Einstein Constants

and

Differential Topology

Claude LeBrun
Stony Brook University

“Differentialgeometrie im Großen,”
Mathematisches Forschungsinstitut Oberwolfach,
4. Juli 2025

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— A. Einstein, to G. Gamow

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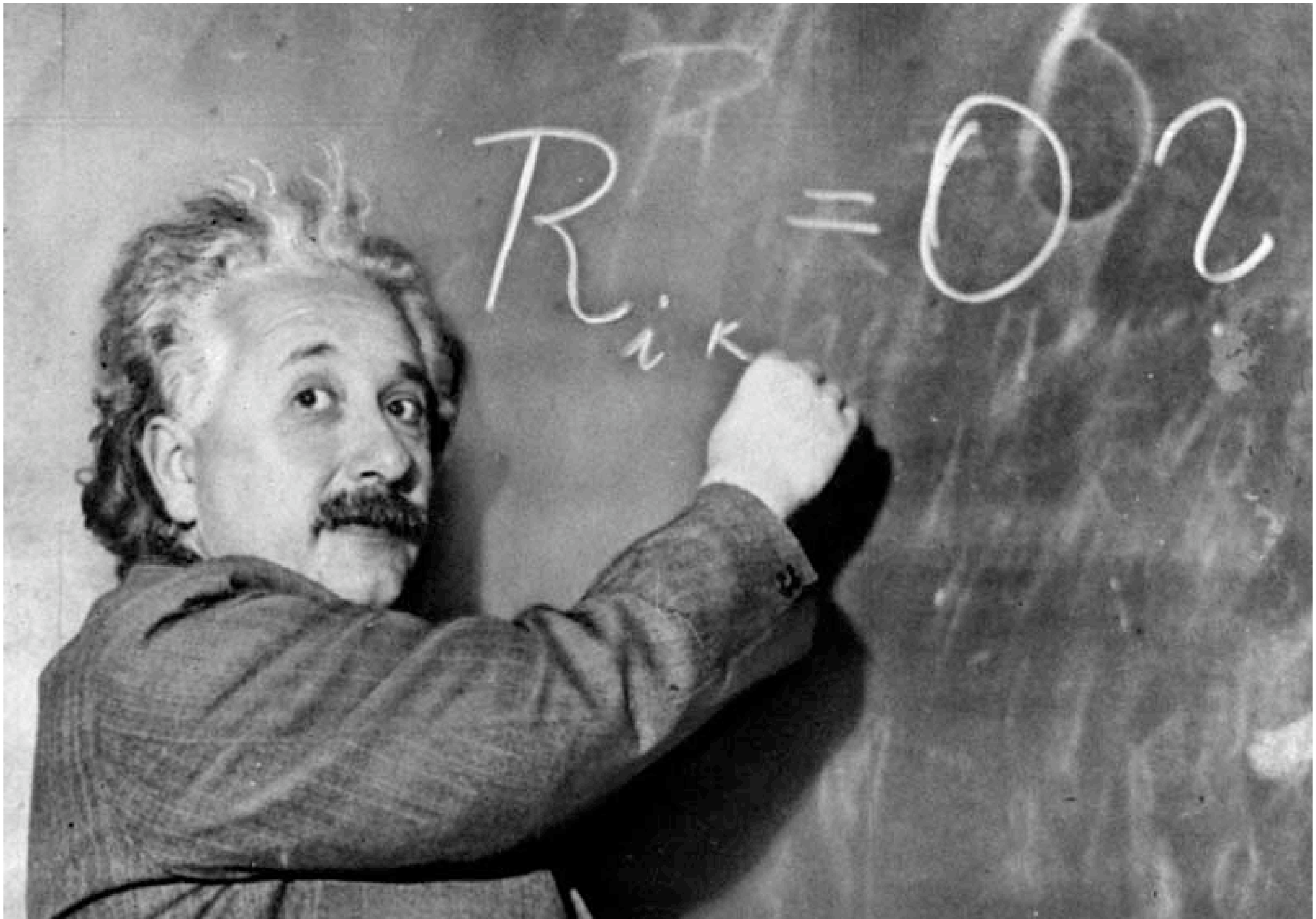

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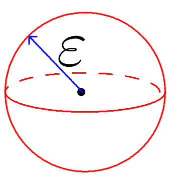

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For the purposes of this talk...

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Ergebnisse der Mathematik und ihrer Grenzgebiete
3. Folge · Band 10
A Series of Modern Surveys in Mathematics

Arthur L. Besse

Einstein Manifolds

Springer-Verlag Berlin Heidelberg GmbH

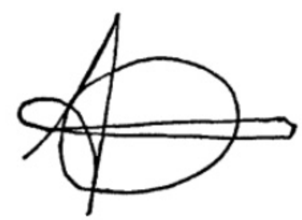
Acknowledgements

Pour rassembler les éléments un peu disparates qui constituent ce livre, j'ai dû faire appel à de nombreux amis, heureusement bien plus savants que moi. Ce sont, entre autres, Geneviève Averous, Lionel Bérard-Bergery, Marcel Berger, Jean-Pierre Bourguignon, Andrei Derdzinski, Dennis M. DeTurck, Paul Gauduchon, Nigel J. Hitchin, Josette Houillot, Hermann Karcher, Jerry L. Kazdan, Norihito Koiso, Jacques Lafontaine, Pierre Pansu, Albert Polombo, John A. Thorpe, Liane Valère.

Les institutions suivantes m'ont prêté leur concours matériel, et je les en remercie: l'UER de mathématiques de Paris 7, le Centre de Mathématiques de l'Ecole Polytechnique, Unités Associées du CNRS, l'UER de mathématiques de Chambéry et le Conseil Général de Savoie.

Enfin, qu'il me soit permis de saluer ici mon prédecesseur et homonyme Jean Besse, de Zürich, qui s'est illustré dans la théorie des fonctions d'une variable complexe (voir par exemple [Bse]).

