Einstein 4-Manifolds,

Weyl Curvature, &

Orbifold Limits

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Curvature and Global Shape
Westfälische Wilhelms-Universität Münster
4. August 2023

Joint work with

Joint work with

Tristan Ozuch

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MIT

Gromov '81:

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pre-compactness theorem for closed Riem'n n-manifolds

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— A. Einstein, to G. Gamow

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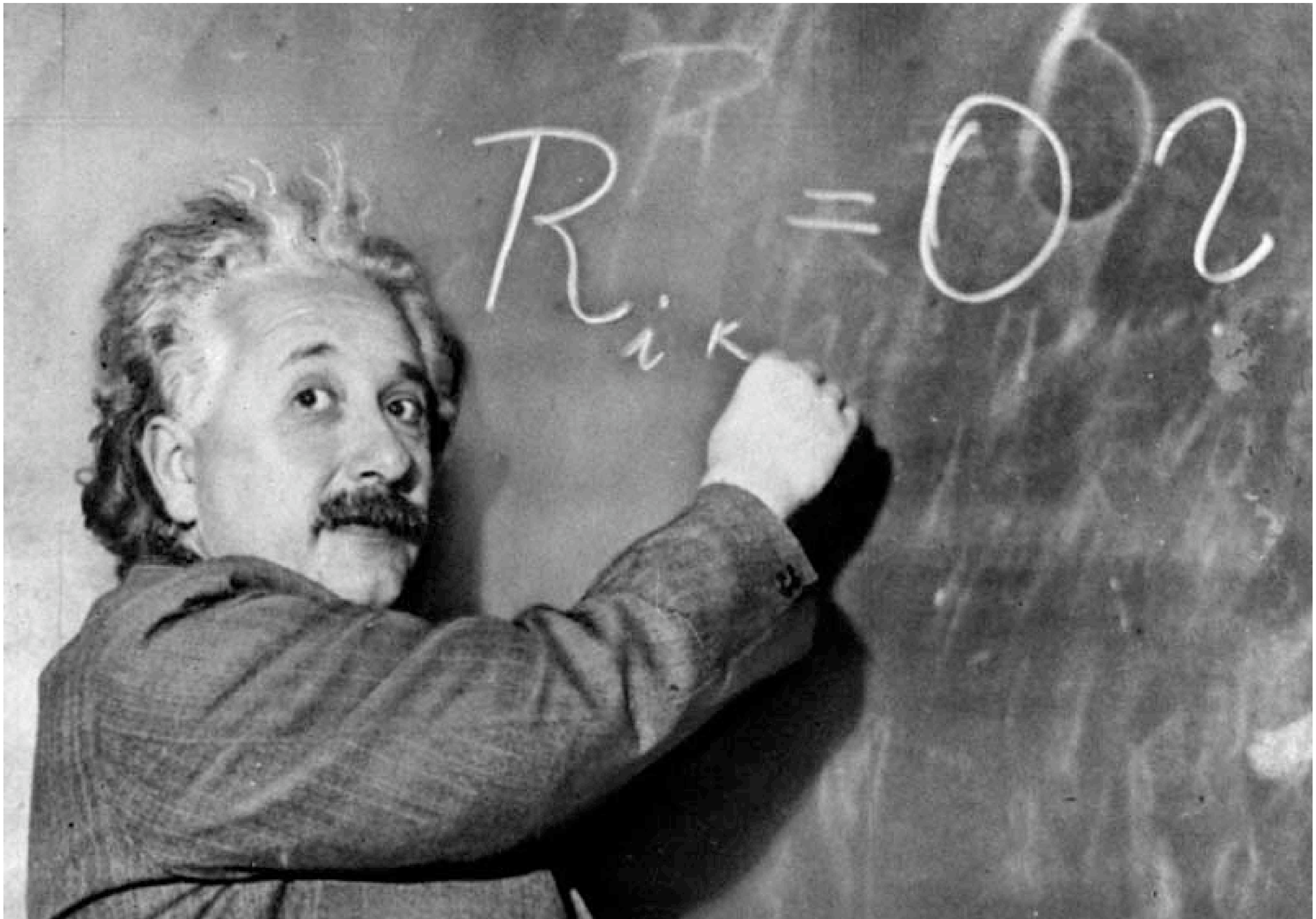

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Has same sign as the *scalar curvature*

$$s = r_j^{\ j} = \mathcal{R}^{ij}{}_{ij}.$$

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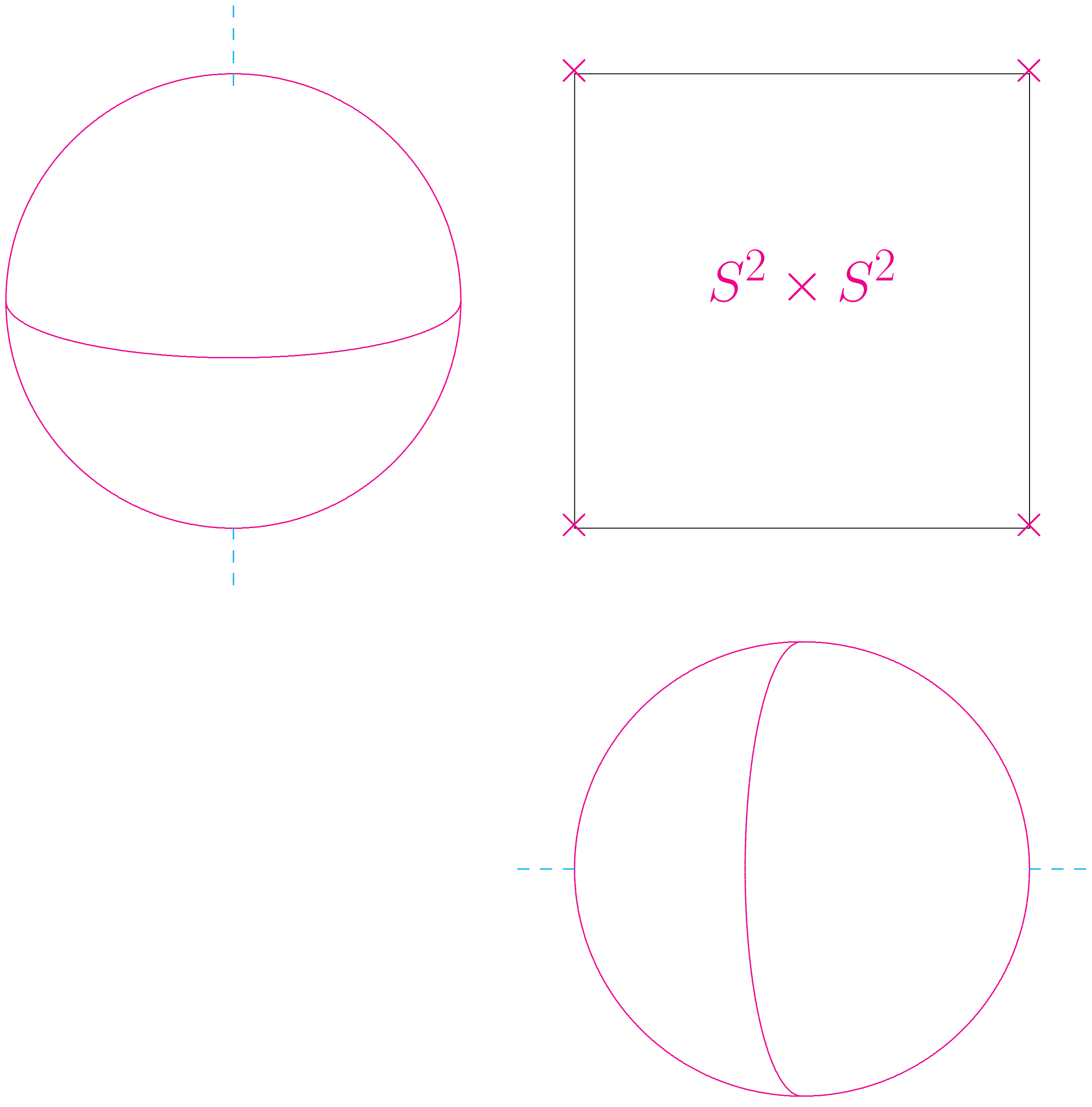

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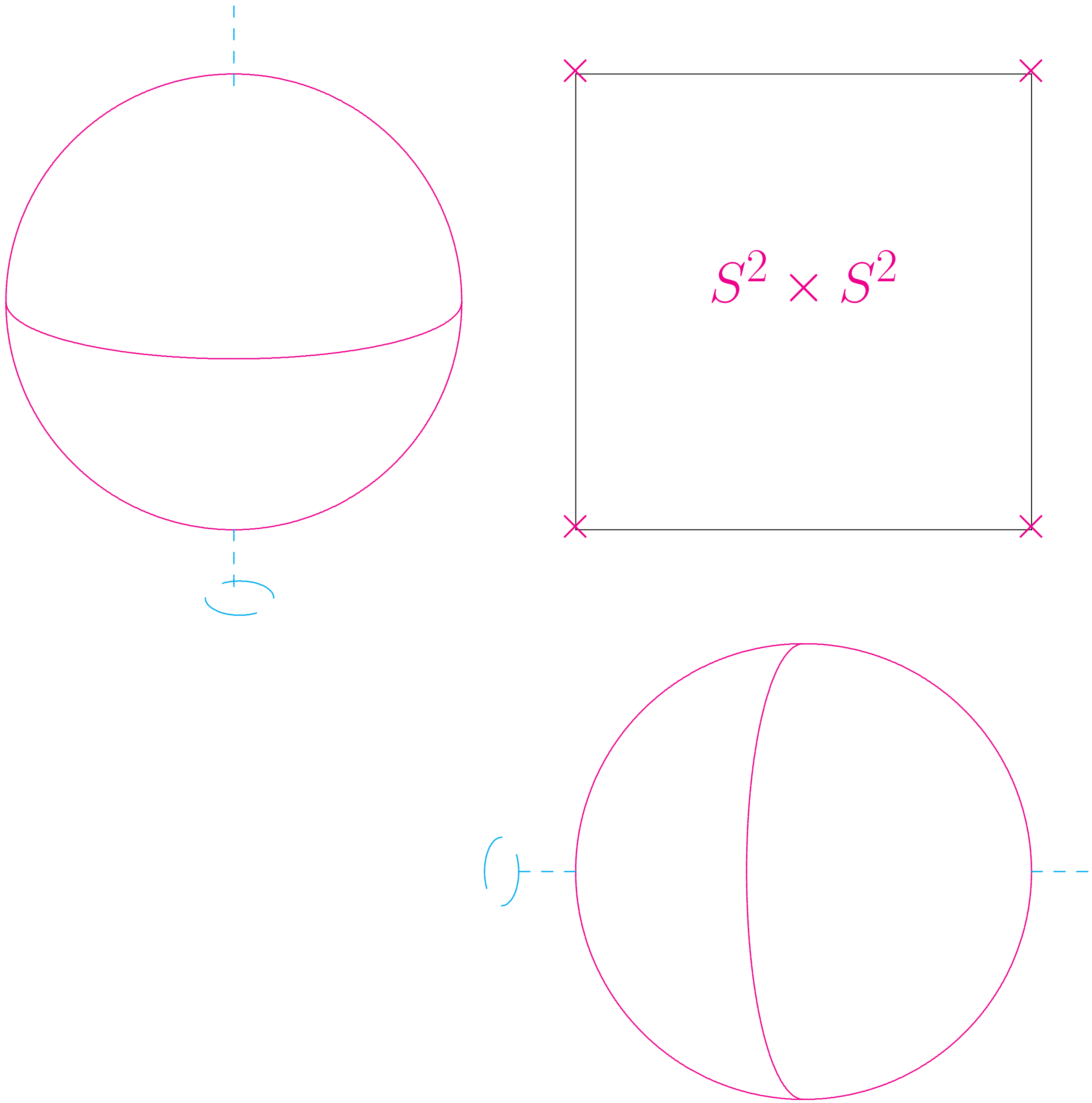

