Einstein Manifolds,

Kähler Metrics, &

Conformal Geometry

Claude LeBrun
Stony Brook University

Special Geometries on Riemannian Manifolds
Montreal, October 13, 2021

Most recent references:

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Bach-Flat Kähler Surfaces

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Journal of Geometric Analysis
30 (2020) 2491–2514

Einstein Manifolds, Self-Dual Weyl Curvature, and Conformally Kähler Geometry

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Mathematical Research Letters
28 (2021) 127–144

And

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Einstein Manifolds, Conformal Curvature, and Anti-Holomorphic Involutions

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Annales Mathématiques du Québec
45(2) (2021) 391–405

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"...the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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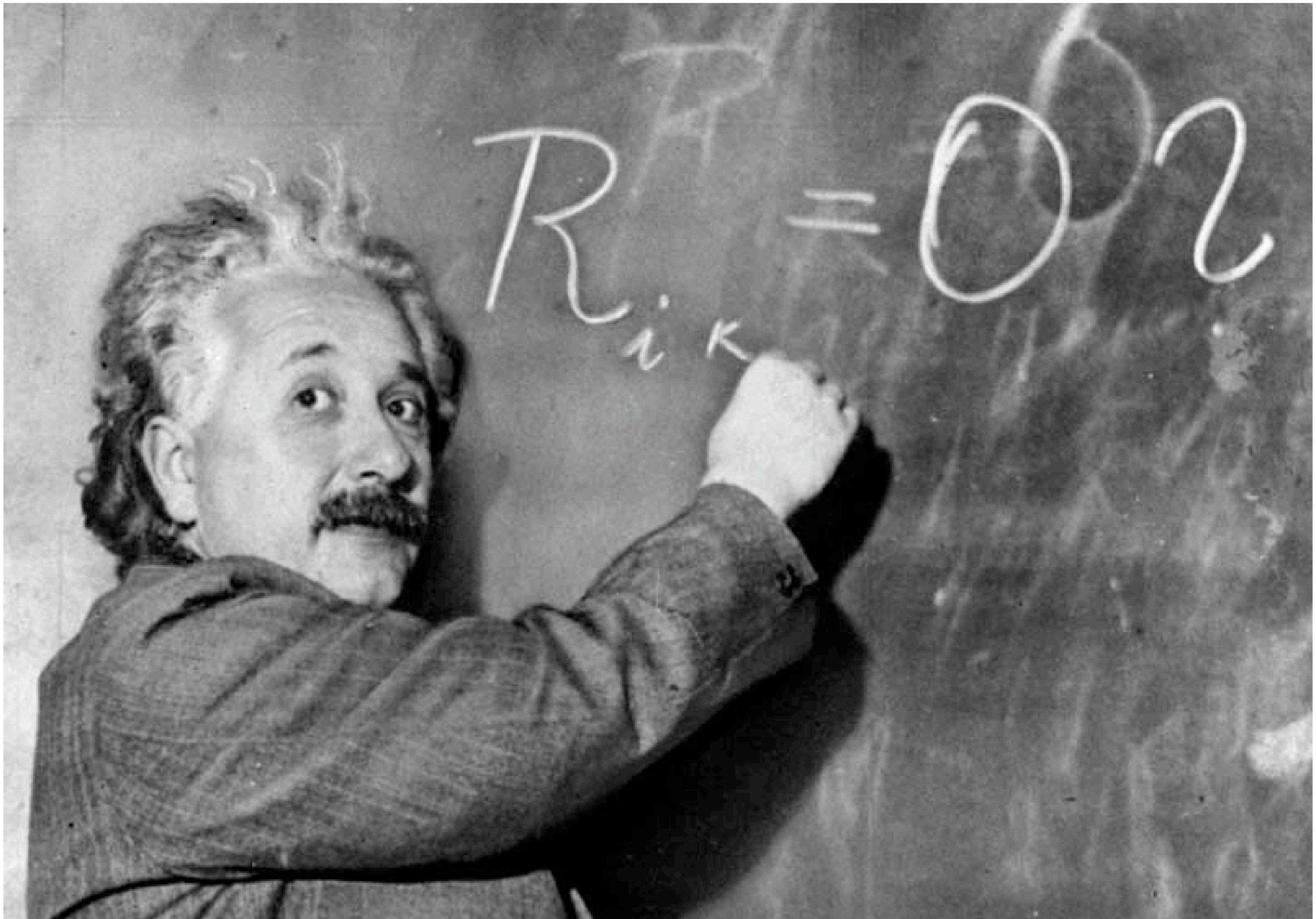

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Has same sign as the *scalar curvature*

$$s = r_j{}^j = \mathcal{R}^{ij}{}_{ij}.$$

Dimension Four is Exceptional

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When $n = 4$, Einstein metrics satisfy a remarkable conformally-invariant condition.

On Riemannian n-manifold (M, g),

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$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2} \mathring{r}^{[a}{}_{[c} \delta^{b]}{}_{d]} + \frac{2}{n(n-1)} \, s \, \delta^{a}_{[c} \delta^{b}_{d]}$$

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$W^{a}{}_{bcd}$ unchanged if $g \rightsquigarrow \hat{g} = u^2 g$.

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Proposition. *Assume* $n \geq 4$. *Then* (M^n, g) *locally conformally flat* $\Longleftrightarrow W \equiv 0$.

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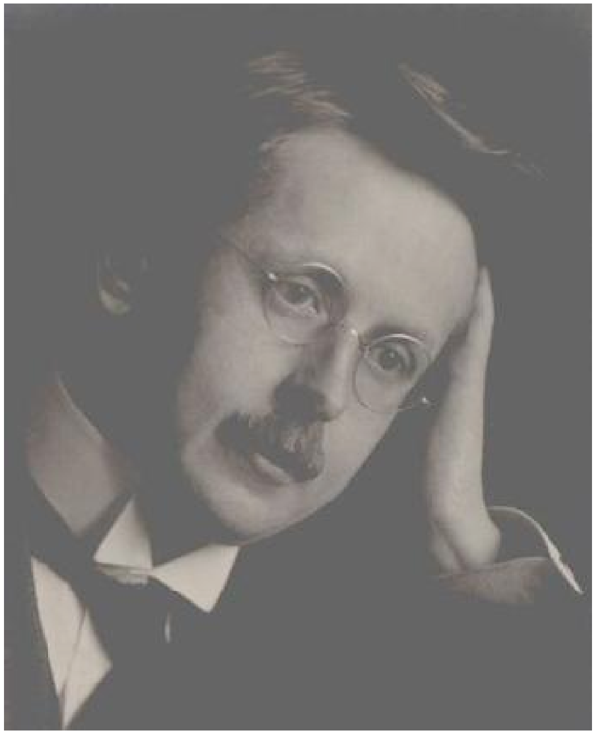


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Measures deviation $[g]$ from conformal flatness.

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Of course, conformally Einstein good enough!

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When $n = 4$, conf. Einstein $\Rightarrow$ critical for $\mathscr{W}$.

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$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W_- + \frac{s}{12} \end{array}\right)$$

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	Λ^{+*}	Λ^{-*}
Λ^+	$W_+ + \frac{s}{12}$	$\mathring{r}$
Λ^-	$\mathring{r}$	$W_- + \frac{s}{12}$

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$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

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Hence

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A case of special interest:

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(M^4, g, J) Kähler.

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$$|W_{+}|^{2} = \frac{s^{2}}{24}$$

Restriction of $\mathscr{W}_+$ to Kähler metrics?

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so any critical point of restriction must be extremal in sense of Calabi.

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on Kähler g with $[\omega] \in H^2(M)$ fixed.