Vôtre,



Arthur Besse

Le Faux, le 15 septembre 1986

Besse en Chandesse

L'Auvergne

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But it turns out that the answer is actually Yes!

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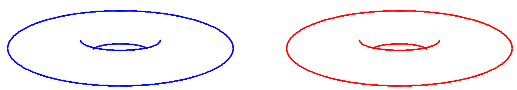

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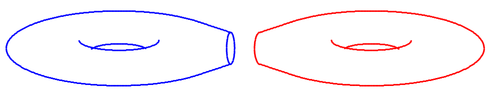

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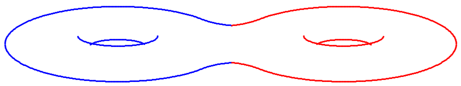

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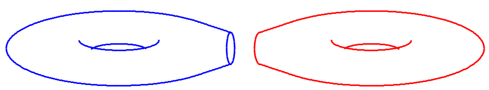

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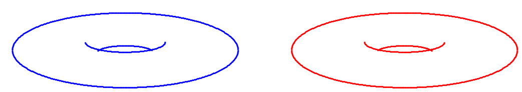

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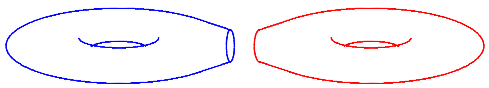

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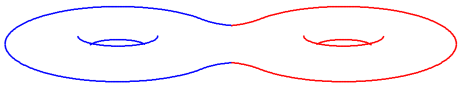

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$\Longleftrightarrow$ they have the same Einstein constant λ.

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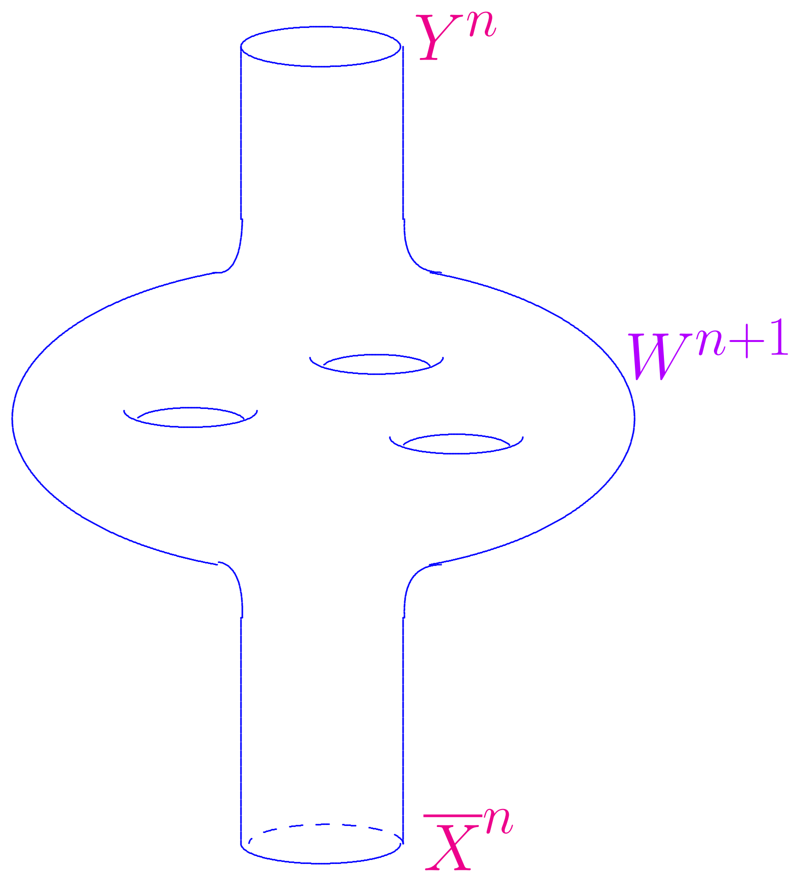
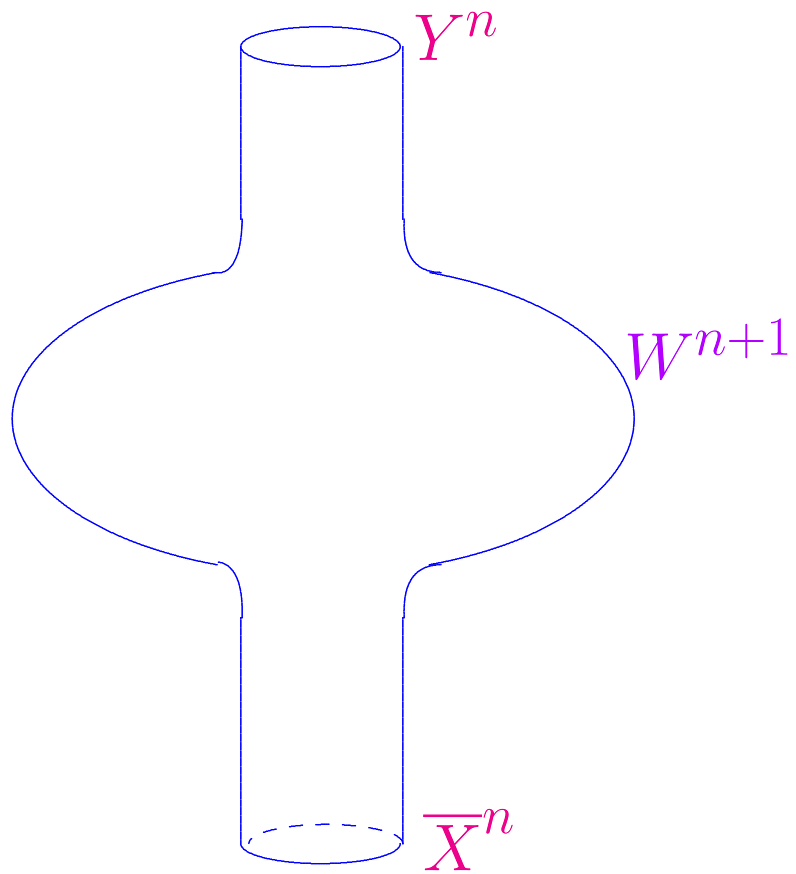
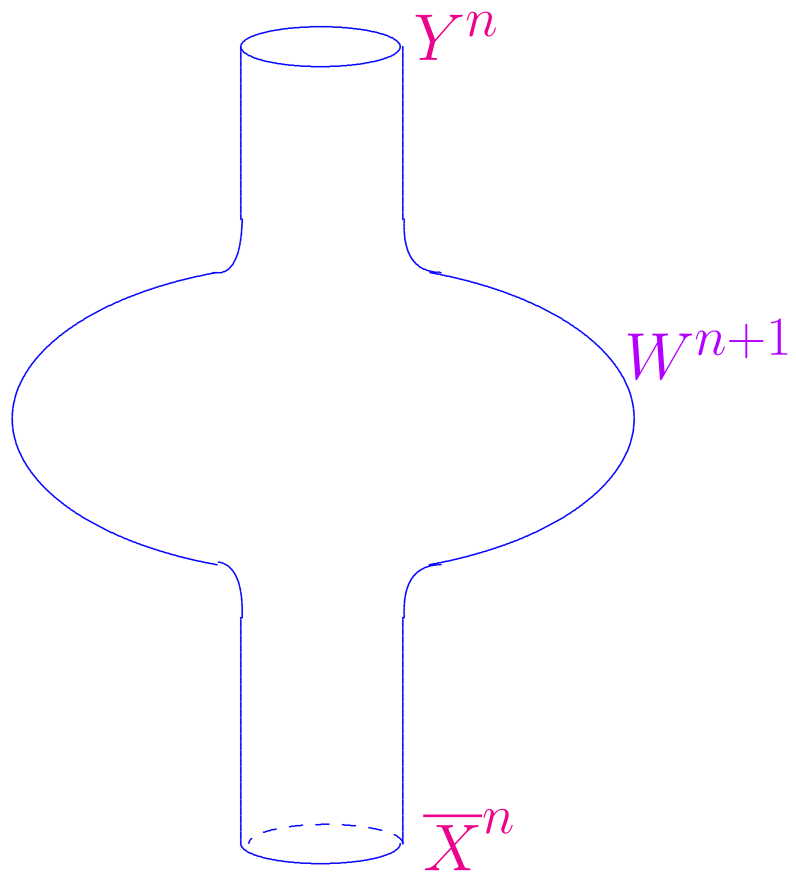