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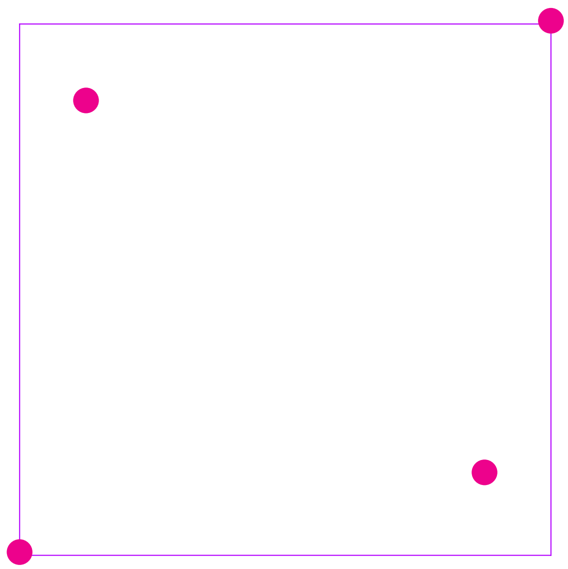

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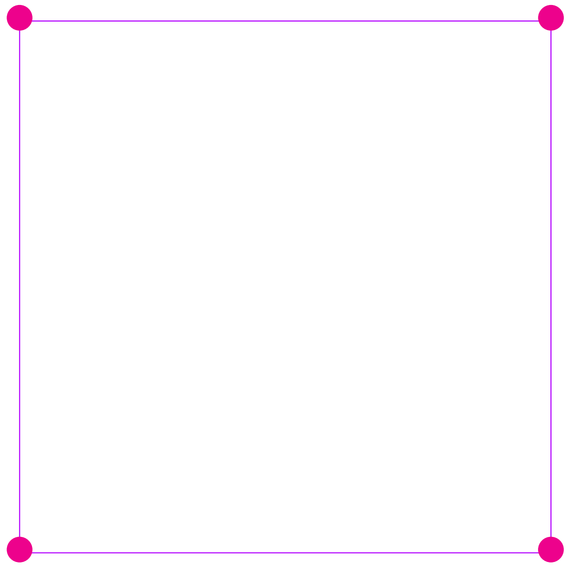

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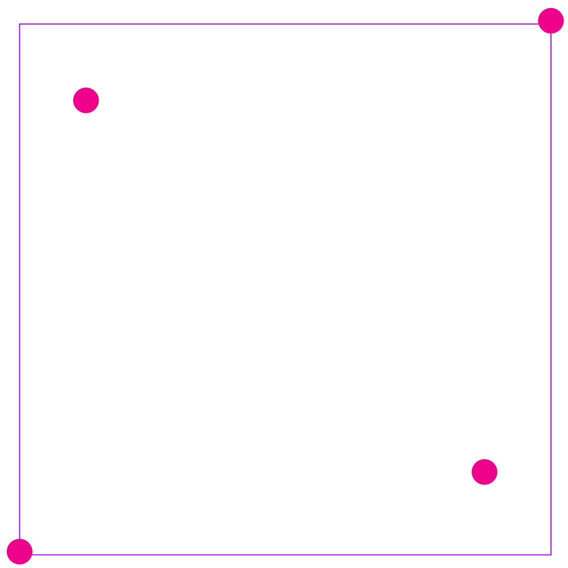

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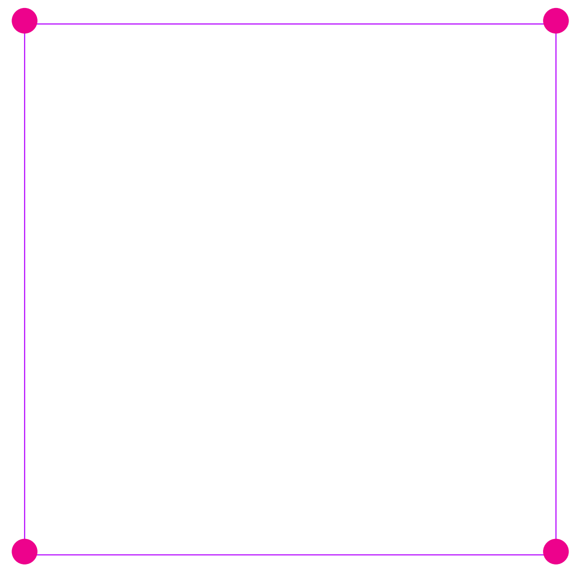

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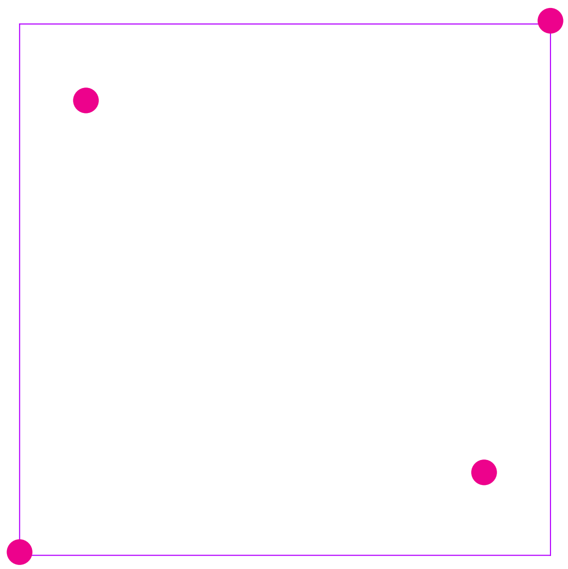

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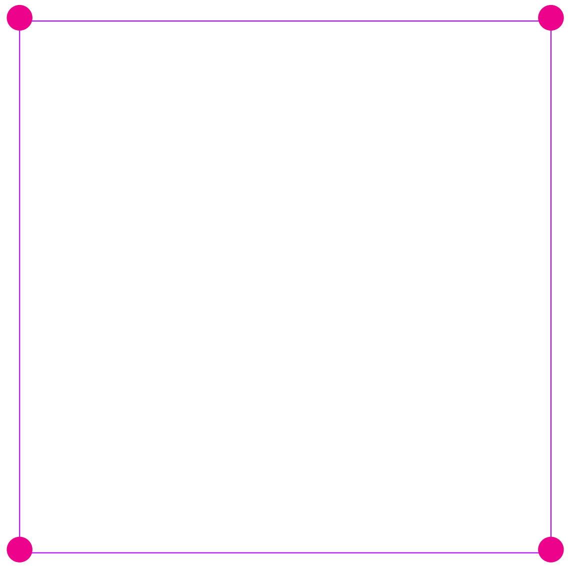

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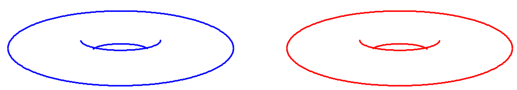

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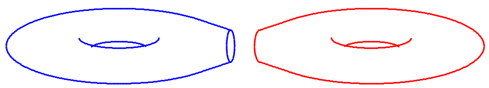

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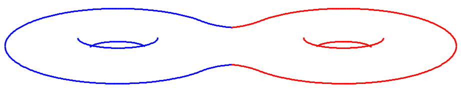

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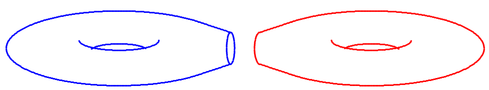

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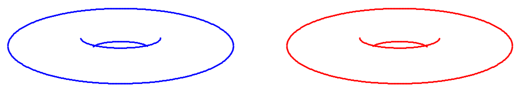

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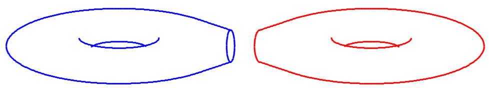

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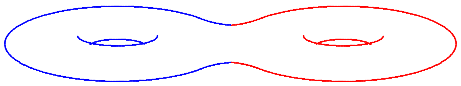

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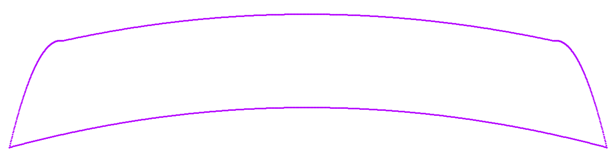

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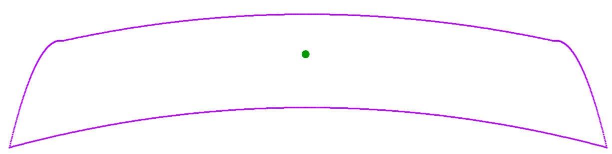

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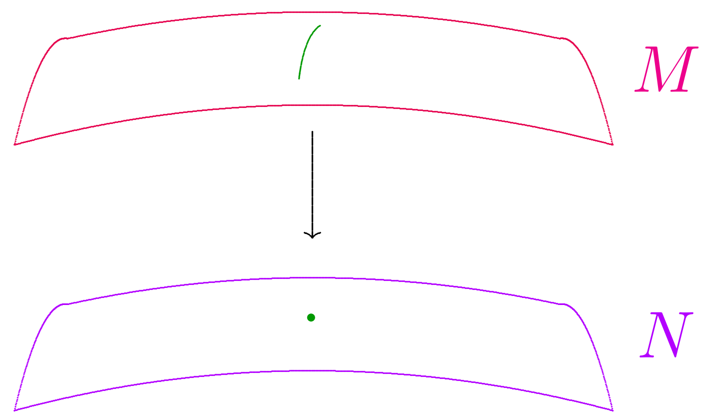

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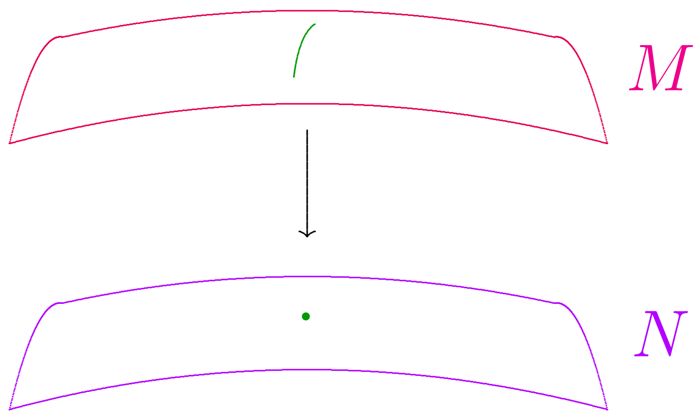
If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$