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Andrzej Derdziński : For Kähler metrics g,

$$B = \frac{1}{12}\left[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\right]$$

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- *$g_t = g + tB$ is Kähler metric for small t.*

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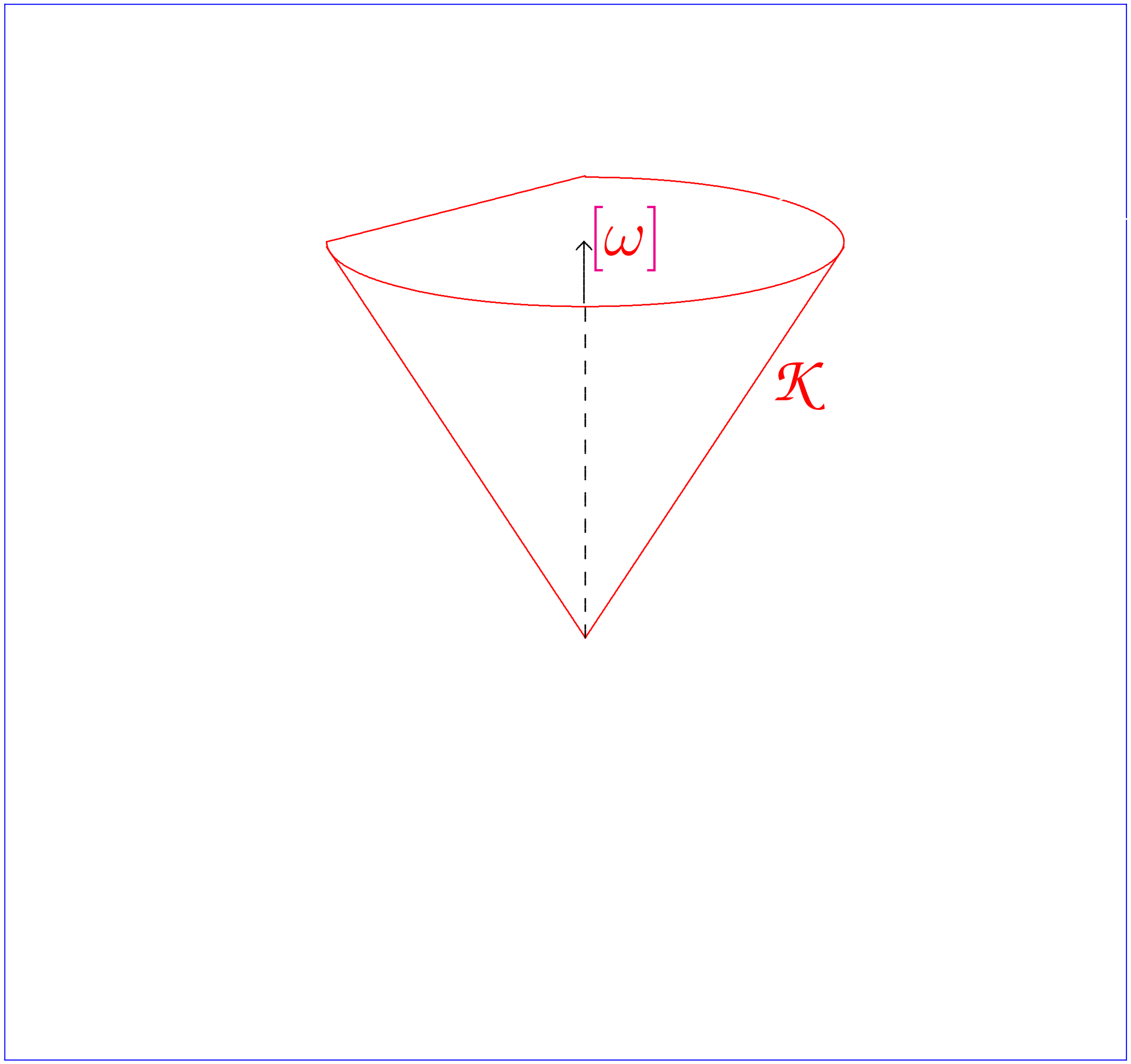
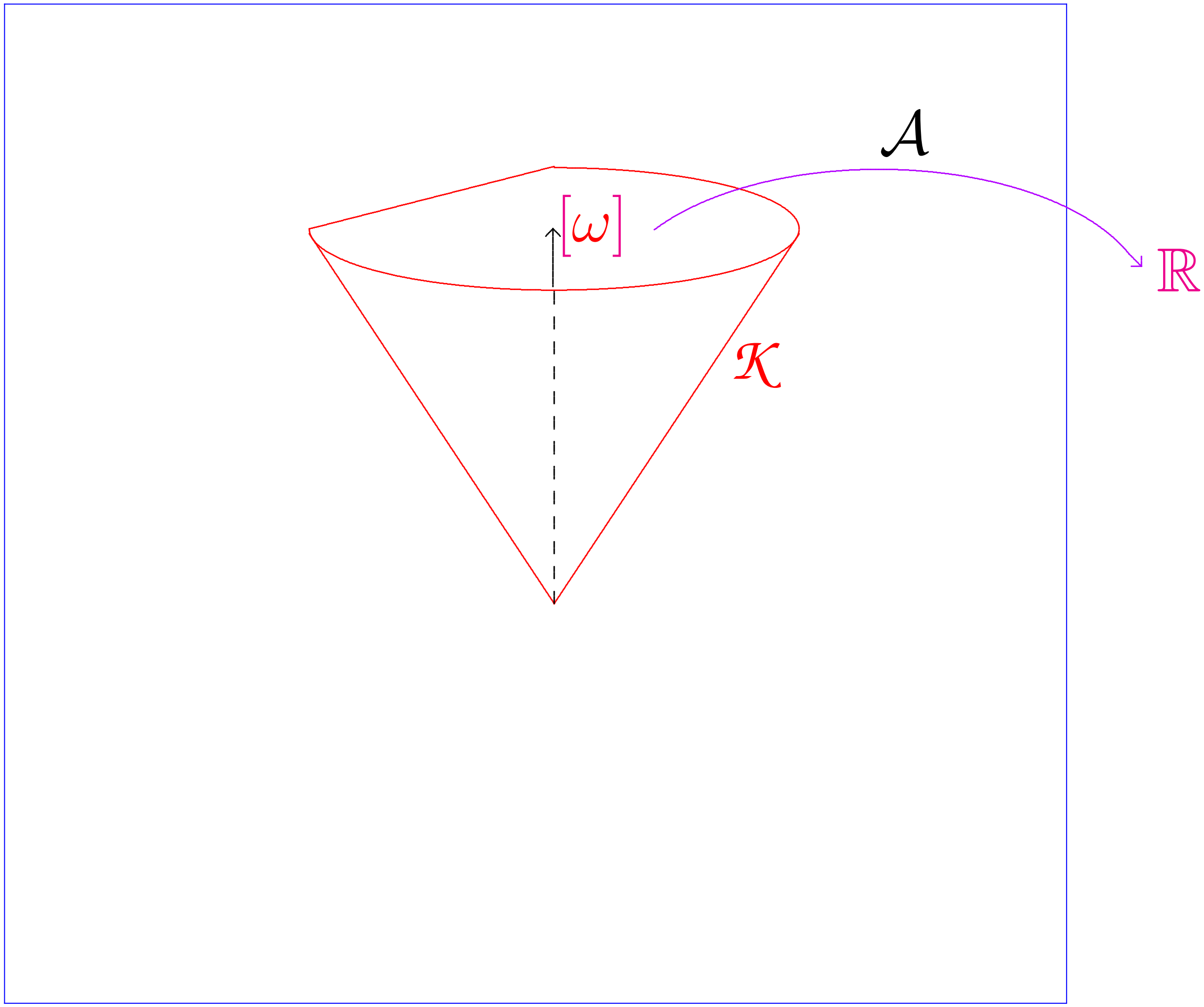
$$=: \mathcal{A}([\omega])$$

where $\mathcal{F}$ is Futaki invariant.



$$\mathcal{K} \subset H^{1,1}(M, \mathbb{R}) \subset H^2(M, \mathbb{R})$$



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$\mathcal{A}$ is function on Kähler cone $\mathcal{K} \subset H^2(M, \mathbb{R})$.