Cobordism

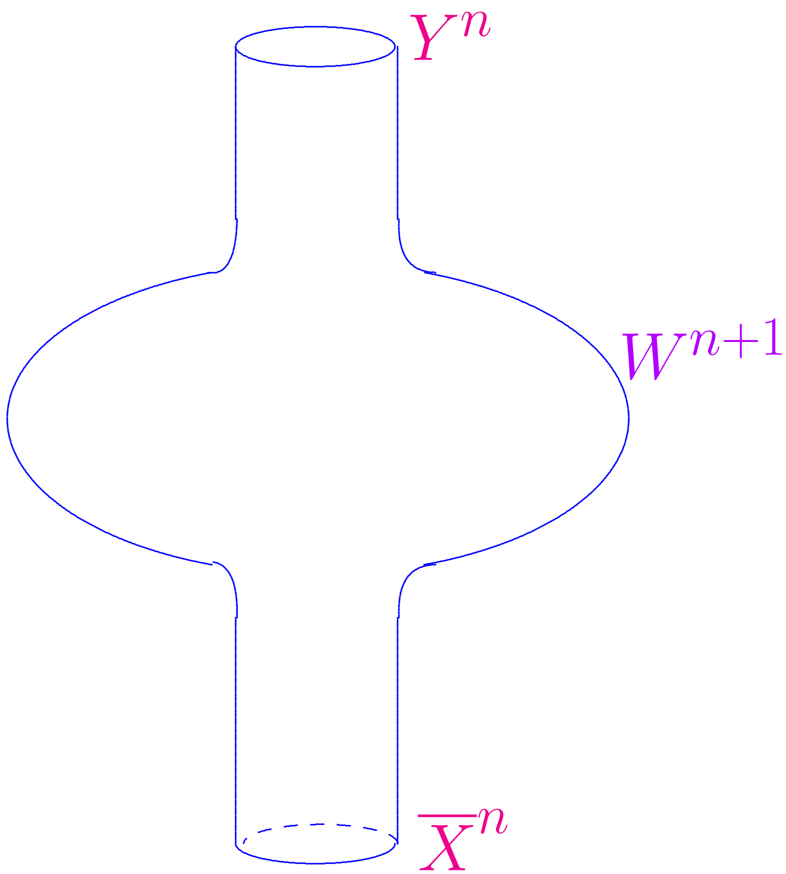


h-Cobordism

if $X \hookrightarrow W$, $Y \hookrightarrow W$ both homotopy equivalences



Smale: Suppose that X^n is h-cobordant to Y^n. If $\pi_1 = 0$ and $n > 4$, then X is diffeomorphic to Y.



