Einstein Manifolds

and

Extremal Kähler Metrics

Claude LeBrun
Stony Brook University
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“...the greatest blunder of my life!”

— A. Einstein, to G. Gamow
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Has same sign as the *scalar curvature*

$$ s = r^j_j = R^{ij} _ { ij}. $$
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in which new \( \mathbb{C}P_1 \) has self-intersection \(-1\).
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**Diffeotypes:** Del Pezzo surfaces. ($\exists J$ with $c_1 > 0$.)
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Full K-E moduli space: Tian, Chen-Wang.
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Of course, $\mathbb{CP}^2$ and $S^2 \times S^2$ also admit K-E metrics with $\lambda > 0$ — namely, obvious homogeneous ones!
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(Matsushima):

$(M, J, g)$ compact K-E $\implies \text{Aut}(M, J)$ reductive.
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Since $\mathbb{CP}_2 \# \overline{\mathbb{CP}_2}$ and $\mathbb{CP}_2 \# 2\overline{\mathbb{CP}_2}$ have non-reductive automorphism groups, no K-E metrics.
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**Theorem** (Chen-LeBrun-Weber ’08). *There is a $\lambda > 0$, conformally Kähler, Einstein metric $h$ on $\mathbb{CP}_2 \# 2\overline{\mathbb{CP}_2}$.*
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Note both of above Einstein metrics are Hermitian.
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- \(M \approx \mathbb{CP}_2 \# 2\overline{\mathbb{CP}}_2\) and \(h\) is a constant times the CLW metric.
Proposition (L ’96). Let $(M^4, J)$ be a compact complex surface, and suppose that $h$ is an Einstein metric on $M$ which is Hermitian with respect to $J$:

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- \(h\) has positive Einstein constant;
- \(g\) is an extremal Kähler metric;
- \(g\) has scalar curvature \(s > 0\); and
- after normalization, \(h = s^{-2}g\).
Ingredients:
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• **Goldberg-Sachs Theorem**
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  \(-T^{1,0}M\) isotropic, integrable, \(\nabla^a(W_+)_{abcd} = 0\)
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- Automorphism group non-trivial, non-semi-simple.
  \[-g \text{ is extremal, } s \text{ non-constant.}\]
Calabi: $\text{Iso}(g) \subset \text{Aut}(M)$ maximal compact.
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**Proposition.** *Up to automorphisms and rescaling, there is exactly one conformally Kähler, Einstein metric $h$ on $M = \mathbb{C}P_2 \# \overline{\mathbb{C}P}_2$, namely the Page metric.*
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**Proposition.** *Up to automorphisms and rescaling, there is exactly one conformally Kähler, Einstein metric \( h \) on \( M = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2} \), namely the Page metric.*

But need new ideas to prove the following...
Theorem 1. *Up to automorphisms and rescaling,*
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Extremal Kähler metrics = critical points of

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where \( g = g_\omega \) for \( J \) and \( [\omega] \in H^2(M, \mathbb{R}) \) fixed.
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\( \nabla^{1,0} s \) is a holomorphic vector field.
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\( J \nabla s \) is a Killing field.
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X.X. Chen: always minimizers.
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Donaldson/Mabuchi/Chen-Tian:
unique in Kähler class, modulo bihomorphisms.
Explicit lower bound:

Any Kähler \((M^4, g, J)\) satisfies

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\frac{1}{32\pi^2} \int s^2 d\mu_g \geq \mathcal{A}([\omega])
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\[
\mathcal{A}([\omega]) := \frac{(c_1 \cdot [\omega])^2}{[\omega]^2} + \frac{1}{32\pi^2} \| \mathcal{F}_{[\omega]} \|^2
\]

where \(\mathcal{F}\) is Futaki invariant.
$(M, J)$ Del Pezzo. $\mathcal{K} \subset H^2(M, \mathbb{R})$ Kähler cone.
Proposition. Suppose that $h$ is an Einstein metric on $M$ which is conformally related to a $J$-compatible Kähler metric $g$ with Kähler class $[\omega] = \Omega \in \mathcal{K}$. 

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**Proposition.** Suppose that \(h\) is an Einstein metric on \(M\) which is conformally related to a \(J\)-compatible Kähler metric \(g\) with Kähler class \([\omega] = \Omega \in \mathcal{K}\). Then \(\Omega\) is a critical point of

\[ \mathcal{A} : \mathcal{K} \to \mathbb{R}. \]
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Moreover, \( g \) is an extremal Kähler metric, and the scalar curvature \( s \) of \( g \) is everywhere positive.

Conversely, if \( \Omega \in \mathcal{K} \) is a critical point of \( \mathcal{A} \), and if \( \omega \in \Omega \) is the Kähler form of an extremal Kähler metric \( g \) with scalar curvature \( s > 0 \), then \( h = s^{-2}g \) is an Einstein metric on \( M \).
$(M, J)$ Del Pezzo. $\mathcal{K} \subset H^2(M, \mathbb{R})$ Kähler cone.

**Proposition.** Suppose that $h$ is an Einstein metric on $M$ which is conformally related to a $J$-compatible Kähler metric $g$ with Kähler class $[\omega] = \Omega \in \mathcal{K}$. Then $\Omega$ is a critical point of $A : \mathcal{K} \to \mathbb{R}$.

Moreover, $g$ is an extremal Kähler metric, and the scalar curvature $s$ of $g$ is everywhere positive.