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$$\frac{1}{32\pi^2}\int s^2 d\mu_g = \frac{(c_1\cdot[\omega])^2}{[\omega]^2} + \frac{1}{32\pi^2}\|\mathcal{F}_{[\omega]}\|^2 =: \mathcal{A}([\omega])$$

where $\mathcal{F}$ is Futaki invariant.

$\mathcal{A}$ is function on Kähler cone $\mathcal{K} \subset H^2(M, \mathbb{R})$.

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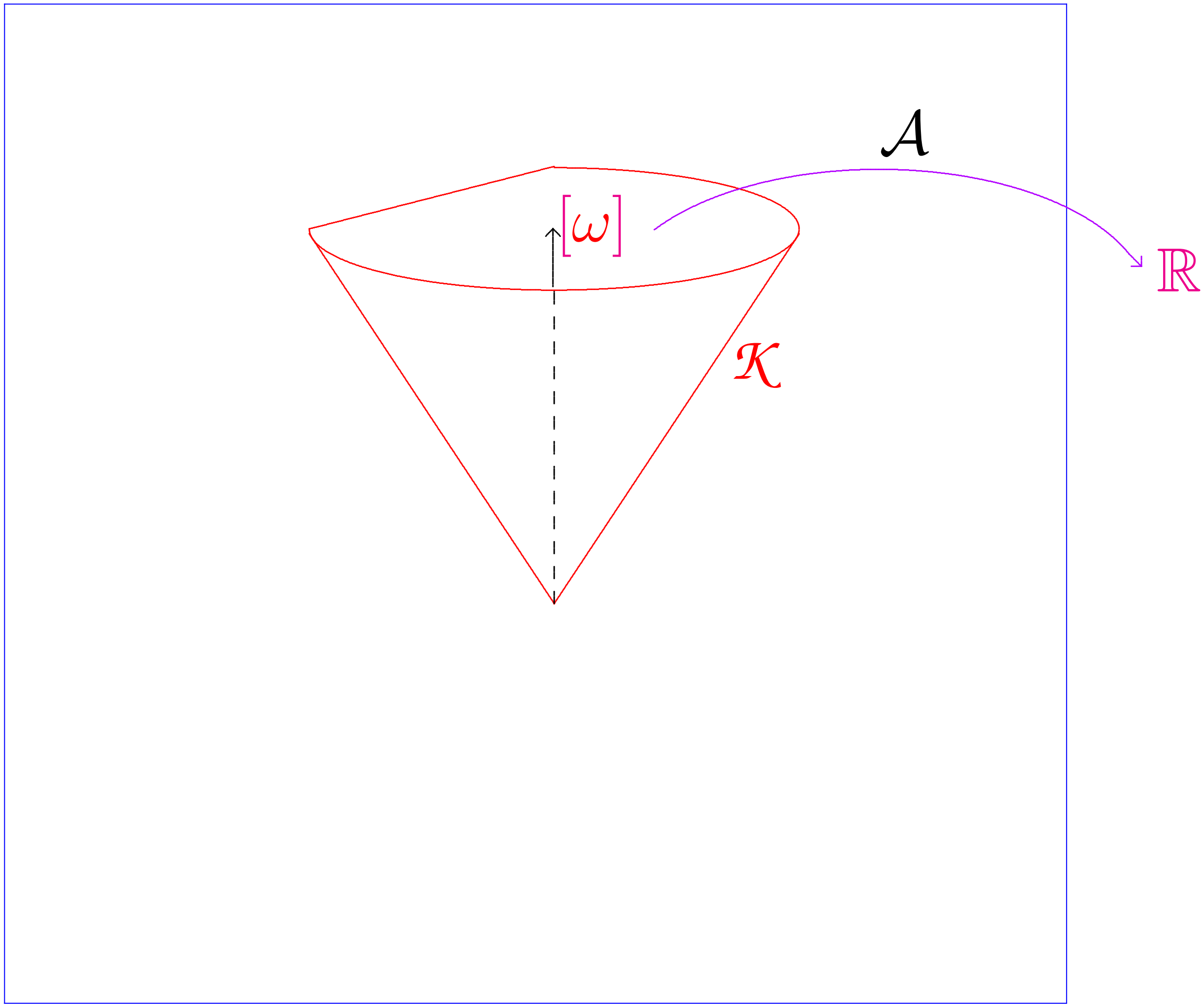

$$\mathcal{K} \subset H^{1,1}(M, \mathbb{R}) \subset H^2(M, \mathbb{R})$$

Restriction of $\mathscr{W}_+$ to Kähler metrics?

On Kähler metrics,

$$\int |W_+|^2 d\mu = \int \frac{s^2}{24} d\mu$$

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Andrzej Derdziński : For Kähler metrics g,

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Global implications?

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Moreover, each case actually occurs.

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I. $s > 0$ *everywhere. Then*

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If not Kähler-Einstein:

I. s *is positive. Then*

$(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. s *is zero. Then*

(M, g, J) *SFK, but not Ricci-flat.*

III. s *changes sign. Then*

$(M, s^{-2}g)$ *double Poincaré-Einstein. Here,* $s = 0$ *defines smooth connected* $\mathcal{Z}^3$, *and* $M - \mathcal{Z}$ *has exactly two components.*

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This happens $\iff$ $c_1 > 0$.

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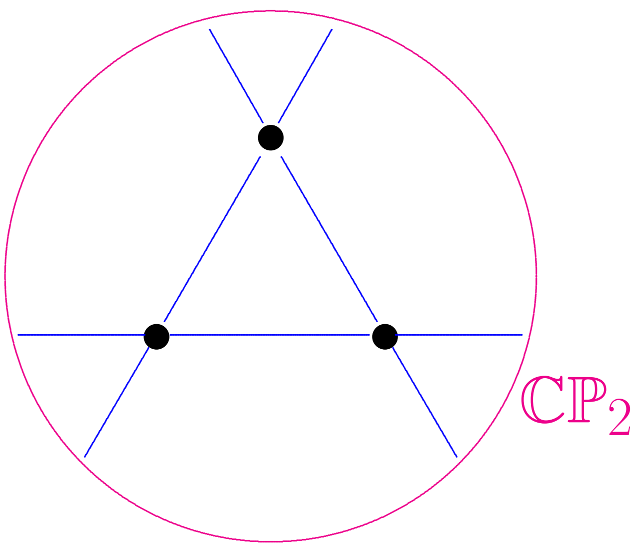

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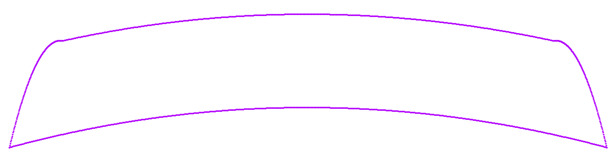

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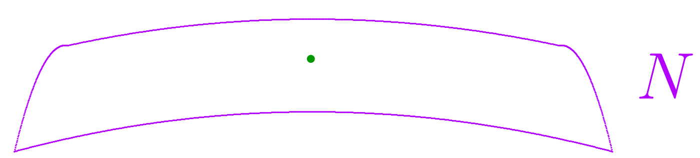