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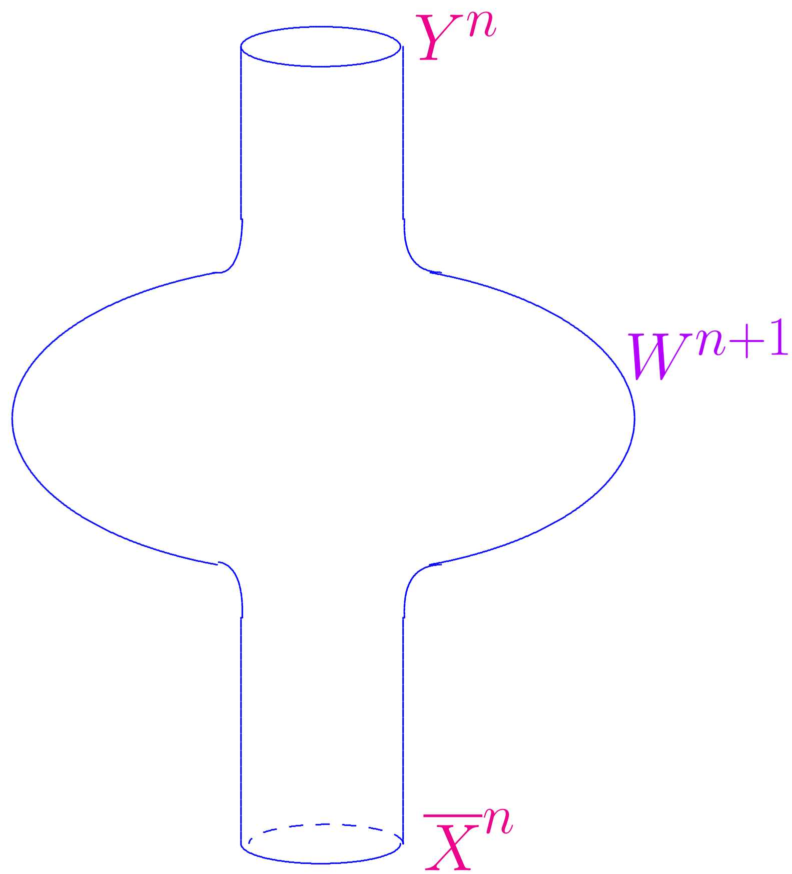

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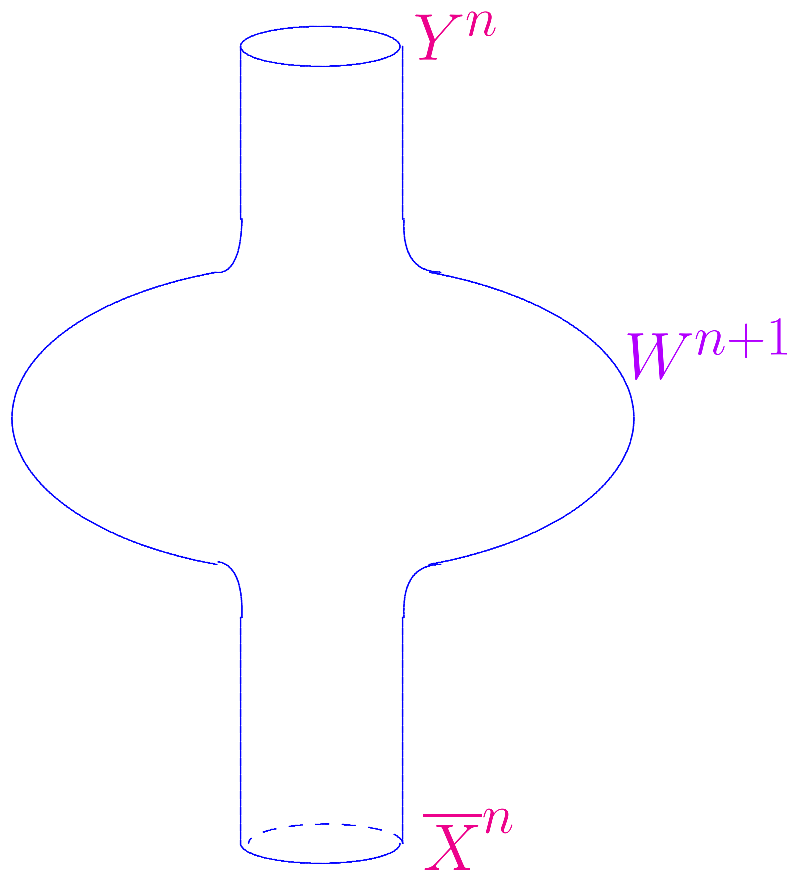
But Smale doesn't apply when $n = 4$!



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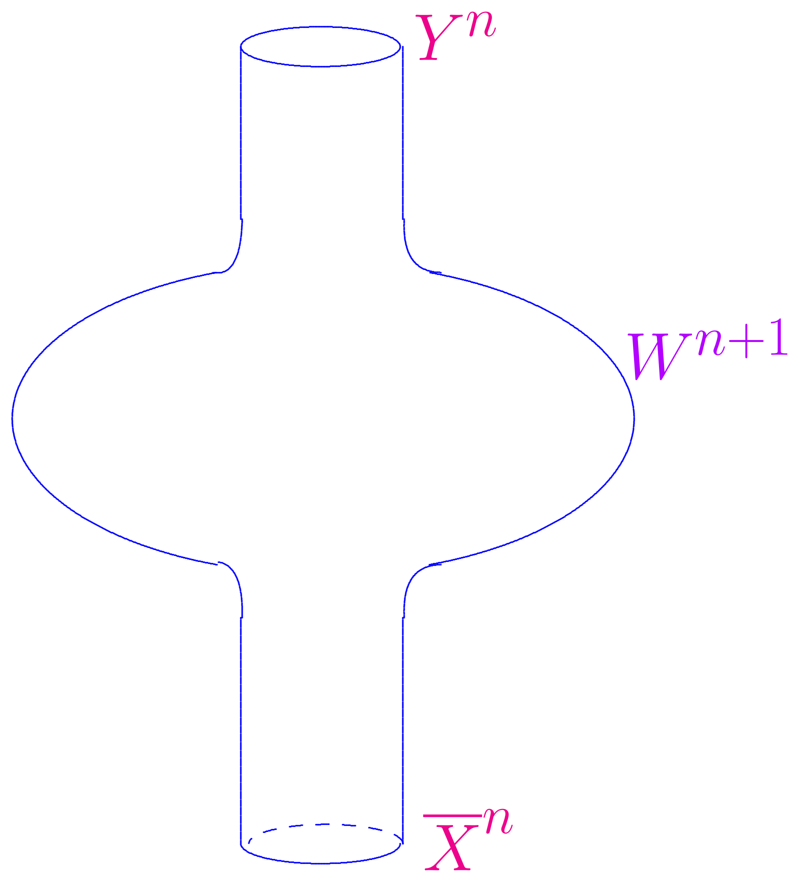