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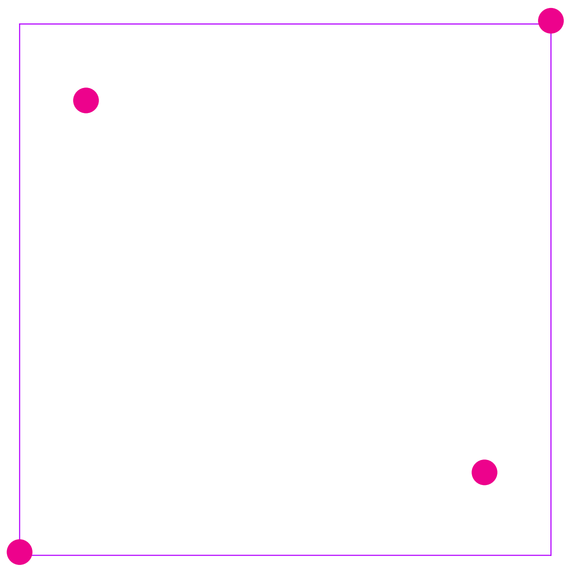

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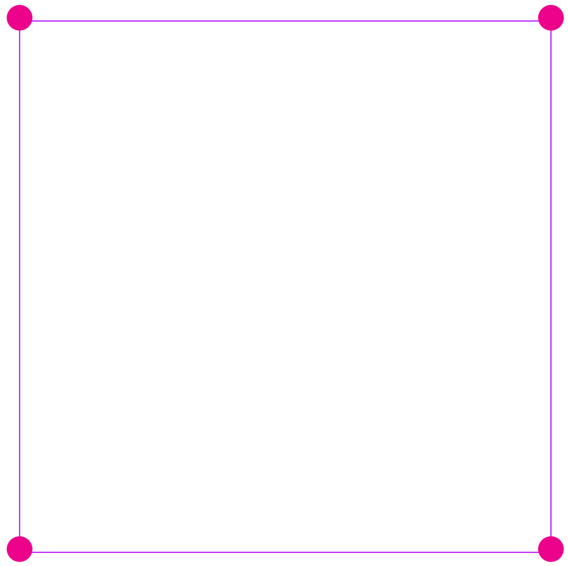

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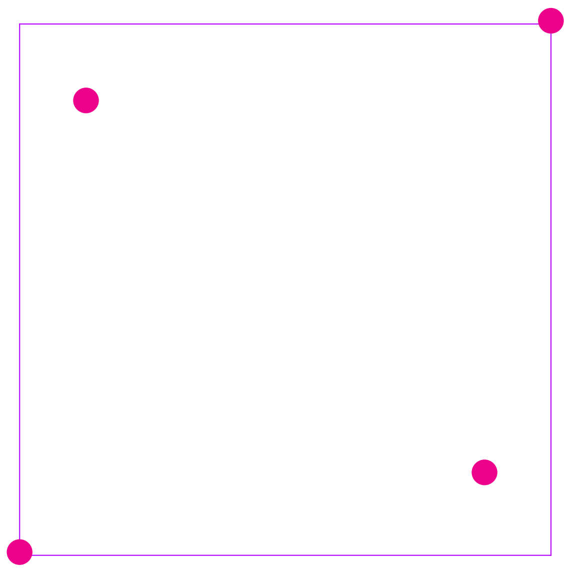

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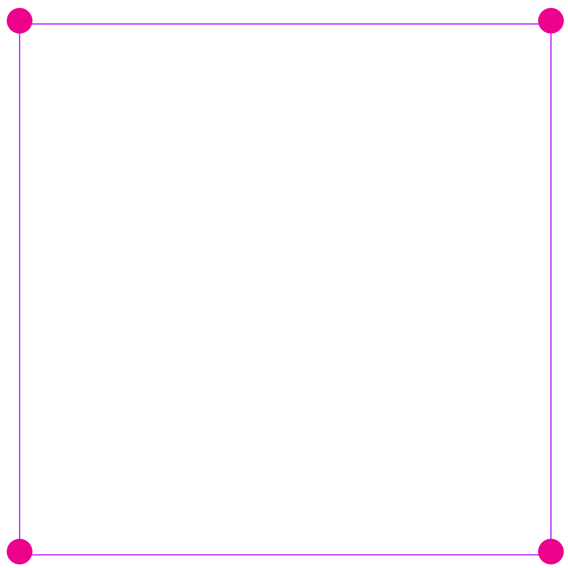

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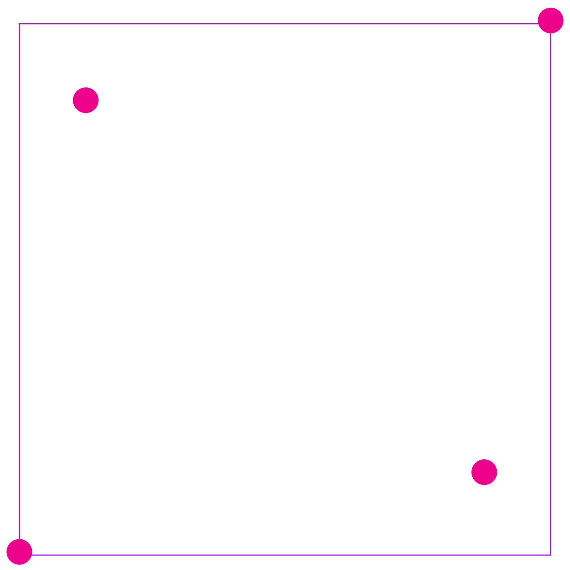

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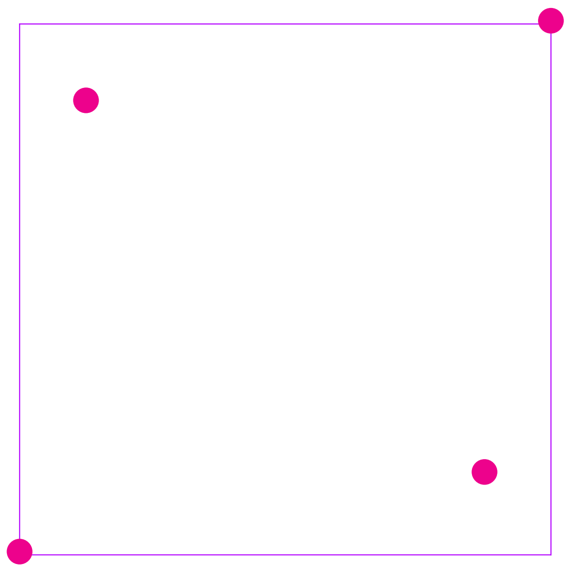

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Existence of these K-E metrics: Tian-Yau '87

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- Can arrange $(U, g_{j_i}) \to (X - \mathfrak{S}, g_\infty)$ smoothly;

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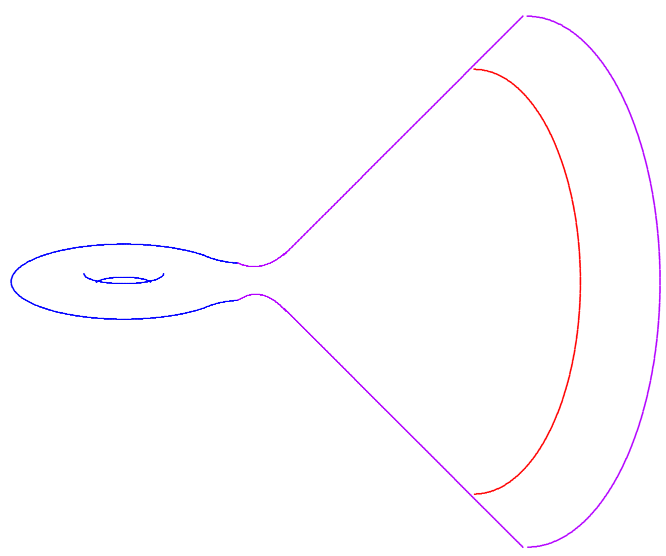

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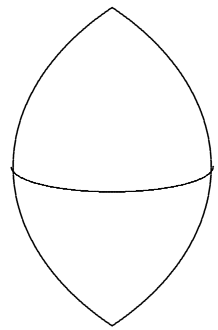

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Goal: Show that this doesn't change anything!

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$$[g] := \{u^2 g \mid u : M \to \mathbb{R}^+\}.$$

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Reversing orientation interchanges $\Lambda^+ \leftrightsquigarrow \Lambda^-$.

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We avoid this question by means of a definition!

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Corollary. *The Odaka-Spotti-Sun classification applies to* (X, g_∞)*.*

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By the Riemannian Goldberg-Sachs Theorem, the Hermitian assumption is equivalent to assuming that the orbifold Einstein metric g_∞ is conformally Kähler.

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This assertion is peculiar to dimension 4.
It is false in all higher dimensions!

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A more transparent proof was then given in L '21.

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Technically hardest when curvature accumulates on many different length-scales, giving rise to a complicated bubble tree.

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By contrast, if $m = 3^2 = 9$, then it is actually a limit of K-E metrics on $\mathbb{CP}_2 \# 8 \overline{\mathbb{CP}}_2$! One A_8 singularity, and two of type $\frac{1}{9}(1,2)$.

And now a word about gravitational instantons. . .

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Example. $\begin{bmatrix} e^{2\pi i/m} & \\ & e^{-2\pi i/m} \end{bmatrix} \in \mathbf{SU}(2)$ generates $\Gamma \cong \mathbb{Z}_m$. Setting

$$w = \frac{1}{2}(z_1^m - z_2^m), \qquad x = \frac{i}{2}(z_1^m + z_2^m), \qquad y = z_1 z_2,$$

then identifies $\mathbb{C}^2/\Gamma$ with

$$w^2 + x^2 + y^m = 0.$$

Felix Klein's Table of Singularities (1884)

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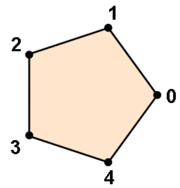

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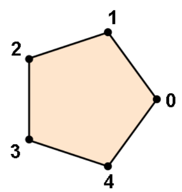

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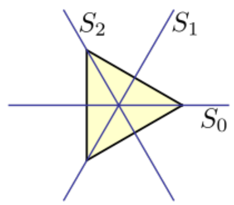

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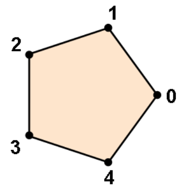

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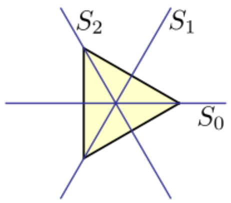

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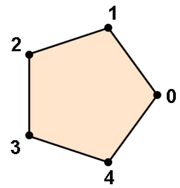

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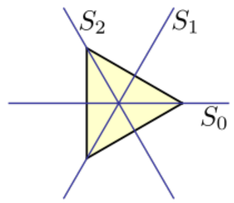

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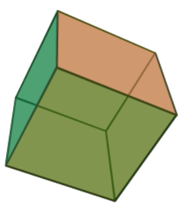

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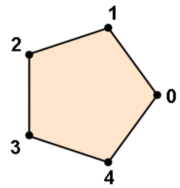

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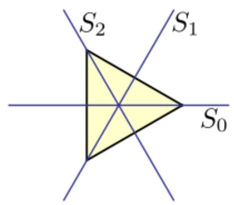

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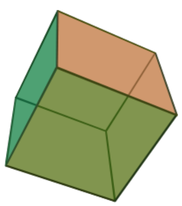