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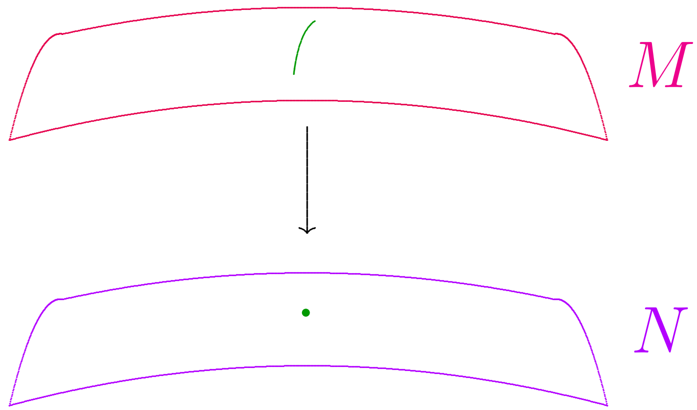

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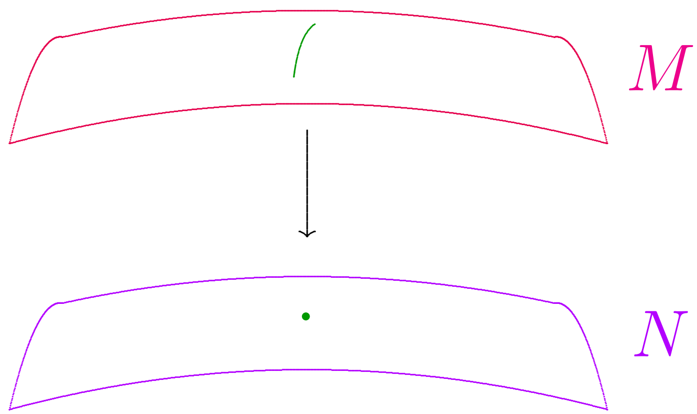
$$M \approx N \# \overline{\mathbb{CP}}_2$$



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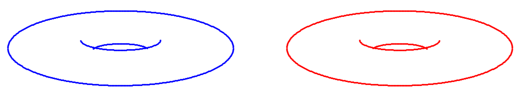

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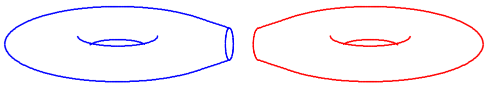

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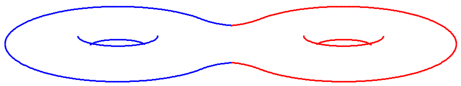

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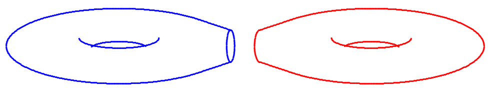

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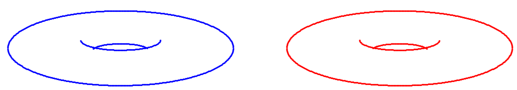

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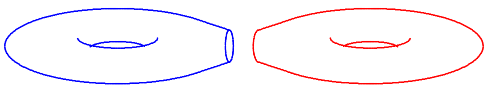

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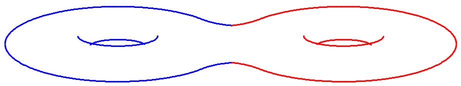

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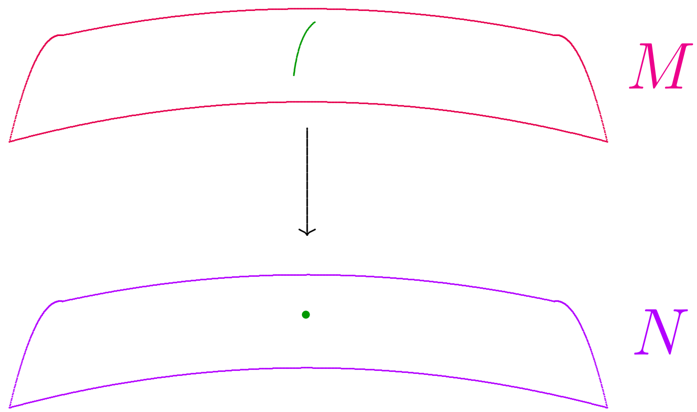

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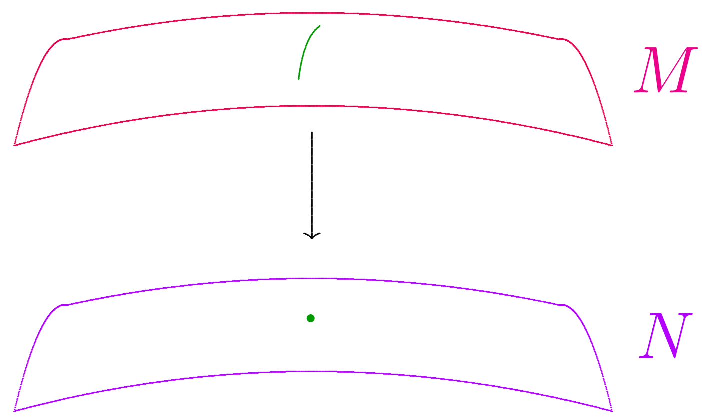