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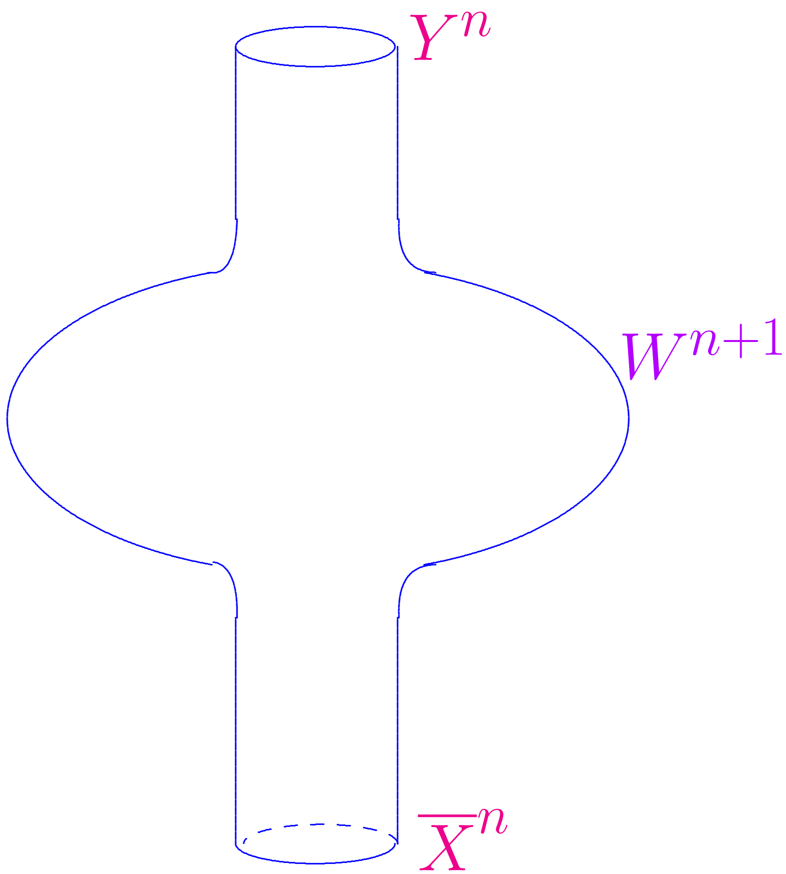
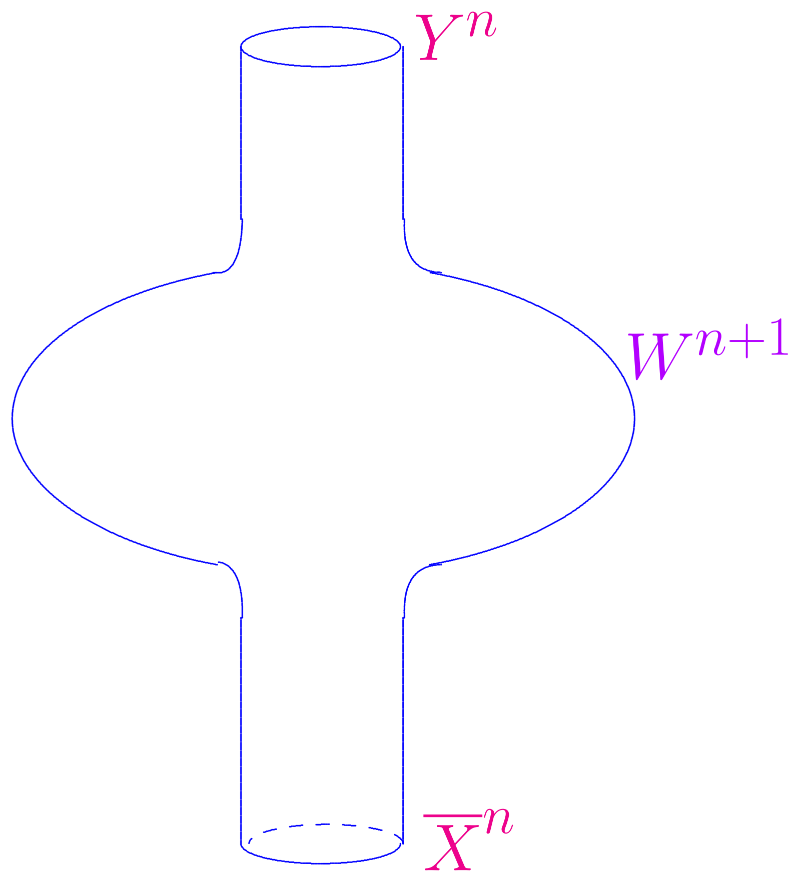
Lemma. *If X^4 and Y^4 are homotopy equivalent and simply connected, then $X \times X$ is actually diffeomorphic to $Y \times Y$.*



Indeed, if W is h-cobordism $X \sim Y$, then

$$(X \times W) \cup_{X \times Y} (W \times Y)$$

is an h-cobordism $(X \times X) \sim (Y \times Y)$.