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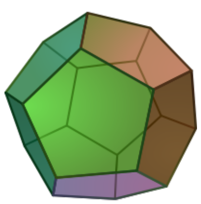

$I^* \quad \longleftrightarrow \quad w^2 + x^3 + y^5 = 0$

Prototypical Klein singularity:

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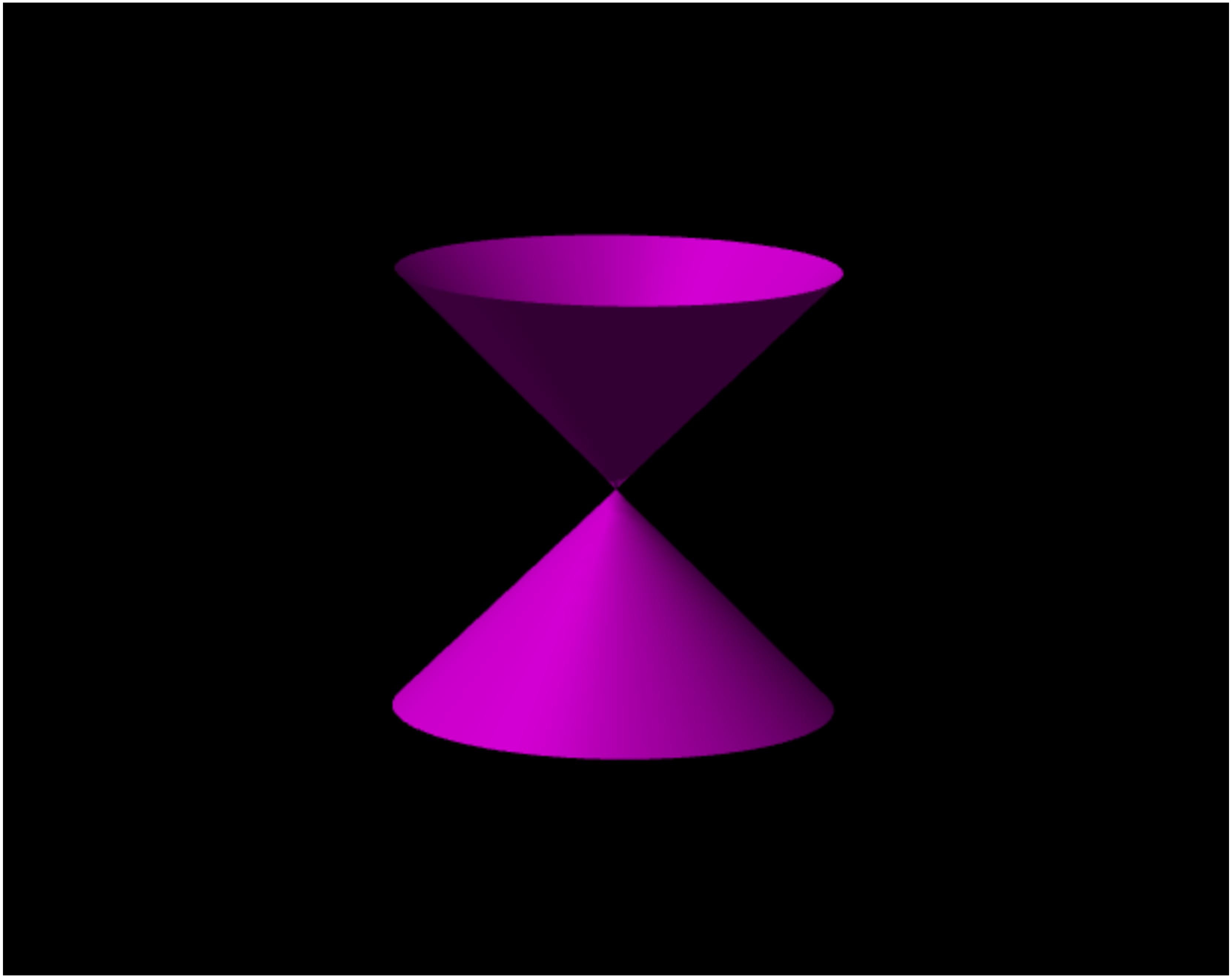

