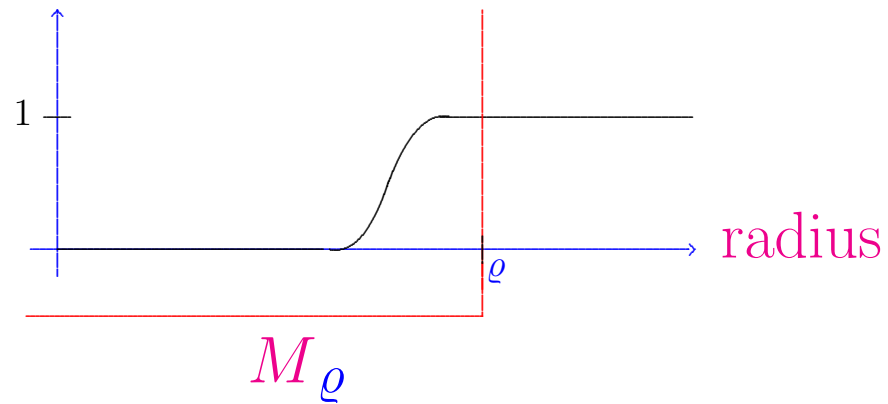
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Compactly supported, because $d\theta = \rho$ near infinity.

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because scalar-flat $\implies \rho \wedge \omega = 0$.

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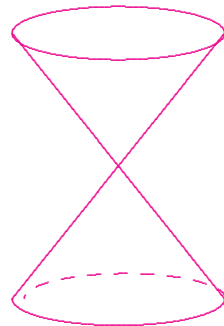
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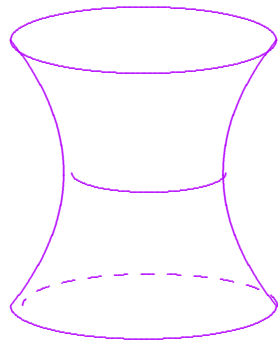
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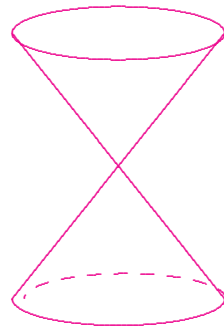
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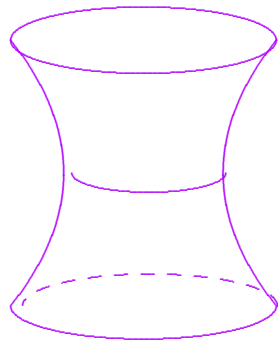
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- General $m \geq 2$: straightforward...
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$$J = J_0 + O(\varrho^{-3}), \quad \nabla J = O(\varrho^{-4})$$

in suitable asymptotic coordinates adapted to g .

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This has some interesting consequences...

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Proof actually shows something stronger!

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so the mass formula implies the claim.

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かいぎ さそ

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以上です。