Lemma. *If X^4 and Y^4 are homotopy equivalent and simply connected, then*

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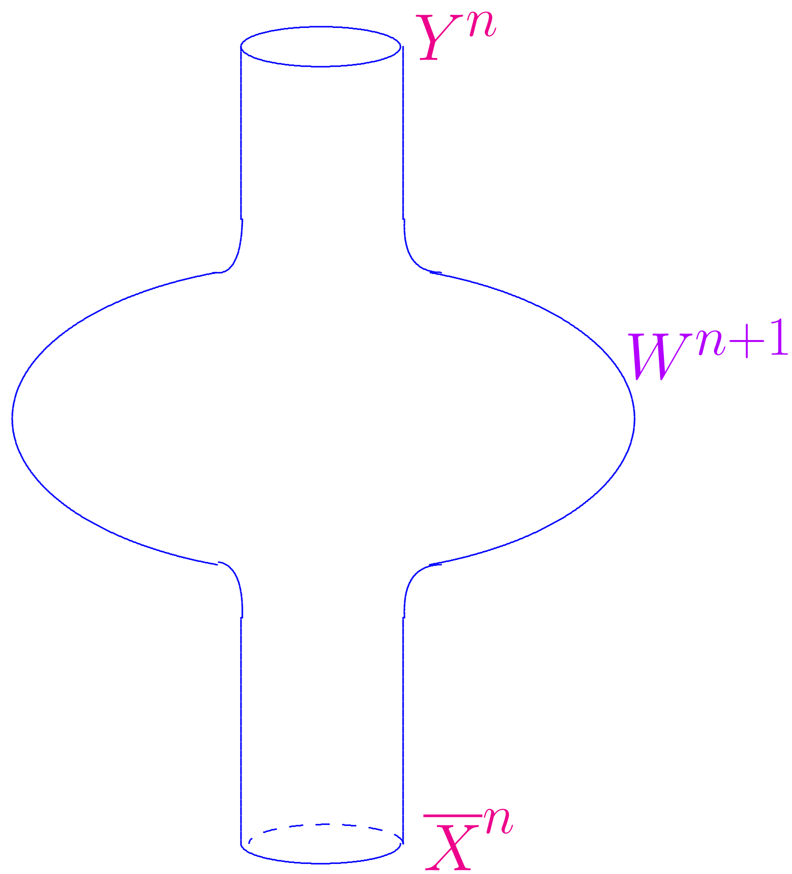

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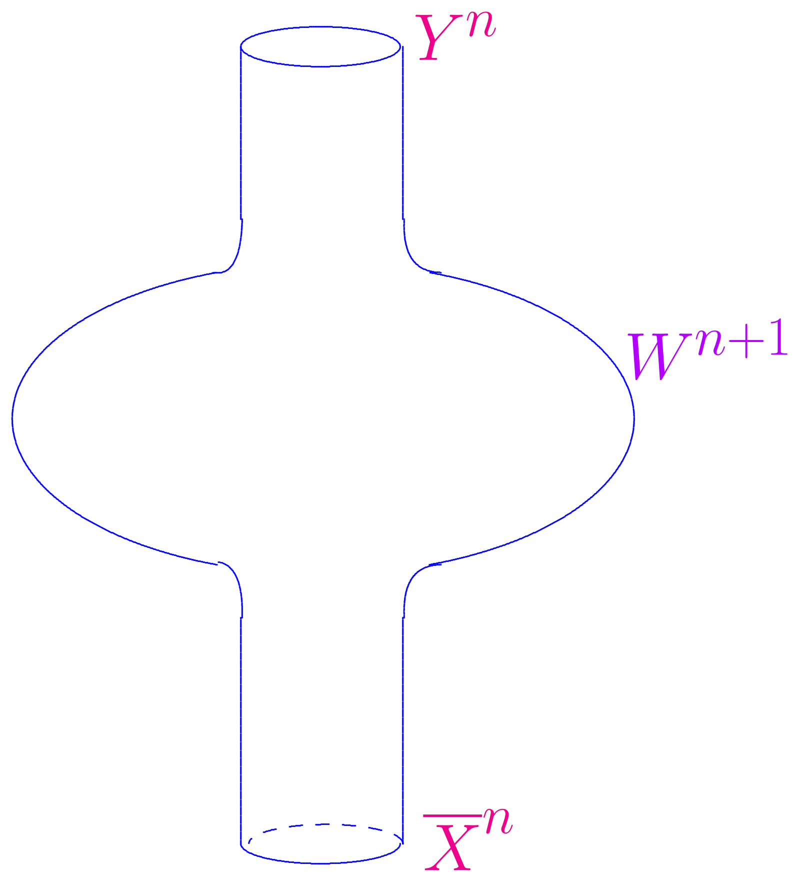

Lemma. *If X^4 and Y^4 are simply connected, non-spin, with $\chi(X) = \chi(Y)$, $\tau(X) = \tau(Y)$, then*

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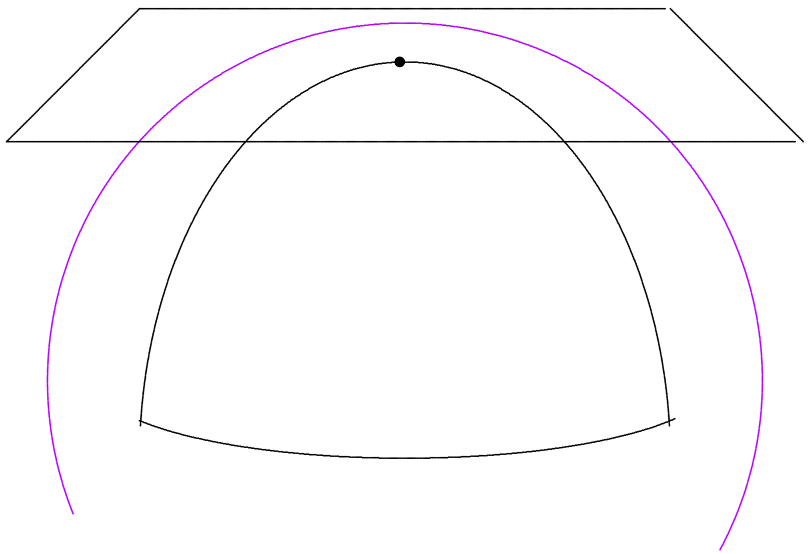