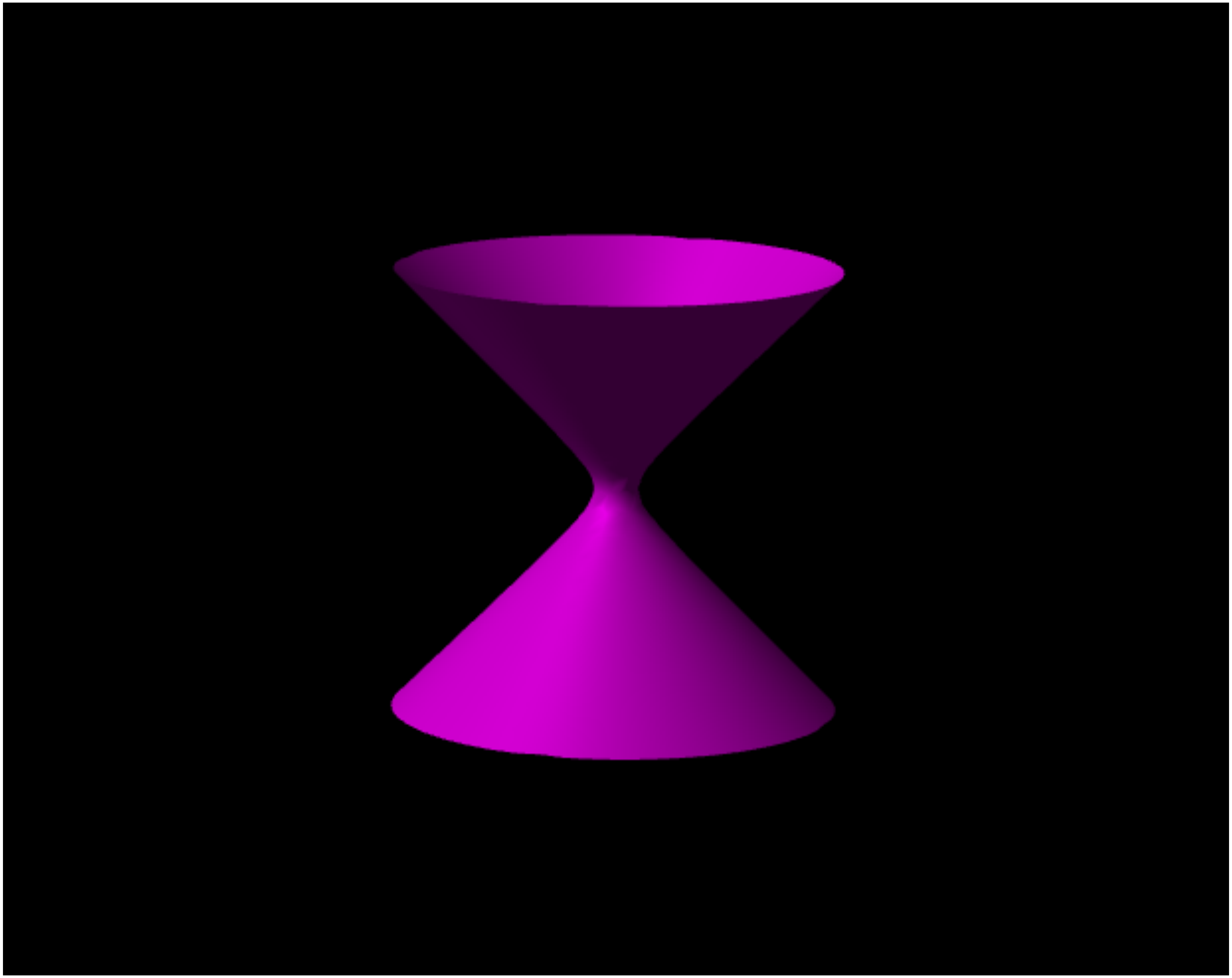

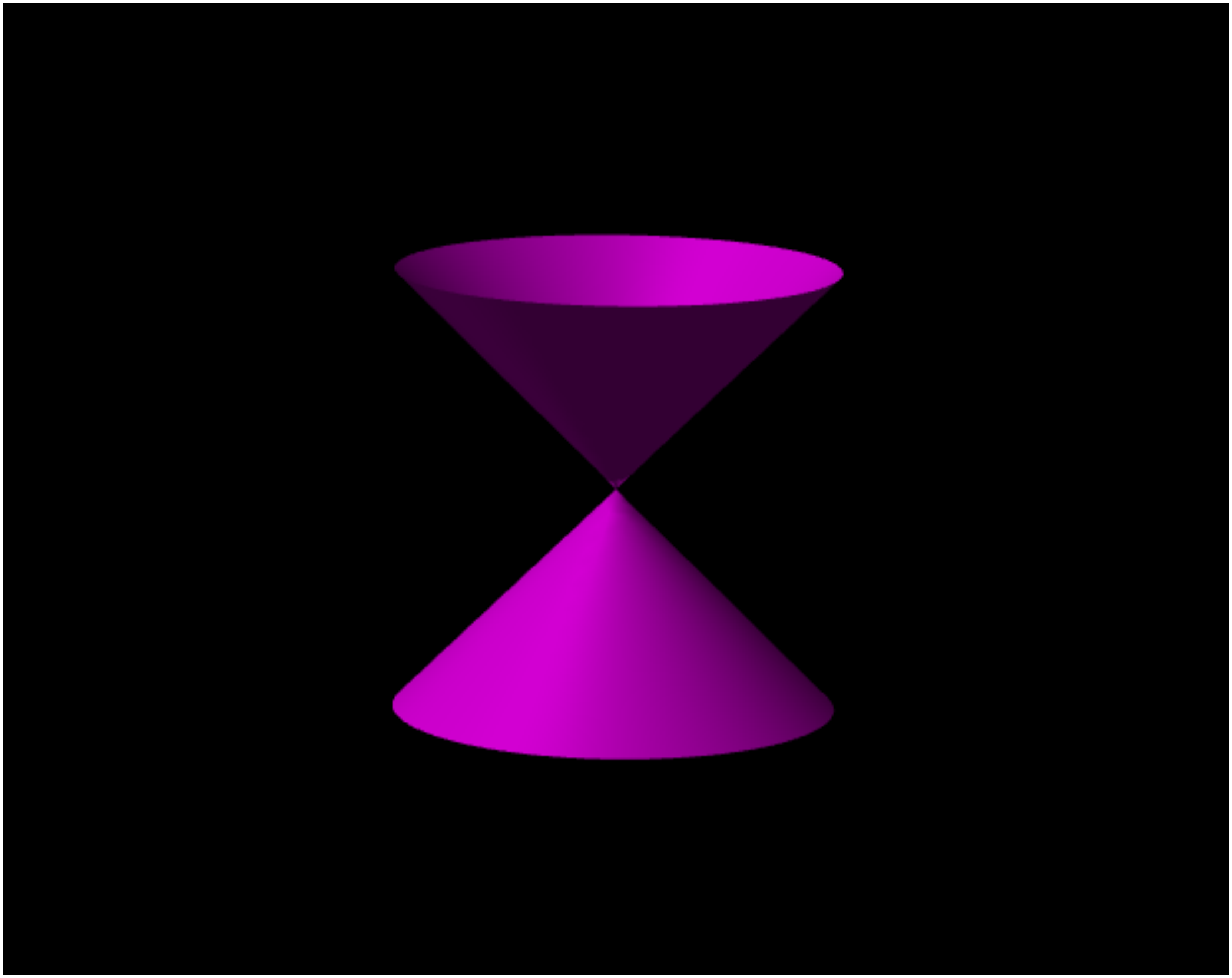

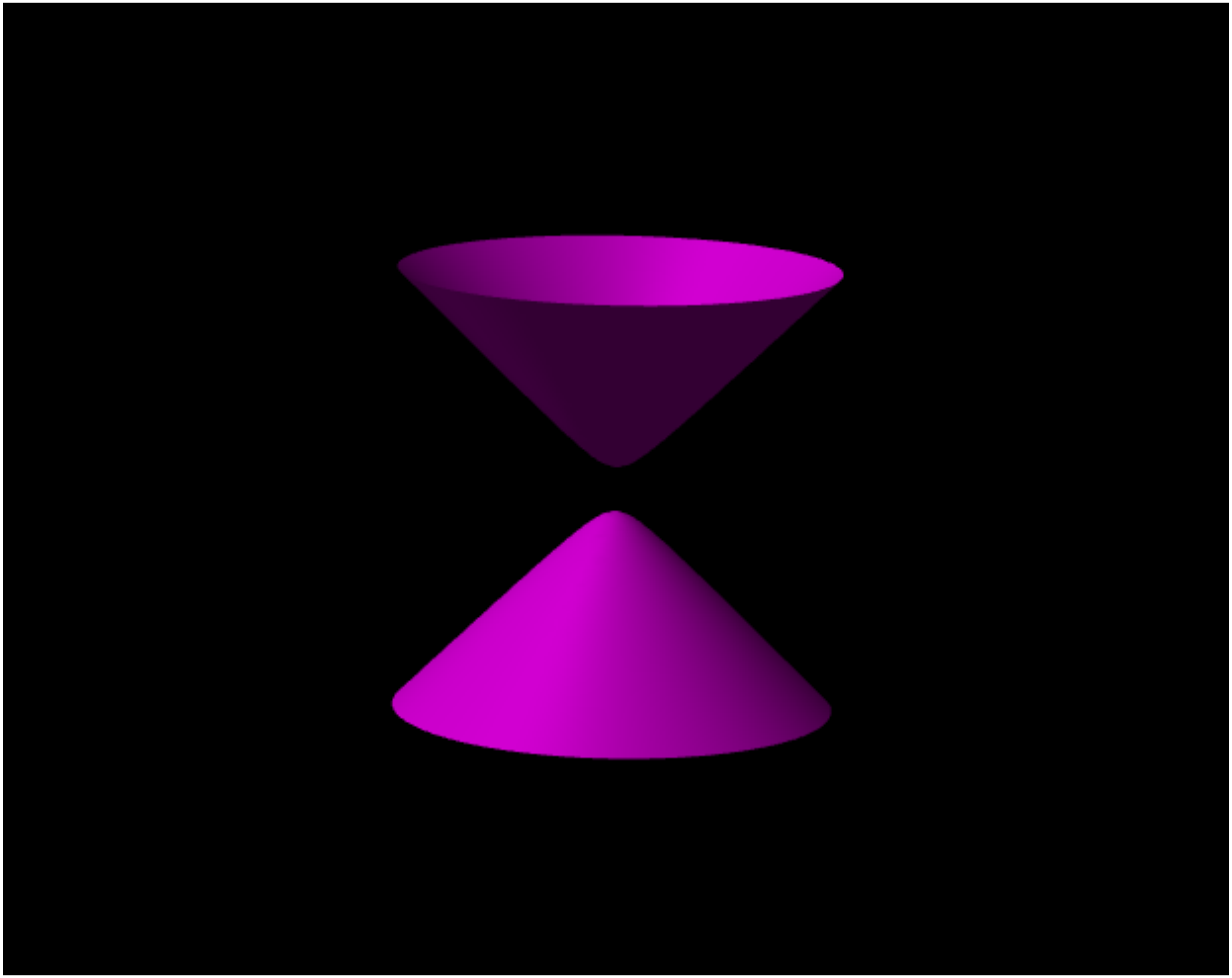

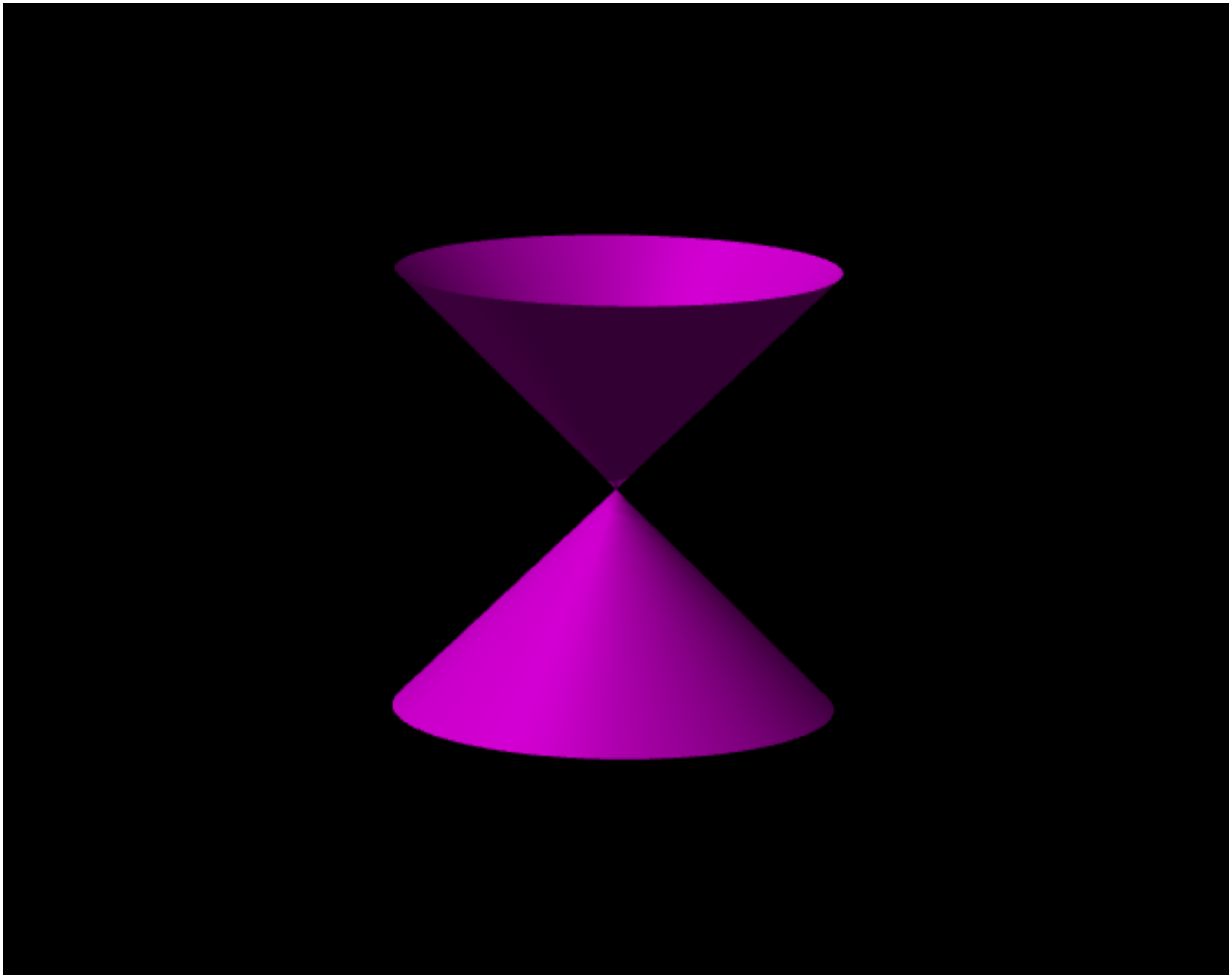