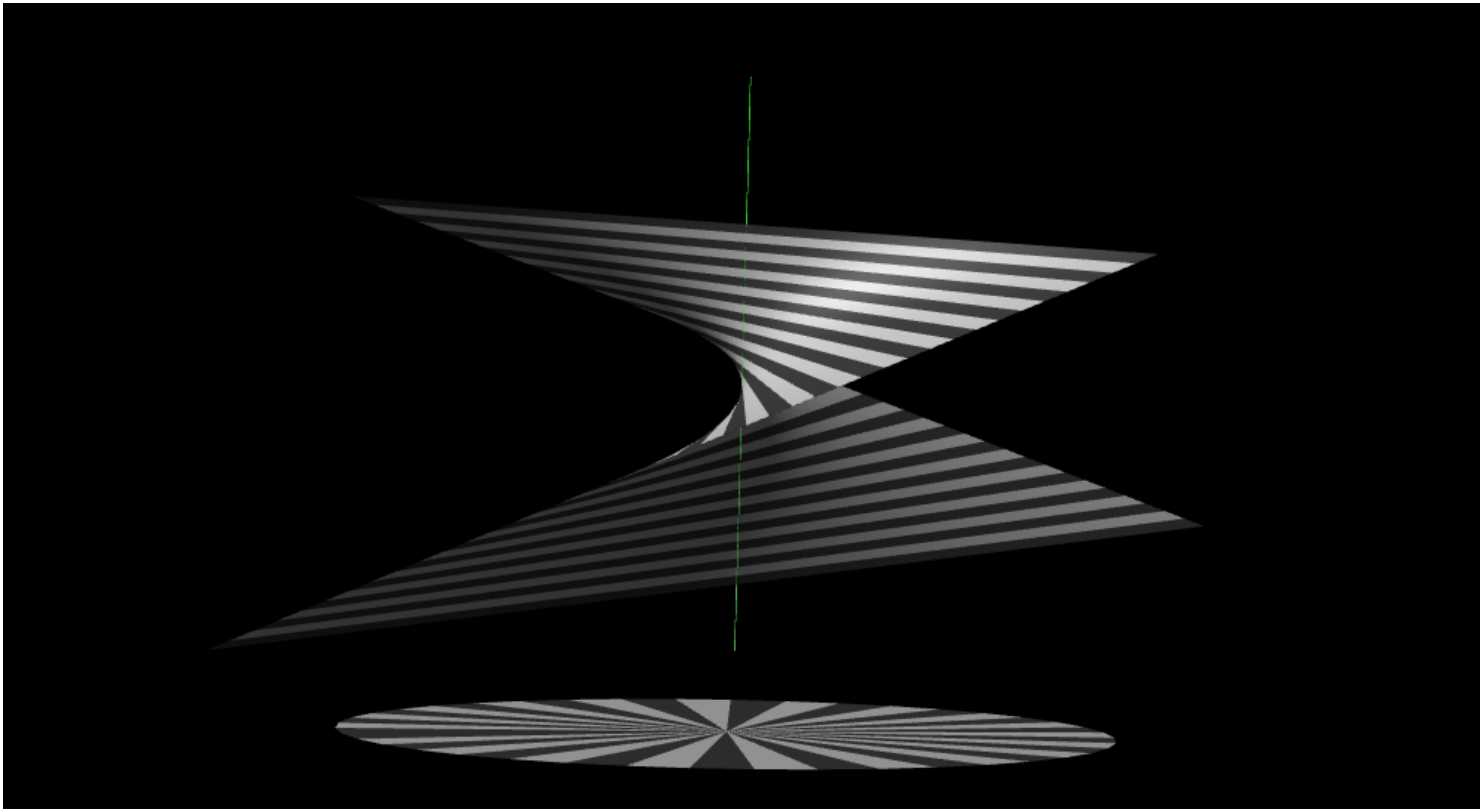

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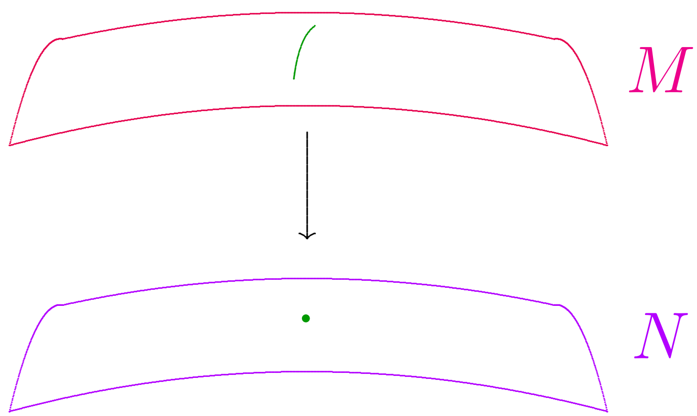

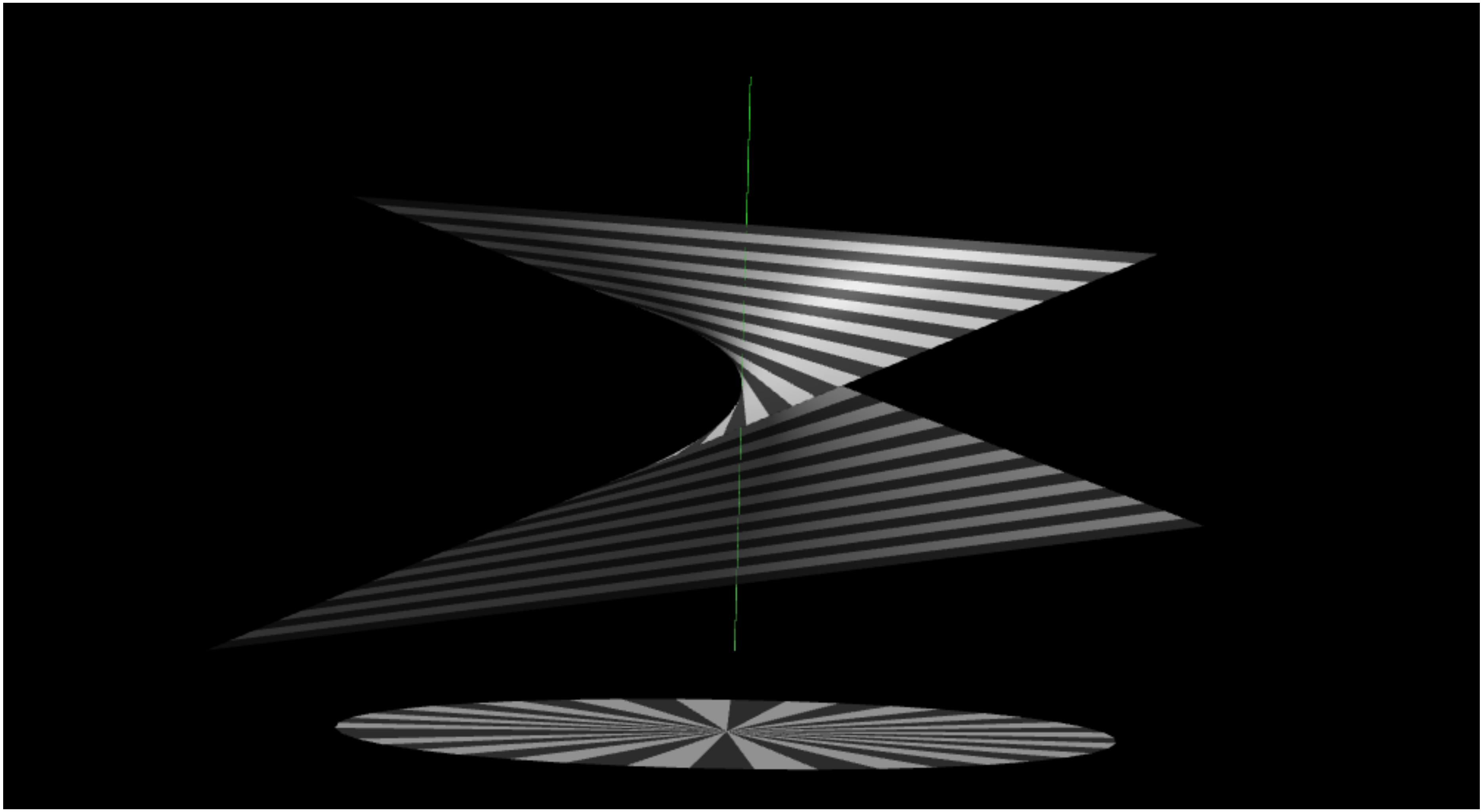

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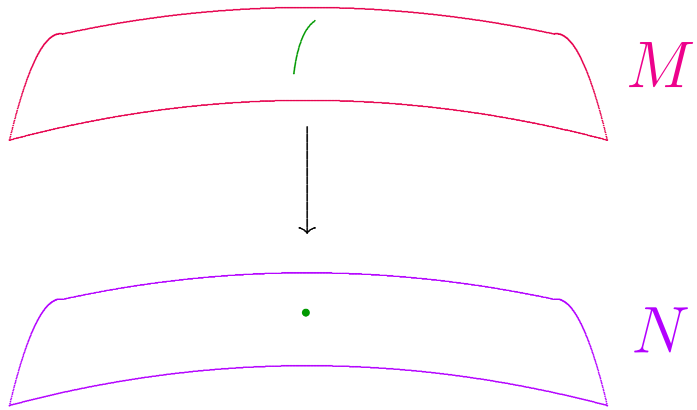