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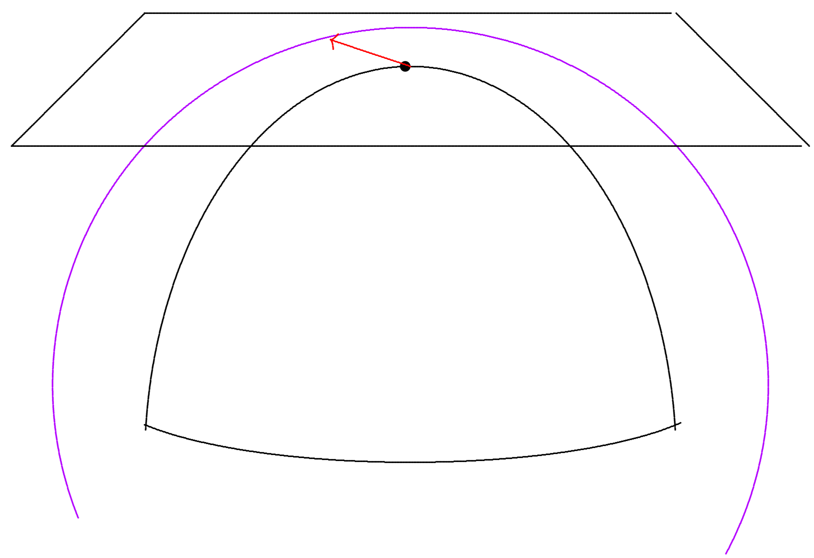

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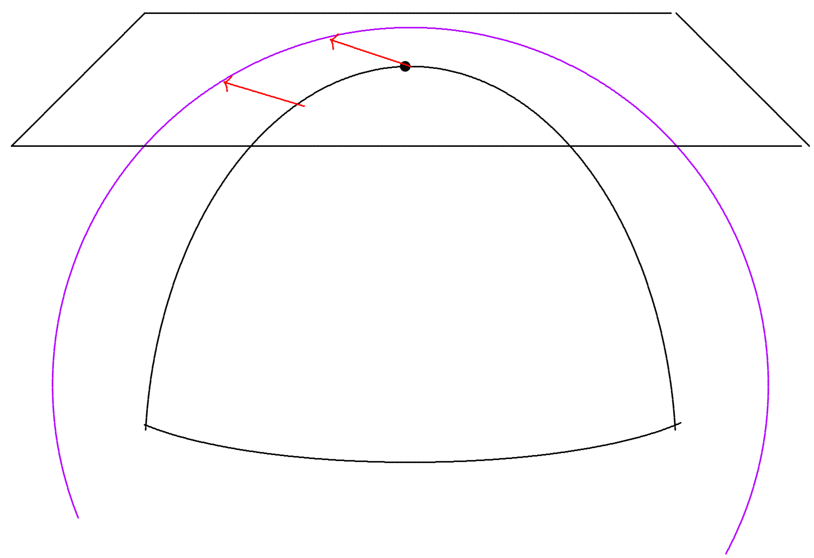

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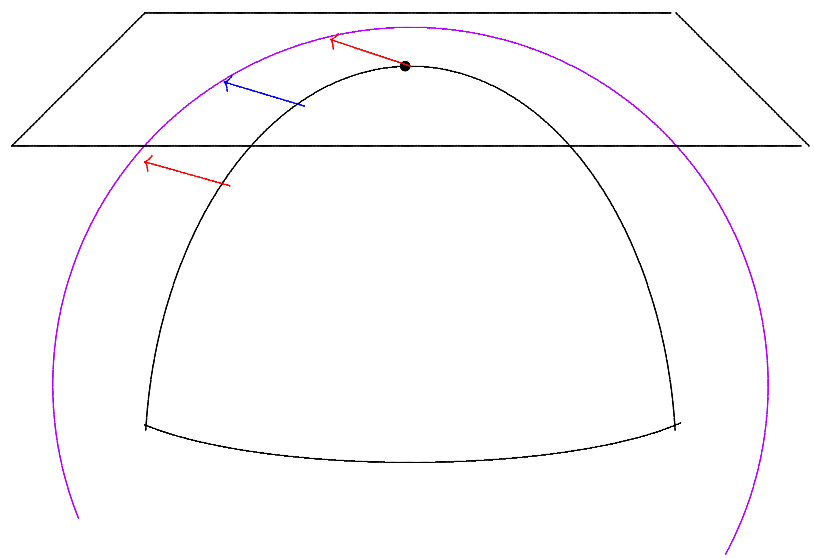

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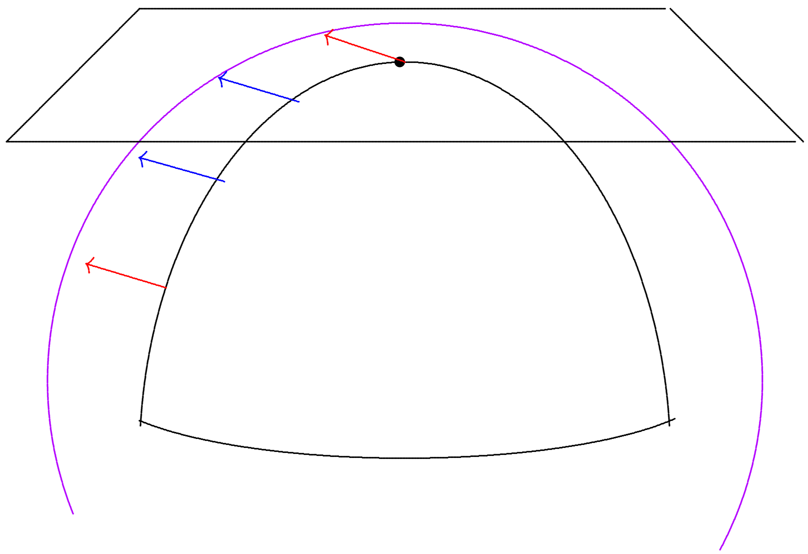