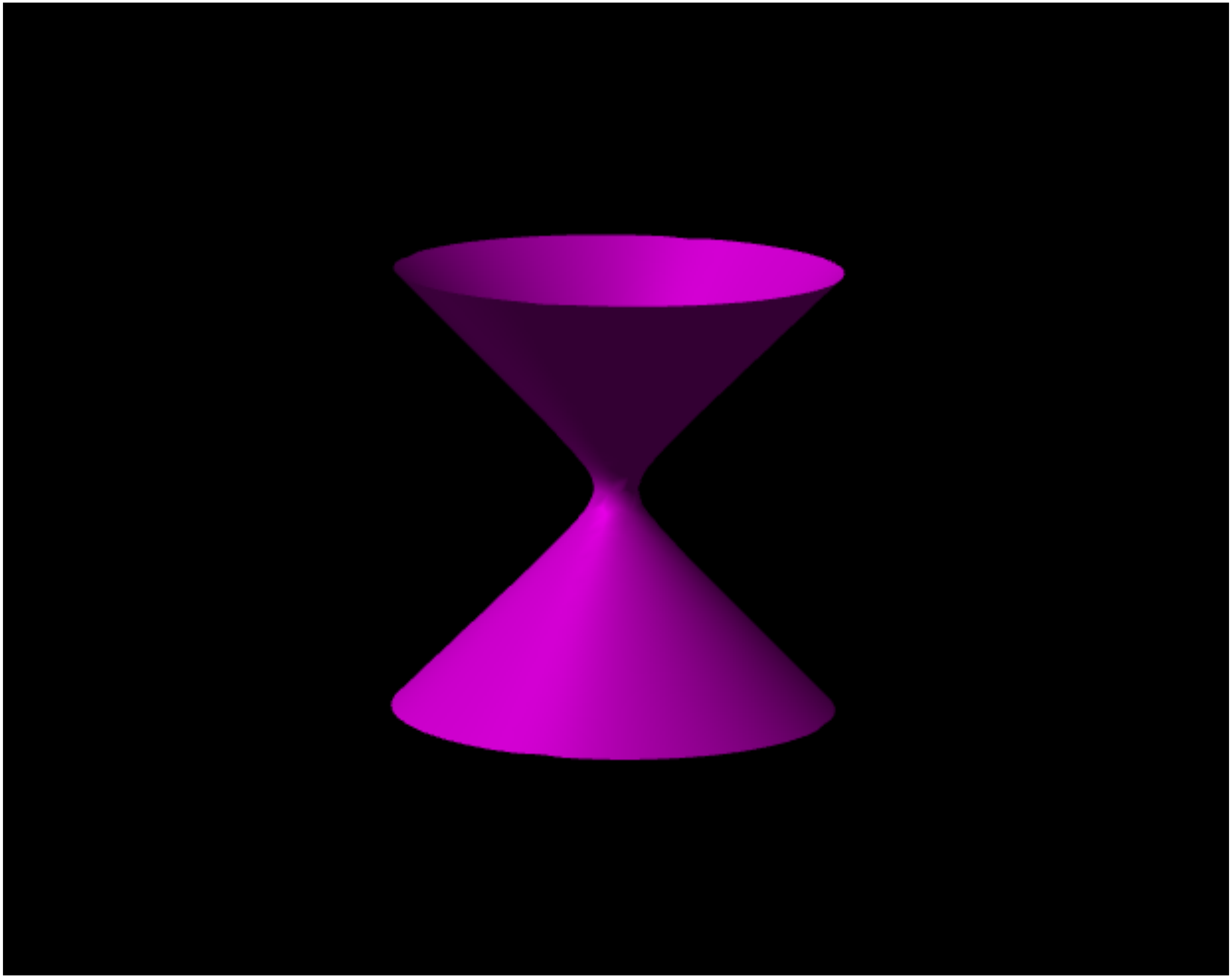

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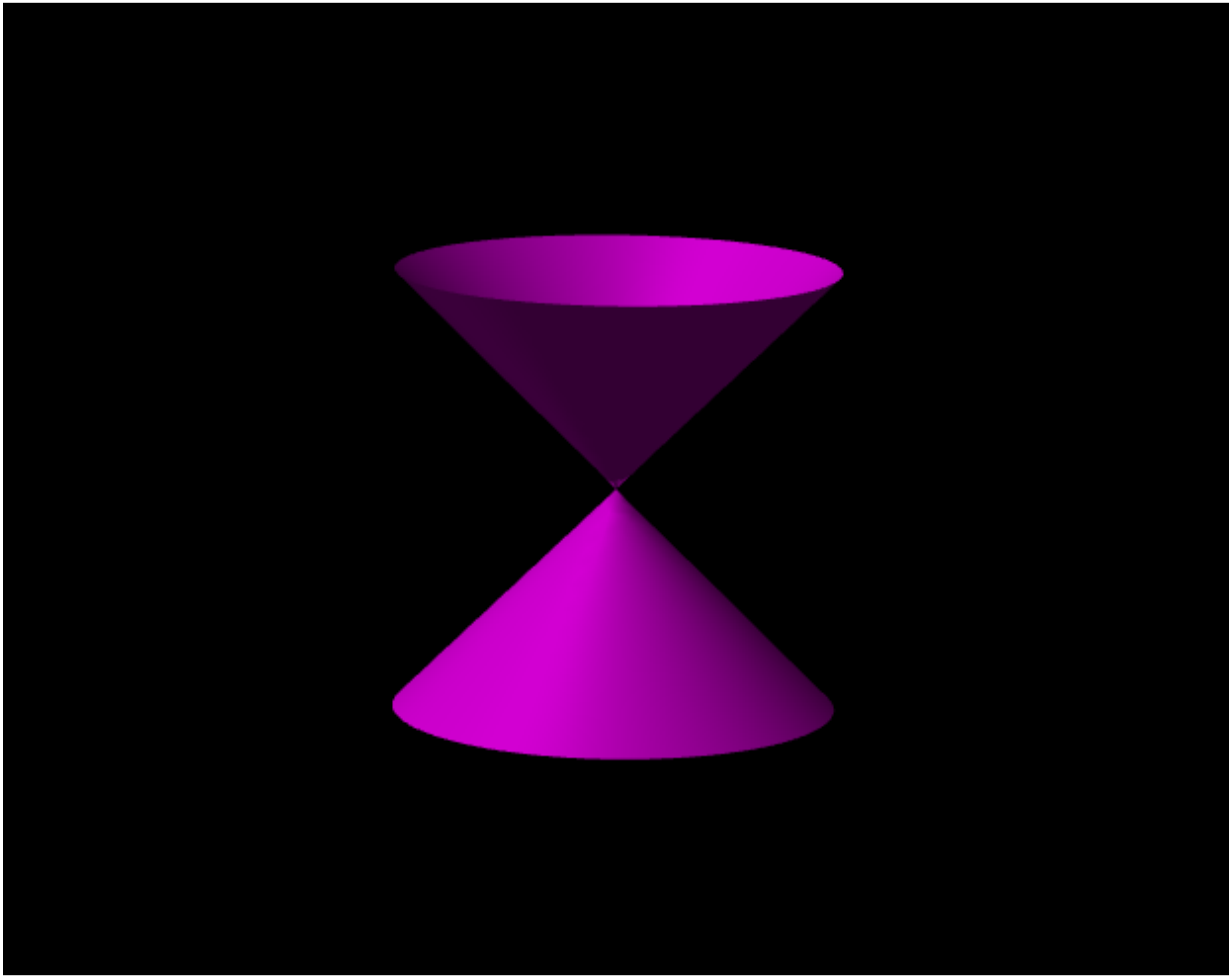