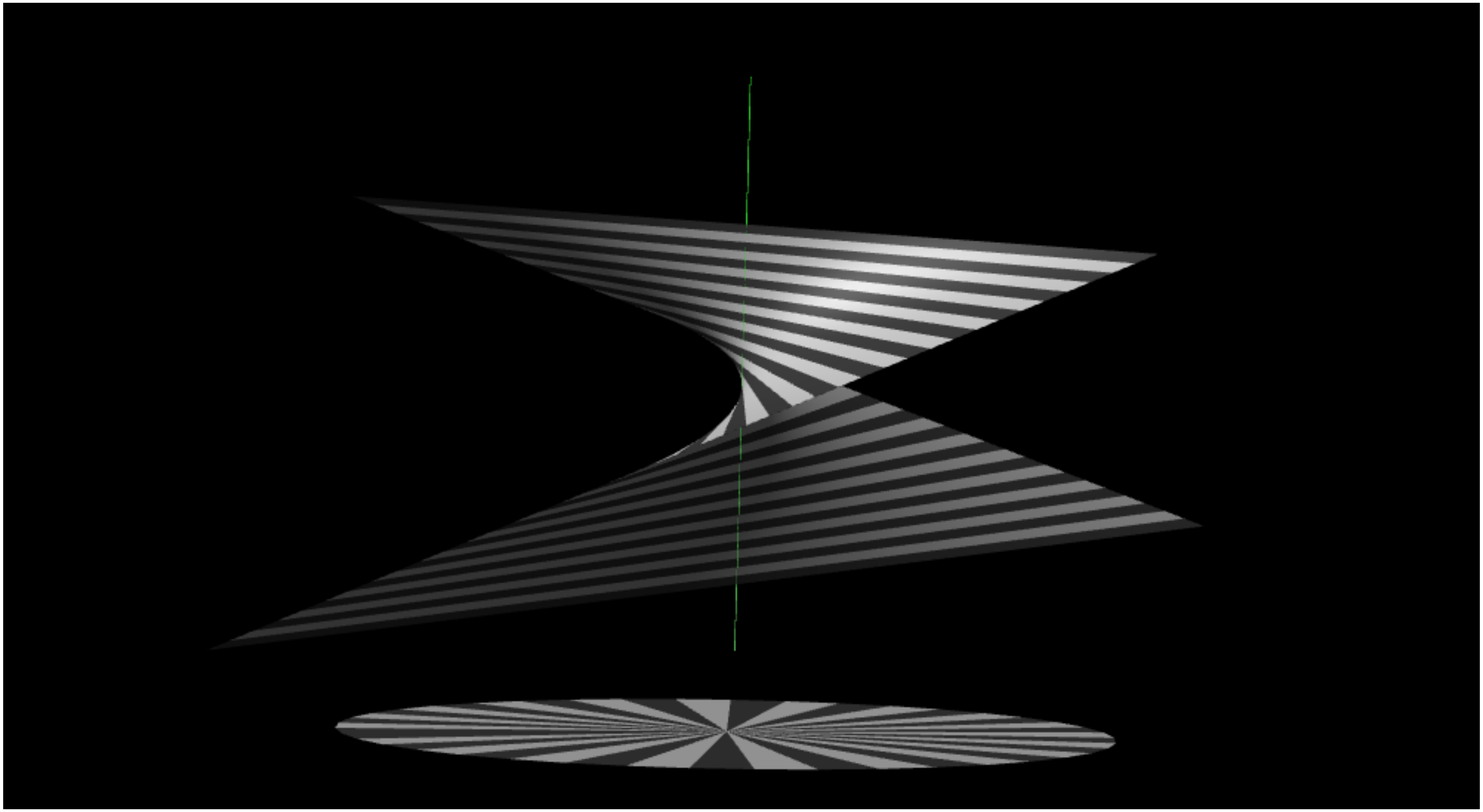

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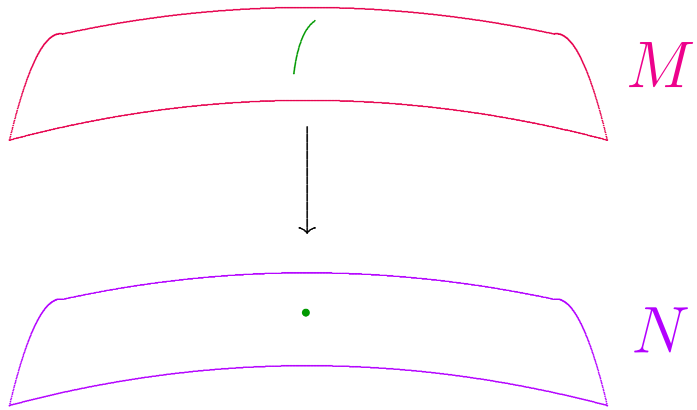
If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

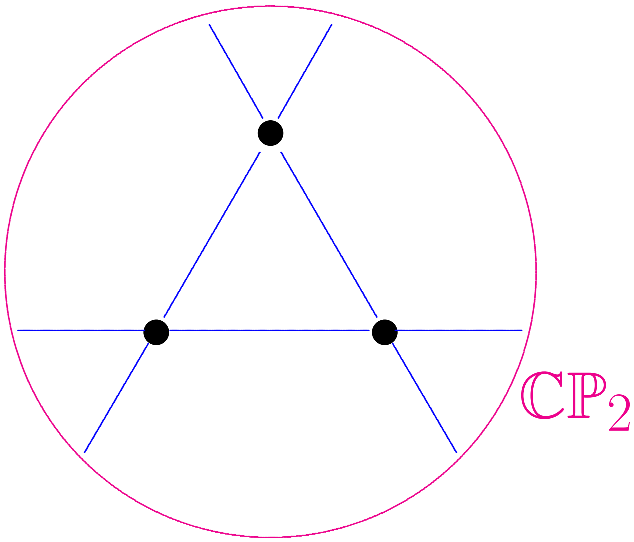
in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.



Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



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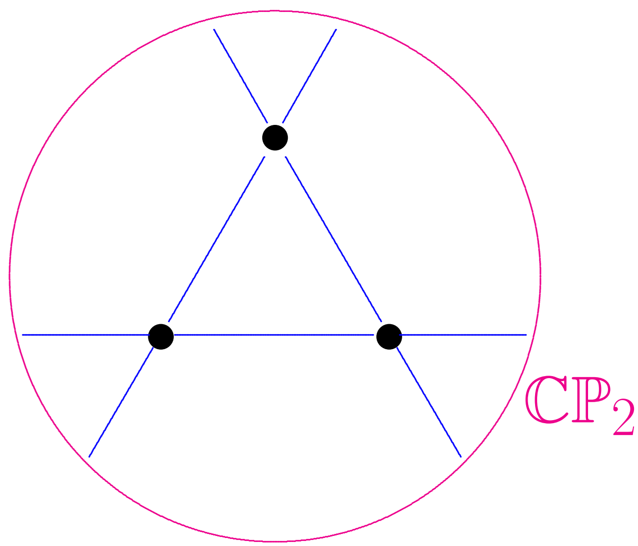

No 3 on a line,

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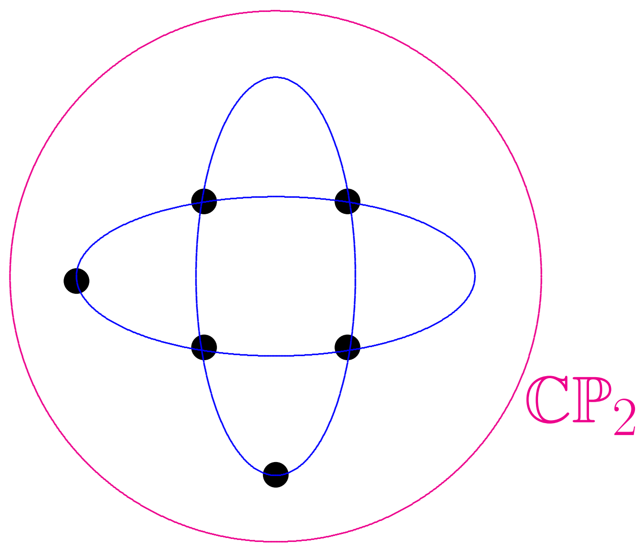

No 3 on a line, no 6 on conic,

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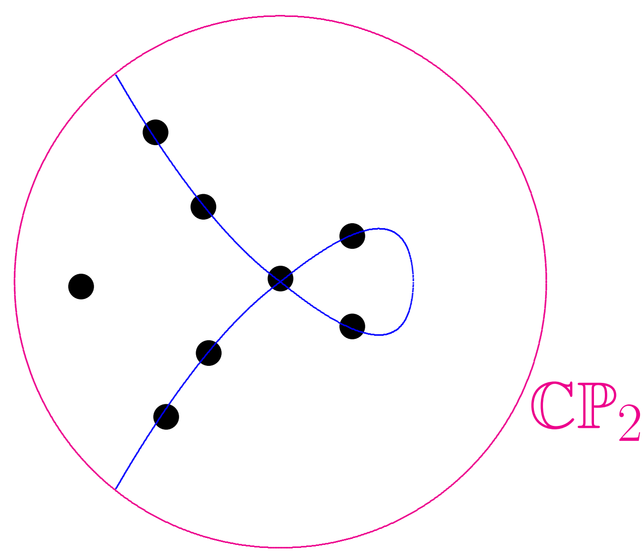

No 3 on a line, no 6 on conic, no 8 on nodal cubic.