Conversely, if $\Omega \in \mathcal{K}$ is a critical point of $A$, and if $\omega \in \Omega$ is the Kähler form of an extremal Kähler metric $g$ with scalar curvature $s > 0$, then $h = s^{-2}g$ is an Einstein metric on $M$.

**Lemma.** For any extremal Kähler $g$ on any Del Pezzo $M$, scalar curvature $s > 0$ everywhere.
$(M, J)$ Del Pezzo. $\mathcal{K} \subset H^2(M, \mathbb{R})$ Kähler cone.

**Proposition.** Suppose that $h$ is an Einstein metric on $M$ which is conformally related to a $J$-compatible Kähler metric $g$ with Kähler class $[\omega] = \Omega \in \mathcal{K}$. Then $\Omega$ is a critical point of

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Moreover, $g$ is an extremal Kähler metric.

Conversely, if $\Omega \in \mathcal{K}$ is a critical point of $A$, and if $\omega \in \Omega$ is the Kähler form of an extremal Kähler metric $g$,

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**Lemma.** For any extremal Kähler $g$ on any Del Pezzo $M$, scalar curvature $s > 0$ everywhere.
Special character of dimension 4:
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\(\Lambda^+\) self-dual 2-forms.
\(\Lambda^-\) anti-self-dual 2-forms.
Riemann curvature of $g$

$\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$
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$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

\[
\begin{array}{cc}
\Lambda^+ & \Lambda^{*+} \\
W_+ + \frac{s}{12} & W_0 \\
\Lambda^- & \Lambda^{*-} \\
\hat{r} & \hat{r}
\end{array}
\]
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where

$s = \text{scalar curvature}$

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![Diagram of Riemann curvature components]

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\[
W_+ = \begin{pmatrix}
-\frac{s}{12} & \frac{s}{6} \\
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\[ |W_+|^2 = \frac{s^2}{24} \]
The Bach Tensor

Conformally invariant Riemannian functional:

\[ \mathcal{W}(g) = \int_{M} |W|_{g}^{2} d\mu_{g}. \]
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\[ \mathcal{W}(g) = 2 \int_M |W_+|^2 d\mu_g - 12\pi^2 \tau(M) \]
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1-parameter family of metrics

$$g_t := g + t\dot{g} + O(t^2)$$
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$$\left. \frac{d}{dt} \mathcal{W}(g_t) \right|_{t=0} = -\int \dot{g}^{ab} B_{ab} d\mu_g$$
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is the Bach tensor of $g$. 
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\[ \nabla^a B_{ab} = 0 \]
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Conformally Einstein \( \implies B = 0 \)
Restriction of $\mathcal{W}$ to Kähler metrics?
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Now for an extremal Kähler metric

$$B = \frac{1}{12} \left[ s\dot{r} + 2\text{Hess}_0(s) \right]$$
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Now for an extremal Kähler metric

$$B = \frac{1}{12} \left[ s\dot{r} + 2\text{Hess}_0(s) \right]$$

and corresponds to harmonic primitive $(1, 1)$-form

$$\psi := B(J\cdot, \cdot) = \frac{1}{12} \left[ s\rho + 2i\partial\bar{\partial}s \right]_0$$
Restriction of $\mathcal{W}$ to Kähler metrics.
Restriction of $\mathcal{W}$ to Kähler metrics.

Hence if $g$ is extremal Kähler metric,

$$g_t = g + tB$$

is a family of Kähler metrics,
Restriction of $\mathcal{W}$ to Kähler metrics.

Hence if $g$ is extremal Kähler metric,

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$$\left. \frac{d}{dt} \mathcal{W}(g_t) \right|_{t=0} = \int \dot{g}^{ab} B_{ab} \ d\mu_g$$

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So the critical points of restriction of $\mathcal{W}$ to \{Kähler metrics\} also have $B = 0$!
Bach-flat Kähler metrics?
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If \((M^4, J, g)\) Kähler, \(s^{-1}W_+\) parallel.
Bach-flat Kähler metrics?

If $(M^4, J, g)$ Kähler, $s^{-1}W_+$ parallel. Hence

$$\nabla^a(s^{-1}W_+)^{abcd} = 0.$$
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Conformally invariant, with appropriate weight!
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If \(g\) Bach-flat, \(h = s^{-2}g\) Einstein satisfies

\[
0 = \bar{\mathring{r}}^{cd}(W_+)_{abcd}
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If \(g\) Bach-flat, \(h = s^{-2}g\) Einstein satisfies
\[
0 = \hat{r}^{cd}(W_+)_{acbd}
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and so Einstein when \(s \neq 0\).
Bach-flat Kähler metrics?

If \((M^4, J, g)\) Kähler, \(s^{-1}W_+\) parallel. Hence
\[
\nabla^a(s^{-1}W_+)_{abcd} = 0.
\]

Conformally invariant, with appropriate weight!

Hence \(h = s^{-2}g\) satisfies
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where defined.

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Del Pezzo case: \(s \neq 0\) everywhere!
To prove uniqueness results, show that

\[ \mathcal{A} : \mathcal{K} \to \mathbb{R} \]

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Necessary calculations also led to new existence proof. . .
Theorem B. There is a Kähler metric $g$ on $\mathbb{CP}_2 \# 2\overline{\mathbb{CP}}_2$ which is conformal to an Einstein metric.
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Then there is an extremal Kähler metric $g$ on $M$ with Kähler form $\omega \in [\omega]$. 
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Theorem B follows.
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  - Yamabe trick + Gauss-Bonnet…

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  - Symplectic 2-spheres \(\rightsquigarrow\) Lagrangian 2-spheres