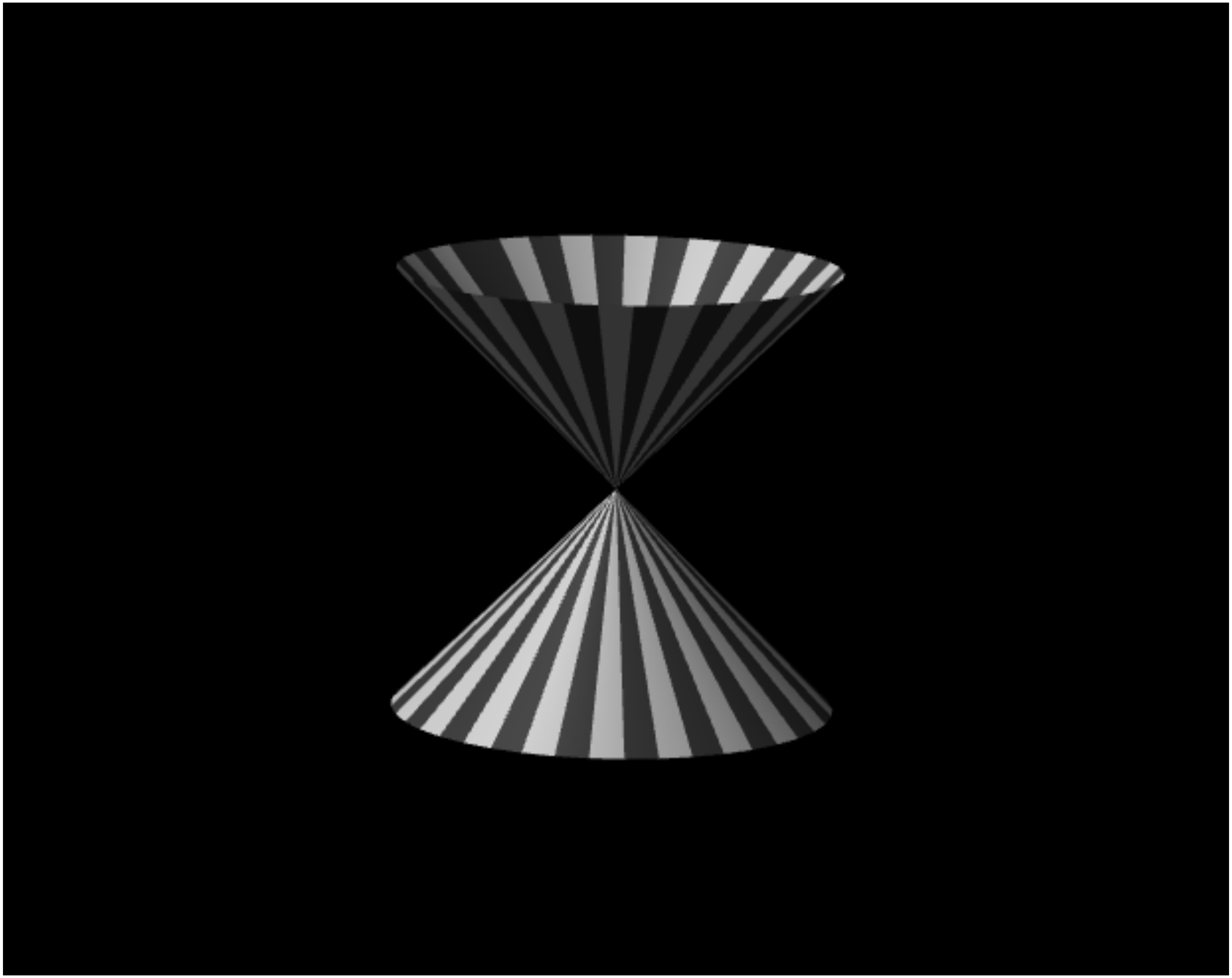

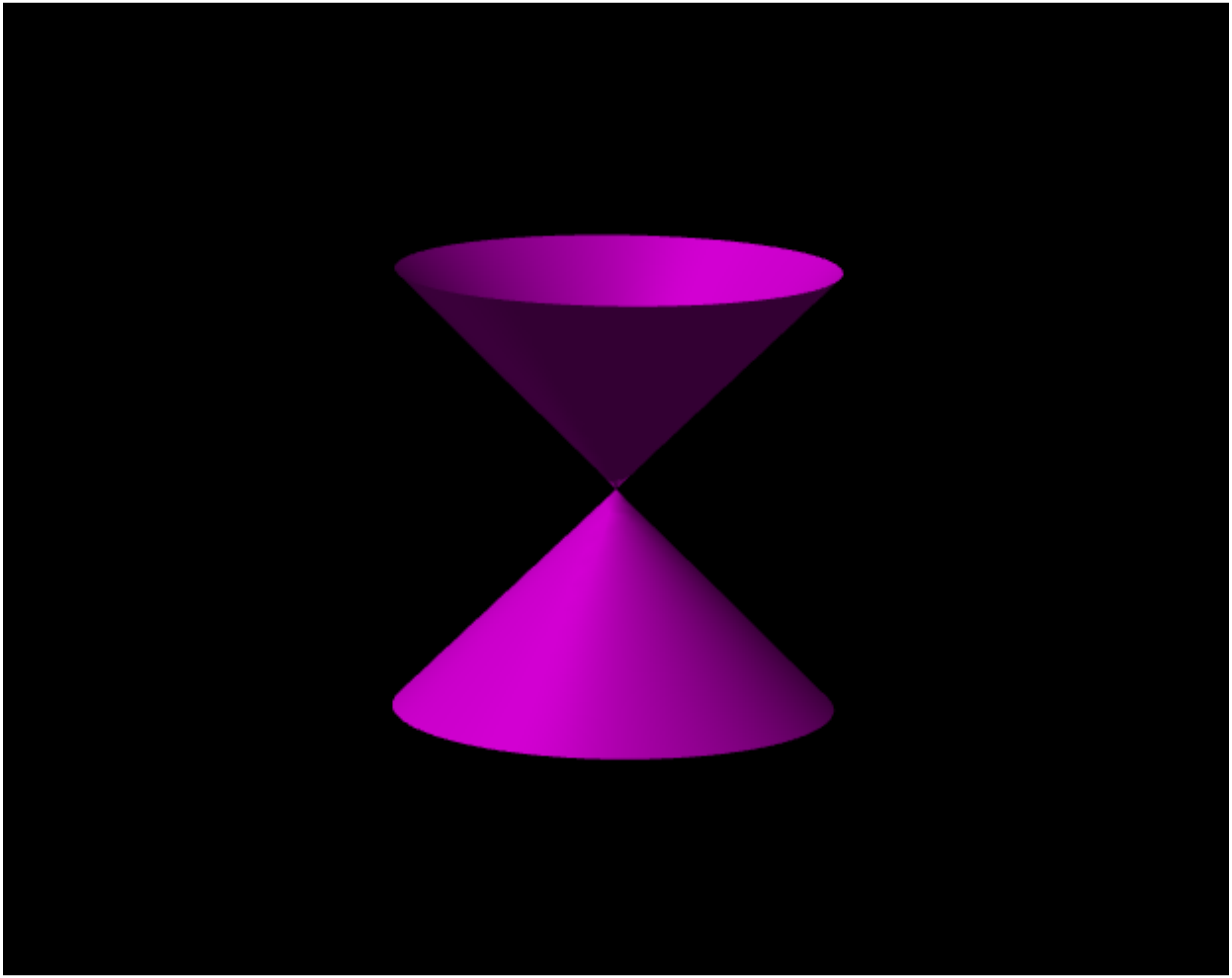

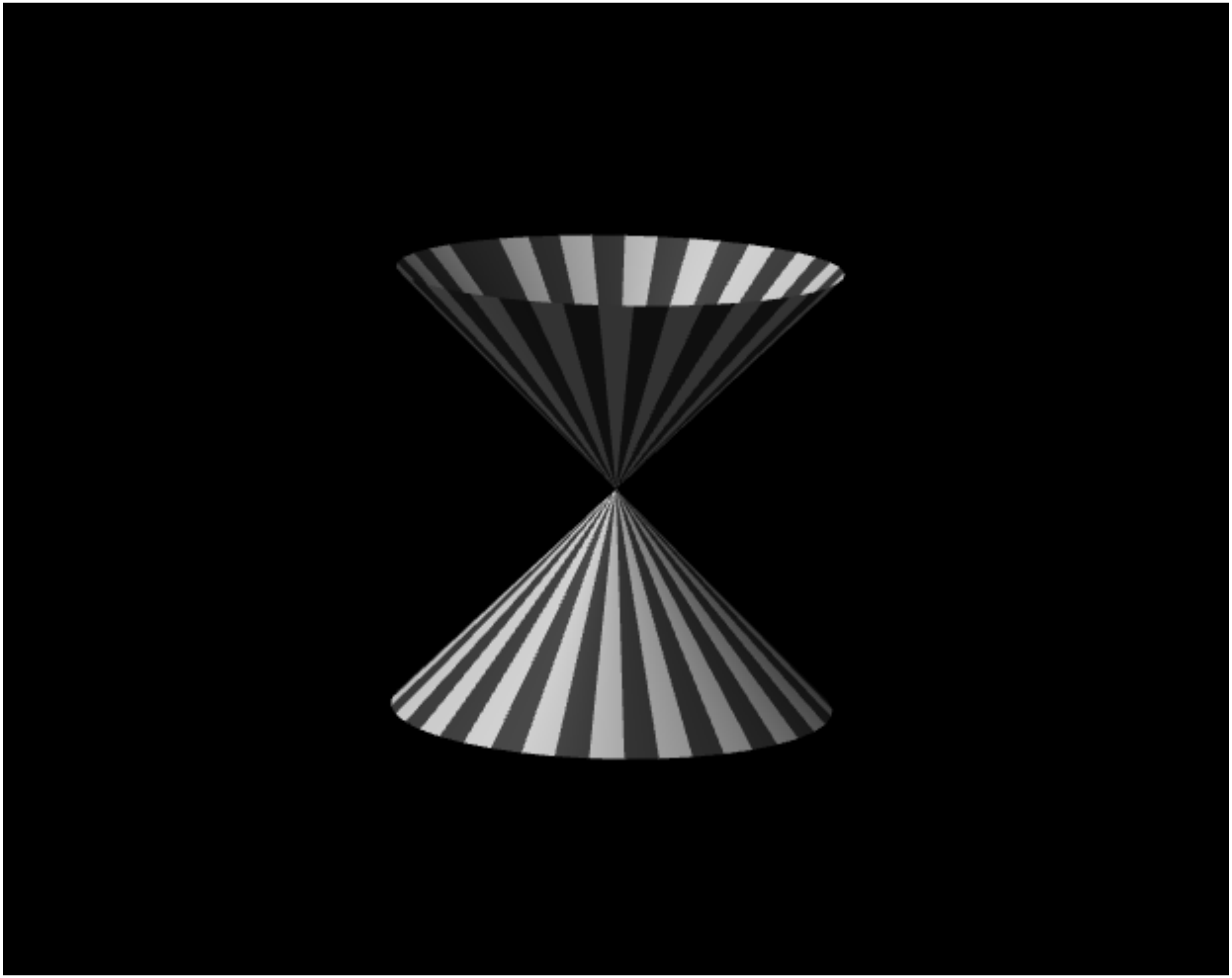

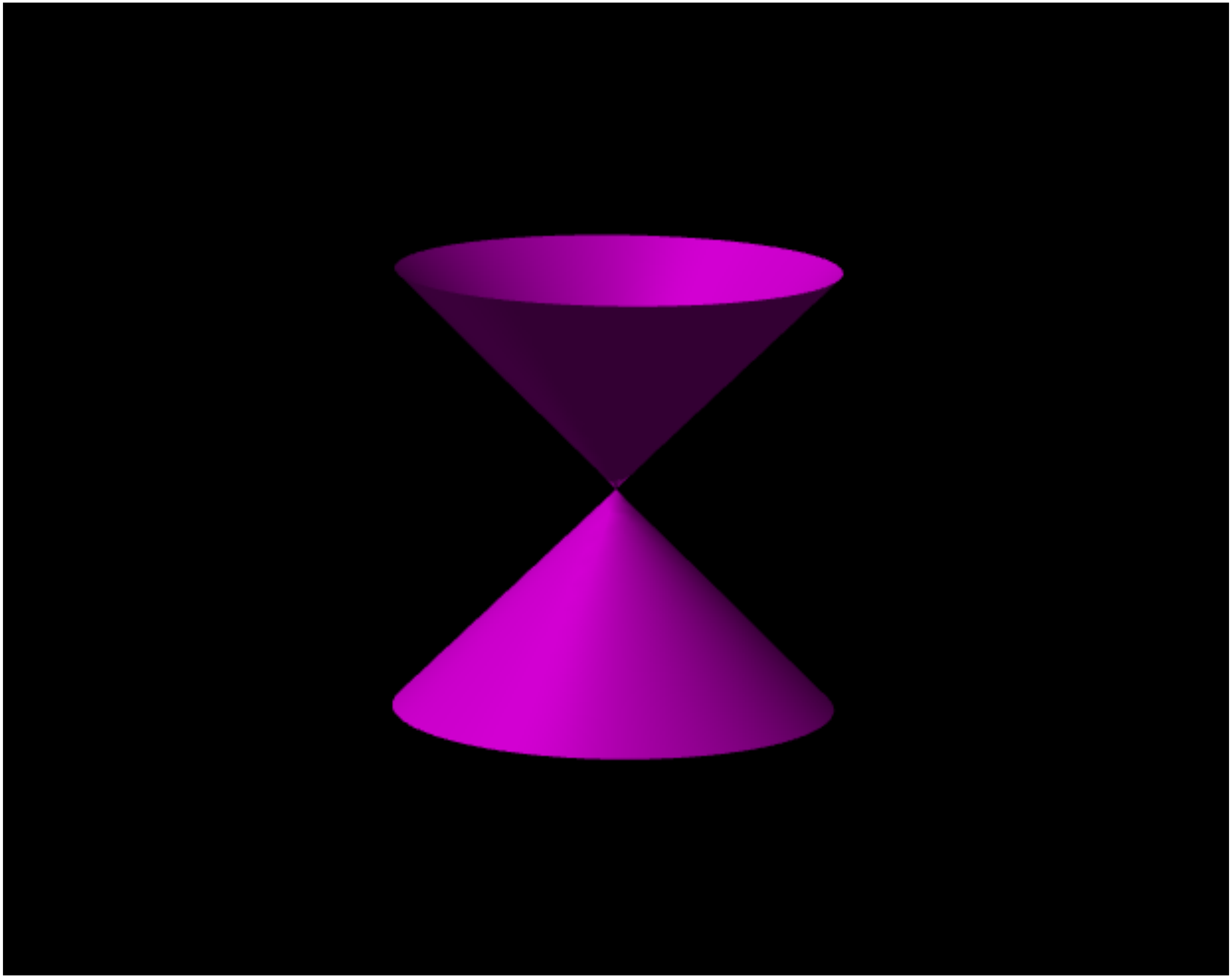

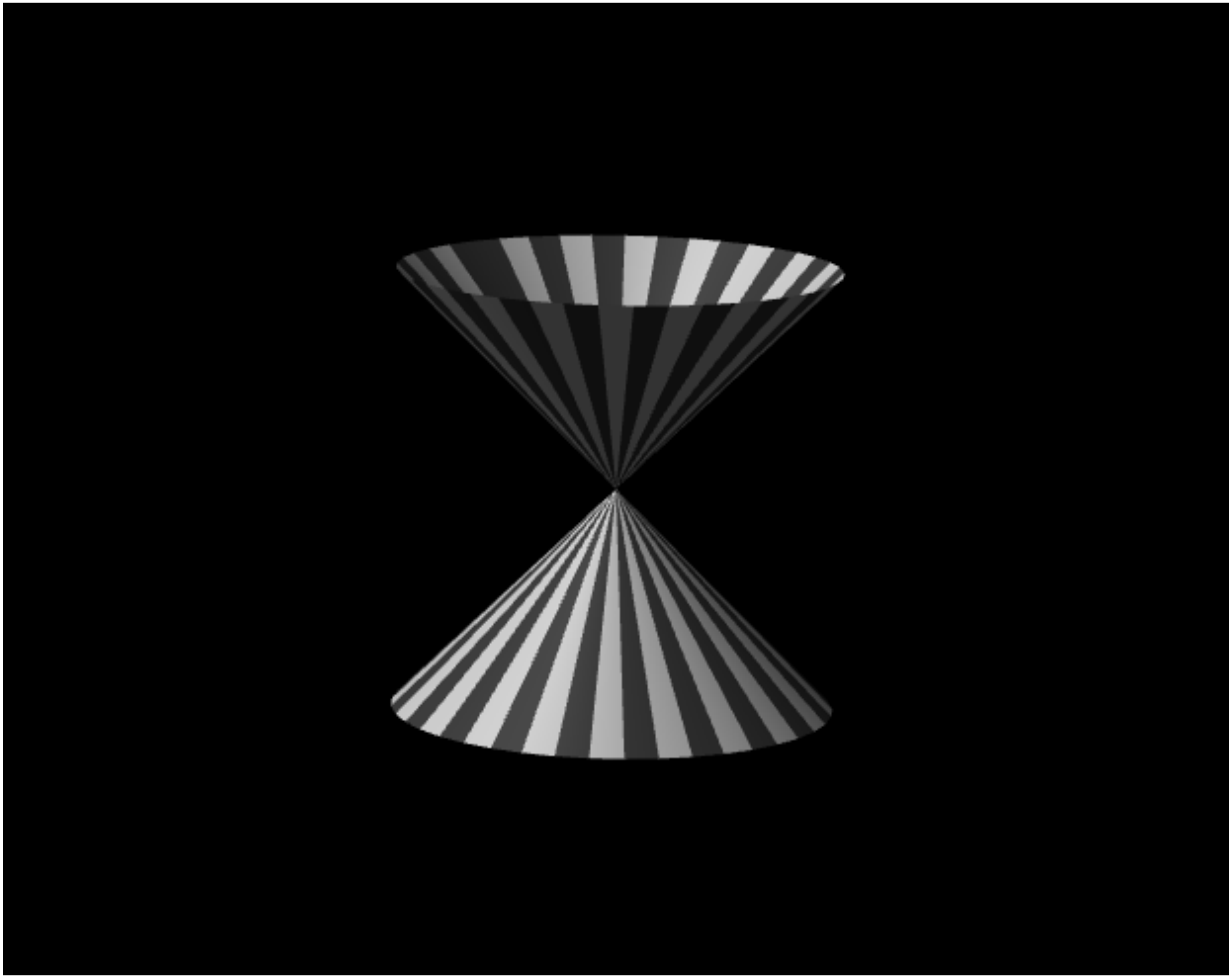

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- Smooth it, by deformation:

$$w^2 + x^2 + y^2 = \epsilon$$

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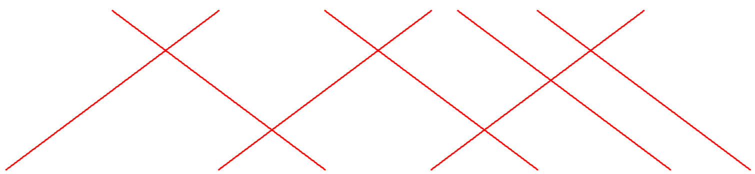