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Existence: Page

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Existence: Page-Derdziński,

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One reason this seems satisfying. . .

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But this is not needed in above result.

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Symplectic type:
Harmonic self-dual 2-form ω is everywhere $\neq 0$.

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Absolute minimizers among conformal classes $[g]$ of symplectic type, when h is K-E.

Also true in non-K-E cases, among toric conformal classes $[g]$ of symplectic type.

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Understand all Einstein metrics on del Pezzos.

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Is Einstein moduli space connected?

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Exactly one connected component of moduli space!

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everywhere on M, *for* ω *an arbitrary non-trivial global self-dual harmonic 2-form.*

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Corollary. *These known Einstein metrics on any del Pezzo* M^4 *sweep out exactly one connected component of the Einstein moduli space* $\mathscr{E}(M)$*.*

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Kähler $\Longrightarrow \Lambda^+ = \mathbb{R}\omega \oplus \Re e \Lambda^{2,0}$

$$W^+ = \text{trace-free part of } \begin{bmatrix} 0 & & \\ & 0 & \\ & & \frac{s}{4} \end{bmatrix}$$

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$$W^+ = \begin{bmatrix} -\frac{s}{12} & & \\ & -\frac{s}{12} & \\ & & \frac{s}{6} \end{bmatrix}$$

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for these metrics & conformal rescalings:

$$g \rightsquigarrow h = f^2 g \quad \Longrightarrow \quad \det(W^+) \rightsquigarrow f^{-6} \det(W^+).$$

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method also proves more general results.

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L (2021b): related classification result.

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Let $\alpha \geq \beta \geq \gamma$ be eigenvalues of W^+:

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Claim: (M, h) compact Einstein $\Longrightarrow$ J integrable.

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$$W^+ : \Lambda^+ \to \Lambda^+$$

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Corollary. *Every simply-connected compact oriented Einstein* (M^4, h) *with* $\det(W^+) > 0$ *is diffeomorphic to a del Pezzo surface. Conversely, every del Pezzo* M^4 *carries Einstein* h *with* $\det(W^+) > 0$*, and these sweep out exactly one connected component of moduli space* $\mathscr{E}(M)$*.*

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Simply connected hypothesis is essential!

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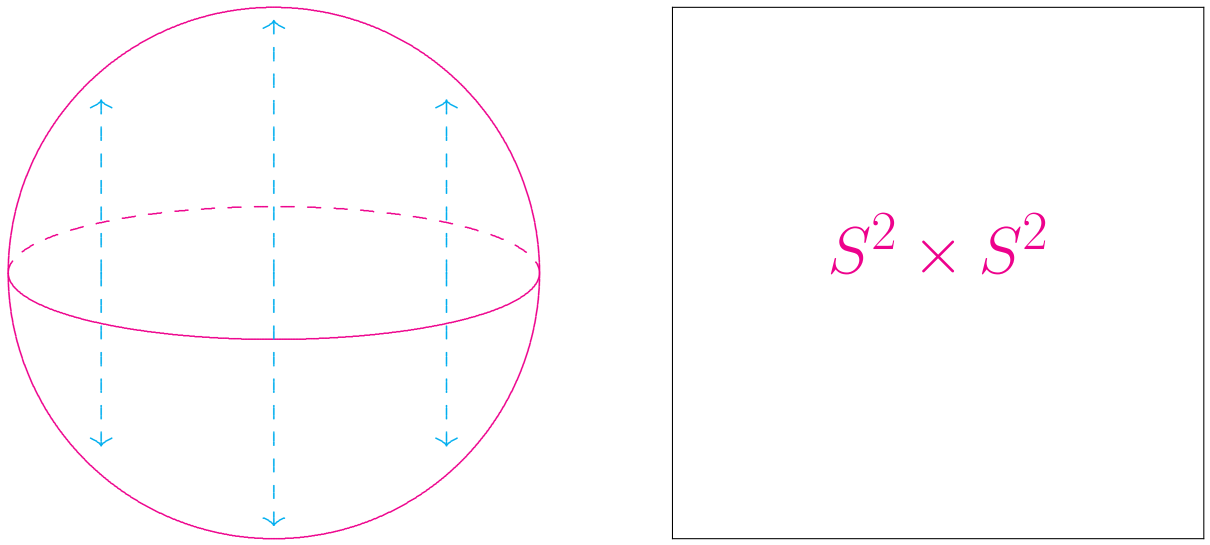

Oriented spin 4-manifold

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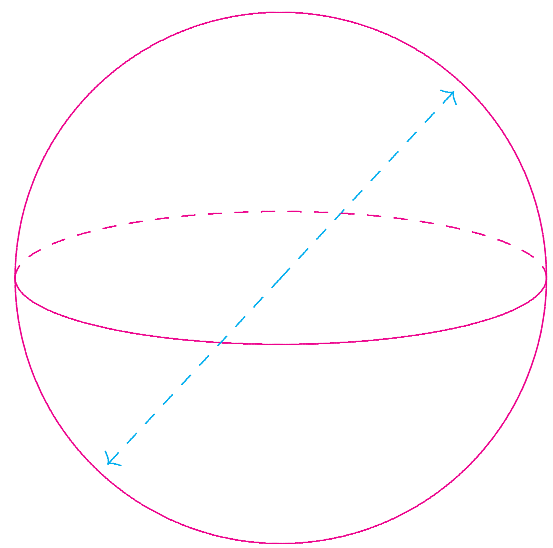

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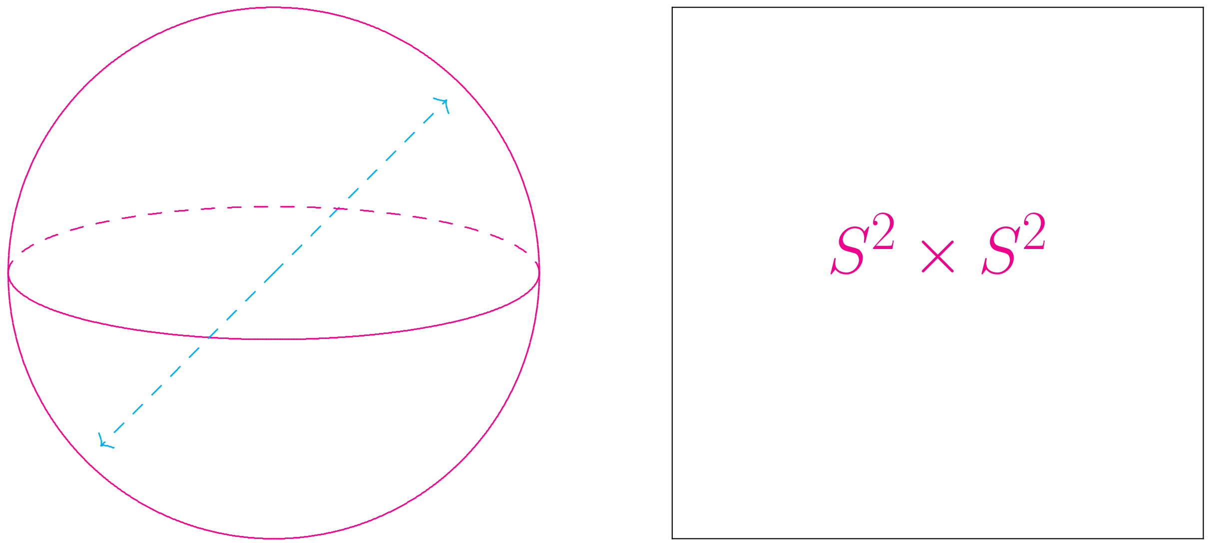
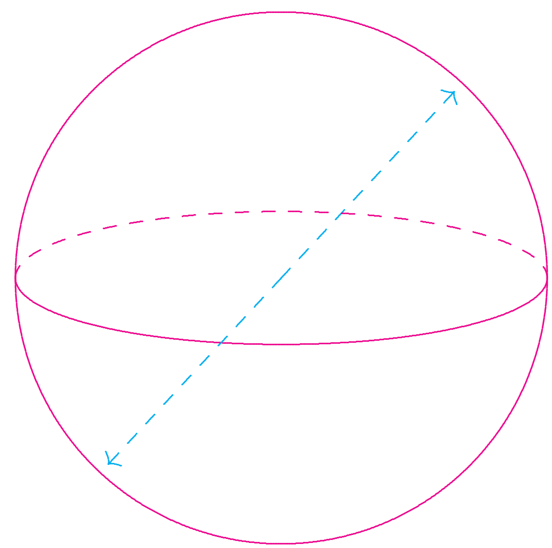


Non-spin 4-manifold

$\mathscr{Q} := (S^2 \times S^2)/\langle \mathfrak{a} \times \mathfrak{a} \rangle$



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Why is $\mathscr{E}_{\det}(M) \subset \mathscr{E}(M)$ open and closed?

Open: $\det(W^+) > 0$.

Closed: $\det(W^+) = \frac{1}{3\sqrt{6}}|W^+|^3$ and $s \geq 0$.

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Theorem E. *Let* (M, h) *be a compact oriented Einstein* 4*-manifold. If*

$$\det(W^+) > -\frac{5\sqrt{2}}{21\sqrt{21}}|W^+|^3$$

everywhere on M*, then actually* $\det(W^+) > 0$*. Consequently, all the results described remain true if we merely impose this ostensibly weaker hypothesis.*

Some indication of the proof:

For clarity, let's just assume $\det(W^{+}) > 0 \ldots$

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as proxy for Einstein equation.

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with $\omega \otimes \omega$, and integrate by parts. This yields:

$$0 = \int_M \left[\langle W^+, \nabla^*\nabla(\omega\otimes\omega)\rangle + \frac{s}{2}W^+(\omega, \omega) - 6|W^+(\omega)|^2 + 2|W^+|^2|\omega|^2\right] f \, d\mu$$

Let $\alpha \geq \beta \geq \gamma$ be eigenvalues of W^+:

$$W^+ = \begin{bmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{bmatrix}$$

$$\alpha + \beta + \gamma = 0$$

$$\alpha > 0, \quad \gamma < 0, \quad \text{if } W^+ \neq 0$$

$$\det(W^+) = \alpha\beta\gamma$$

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Now choose $\omega \in \Gamma\Lambda^+$ so that

$$W_g^+(\omega) = \alpha\,\omega, \quad |\omega|_g \equiv \sqrt{2},$$

after at worst passing to double cover $\hat{M} \to M$.

$$0 = \int_{\hat{M}} \Big[\langle W^+, \nabla^* \nabla (\omega \otimes \omega) \rangle + \frac{s}{2} W^+(\omega, \omega) - 6 |W^+(\omega)|^2 + 2 |W^+|^2 |\omega|^2 \Big] f \, d\mu$$

$$0 = \int_M \Big[\langle W^+, \nabla^*\nabla(\omega\otimes\omega)\rangle + \frac{s}{2}W^+(\omega,\omega) - 6|W^+(\omega)|^2 + 2|W^+|^2|\omega|^2\Big] f \, d\mu$$

$$0 = \int_M \Big[-2W^+(\nabla_e\omega, \nabla^e\omega) - 2W^+(\omega, \nabla^e\nabla_e\omega) + \frac{s}{2}W^+(\omega,\omega) - 6|W^+(\omega)|^2 + 2|W^+|^2|\omega|^2 \Big] f \; d\mu$$

$$0 = \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) - 2\alpha\langle \omega, \nabla^e \nabla_e \omega\rangle + \frac{s}{2}\alpha|\omega|^2 - 6\alpha^2|\omega|^2 + 2|W^+|^2|\omega|^2 \Big] f \, d\mu$$

because

$$W_g{}^+(\omega) = \alpha\omega$$

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$$0 \geq \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) + 2\alpha \langle \omega, \nabla^* \nabla \omega \rangle$$
$$+ \frac{s}{2}\alpha |\omega|^2 - 6\alpha^2 |\omega|^2 + 3\alpha^2 |\omega|^2 \Big] f \; d\mu$$

because

$$|W_g{}^+|^2 \geq \frac{3}{2}\alpha^2$$

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$$|\omega|^2_g = 2 \quad \Longrightarrow \quad (\nabla_e \omega) \perp \omega$$

$$\det(W^+) > 0 \quad \Longrightarrow \quad W^+ \sim \begin{bmatrix} + & & \\ & - & \\ & & - \end{bmatrix}$$

$$0 \geq \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) + 2\alpha \langle \omega, \nabla^* \nabla \omega \rangle + \frac{s}{2}\alpha|\omega|^2 - 3\alpha^2|\omega|^2 \Big] f \; d\mu$$

$$|\omega|_g^2 = 2 \implies (\nabla_e \omega) \perp \omega$$

$$\det(W^+) > 0 \implies W^+(\nabla_e \omega, \nabla^e \omega) \leq 0$$

$$0 \geq \int_M \Big[\qquad 2\alpha\langle\omega, \nabla^*\nabla\omega\rangle$$

$$+\frac{s}{2}\alpha|\omega|^2 - 3\alpha^2|\omega|^2\Big] f \; d\mu$$

$$|\omega|_g^2 = 2 \quad \Longrightarrow \quad (\nabla_e\omega) \perp \omega$$

$$\det(W^+) > 0 \quad \Longrightarrow \quad -W^+(\nabla_e\omega, \nabla^e\omega) \geq 0$$

$$0 \geq \int_M \left[2\alpha \langle \omega, \nabla^* \nabla \omega \rangle + \frac{s}{2} \alpha |\omega|^2 - 3\alpha^2 |\omega|^2 \right] f \, d\mu$$

$$0 \geq \int_M \left[2\langle \omega, \nabla^* \nabla \omega \rangle + \frac{s}{2}|\omega|^2 - 3\alpha|\omega|^2 \right] (\alpha f) \, d\mu$$

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But

$$\alpha f \equiv 1$$

$$0 \geq \int_M \left[2\langle \omega, \nabla^* \nabla \omega \rangle + \frac{s}{2} \alpha |\omega|^2 - 3|\omega|^2 \alpha \right] d\mu$$

$$0 \geq \int_M \left[2\langle \omega, \nabla^* \nabla \omega \rangle - 3W^+(\omega, \omega) + \frac{s}{2}|\omega|^2 \right] d\mu$$

$$0 \geq \int_M \left[\frac{1}{2}|\nabla\omega|^2 + \frac{3}{2}\langle\omega, \left(\nabla^*\nabla - 2W^+ + \frac{s}{3}\right)\omega\rangle\right] d\mu$$

$$0 \geq \int_M \left[\tfrac{1}{2}|\nabla\omega|^2 + \tfrac{3}{2}\langle\omega, \quad (d+d^*)^2 \quad \omega\rangle\right] d\mu$$

Because

$$(d+d^*)^2 = \nabla^*\nabla - 2W^+ + \frac{s}{3}$$

on $\Gamma\Lambda^+$.

$$0 \geq \frac{1}{2} \int_M |\nabla \omega|^2 \, d\mu + 3 \int_M |d\omega|^2 \, d\mu$$

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So $\nabla\omega \equiv 0$, and g is Kähler!

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It's such a pleasure to be here!