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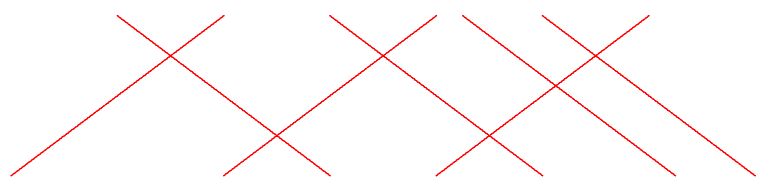

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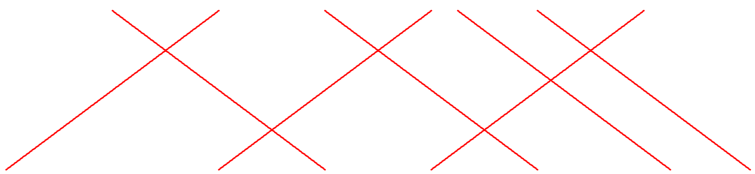

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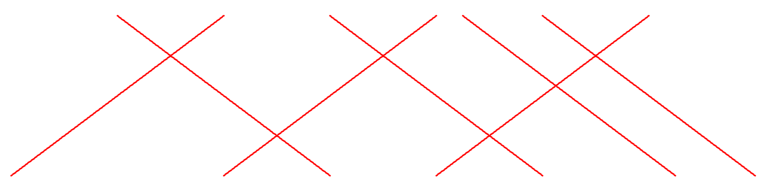

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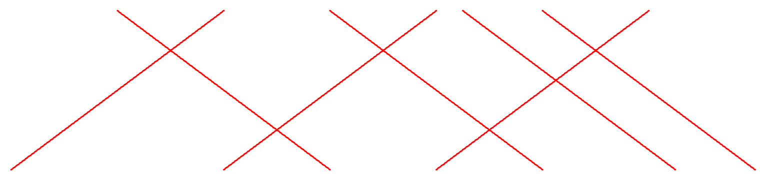
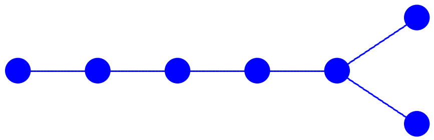


Intersection pattern dual to Dynkin diagram!



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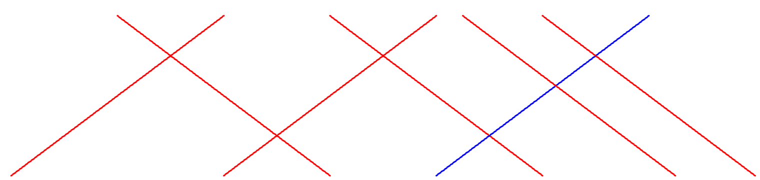

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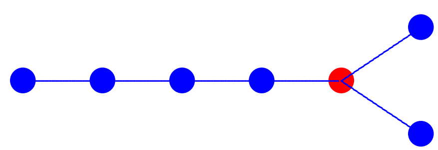

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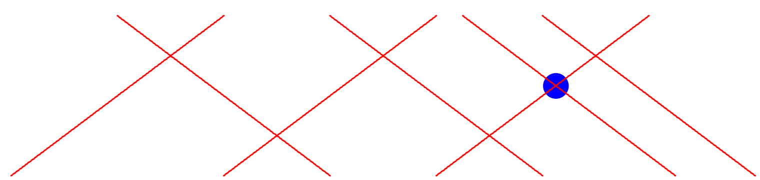

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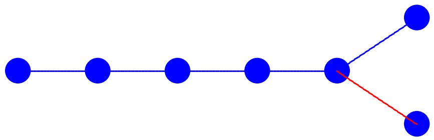

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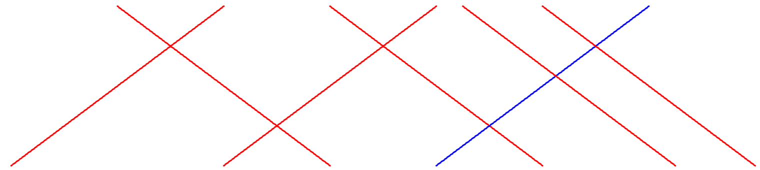

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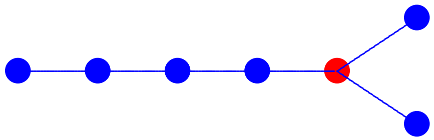

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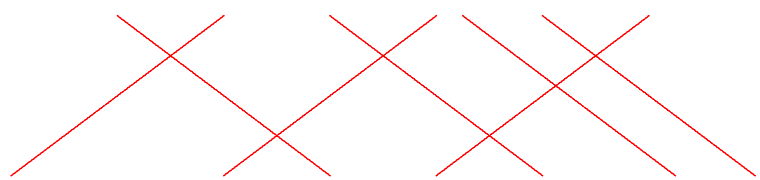

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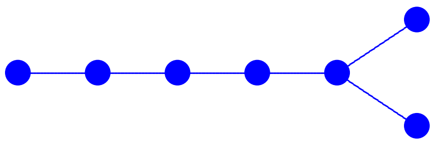
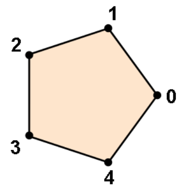
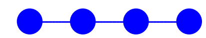
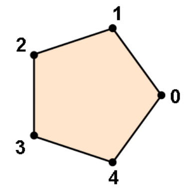




$$\mathbb{Z}_{k+1} \longleftrightarrow A_k$$





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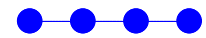

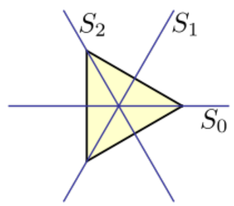

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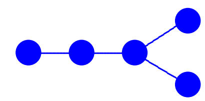

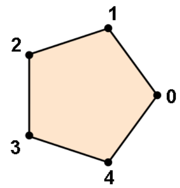

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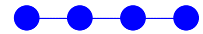

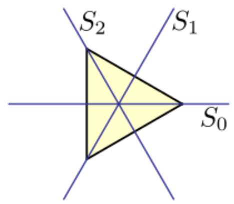

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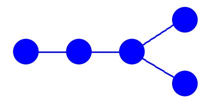

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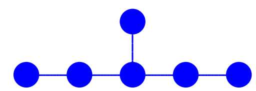

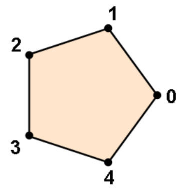

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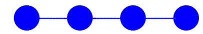

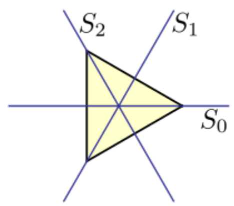

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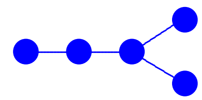

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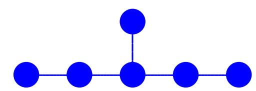

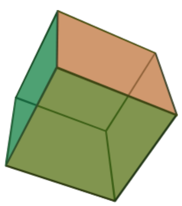

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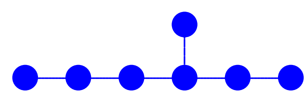

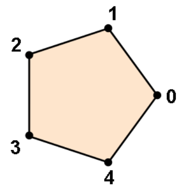

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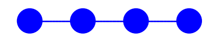

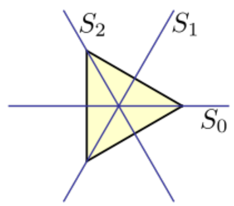

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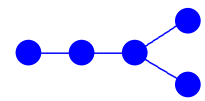

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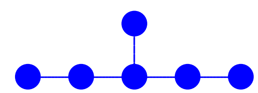

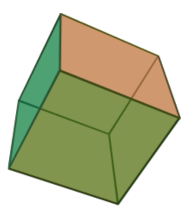

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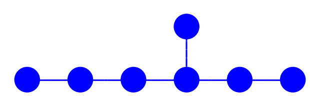

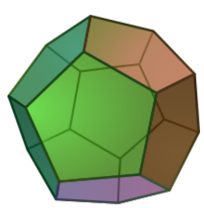

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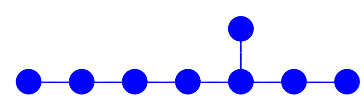

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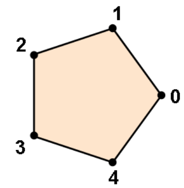
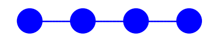
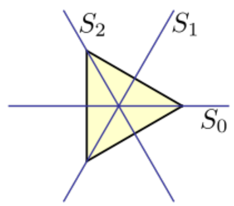
Reproduces Dynkin diagram of crepant resolution!

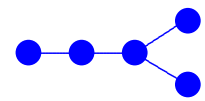


$$\mathbb{Z}_{k+1} \longleftrightarrow A_k$$

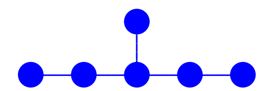


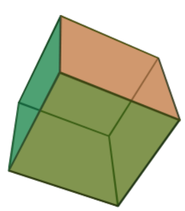


$$\mathrm{Dih}^*_{k-2} \longleftrightarrow D_k$$

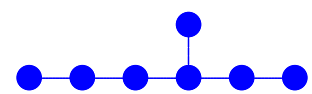


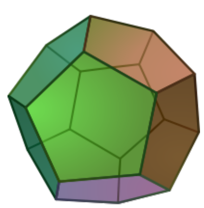
$$T^* \longleftrightarrow E_6$$



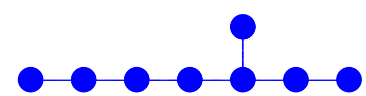


$$O^* \longleftrightarrow E_7$$





$$I^* \longleftrightarrow E_8$$



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With orbifold degenenerations included, exactly parameterized by

$$H^2(\mathbf{Y}) \otimes \mathbb{R}^3 \cong \mathfrak{h} \otimes \mathbb{R}^3,$$

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Cf. Witten's proof of the positive mass theorem.

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