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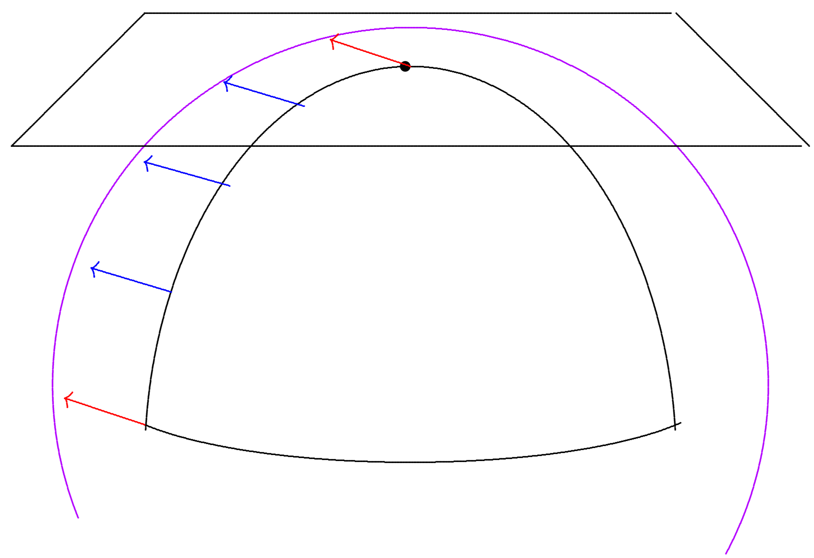

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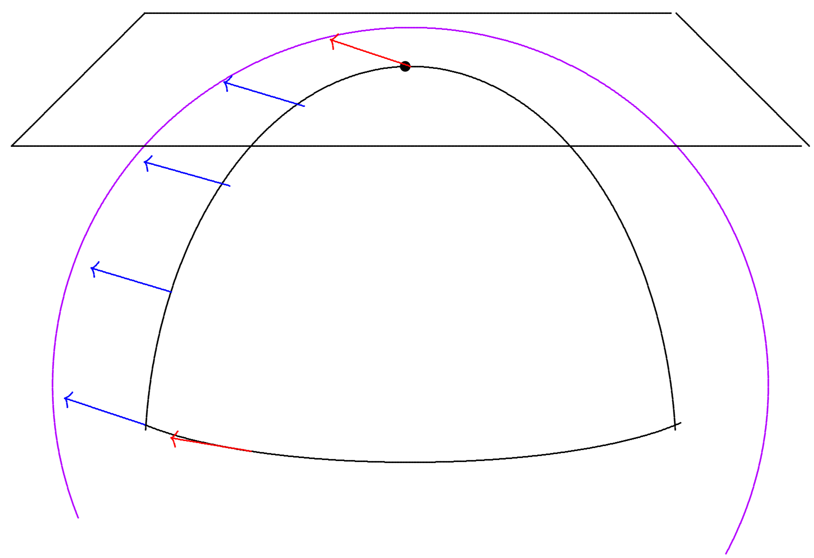

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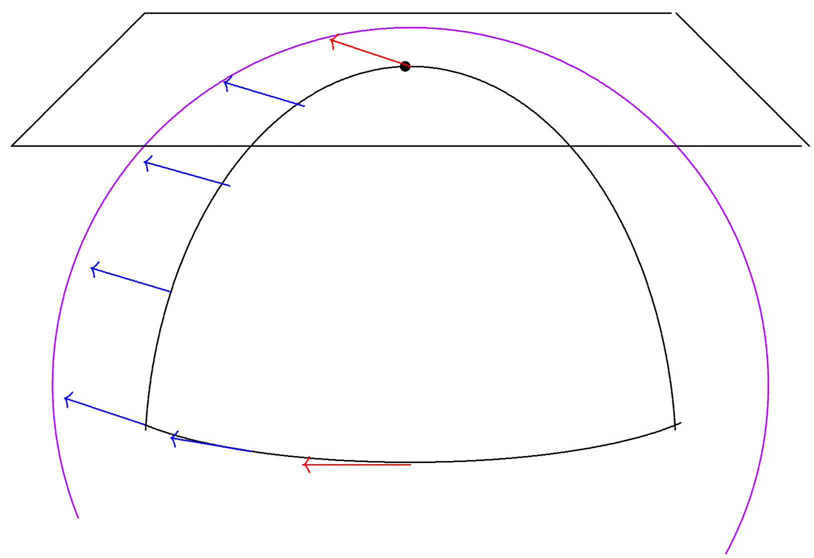

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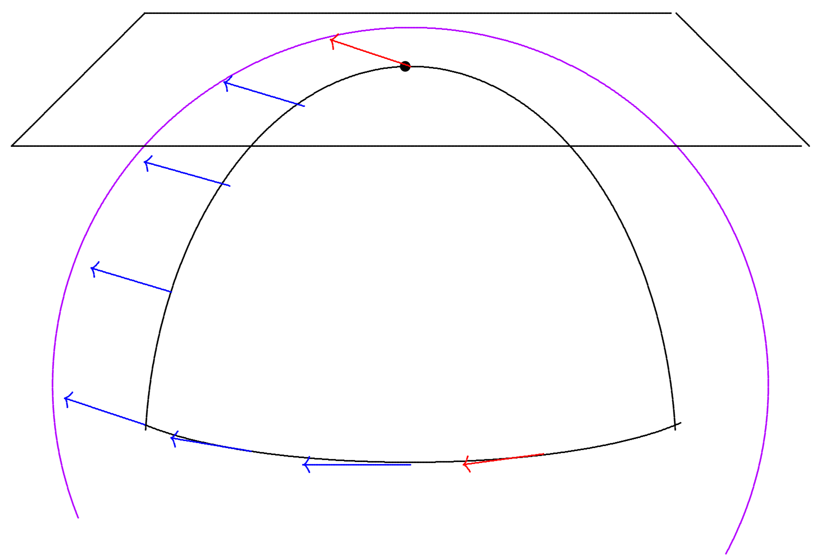

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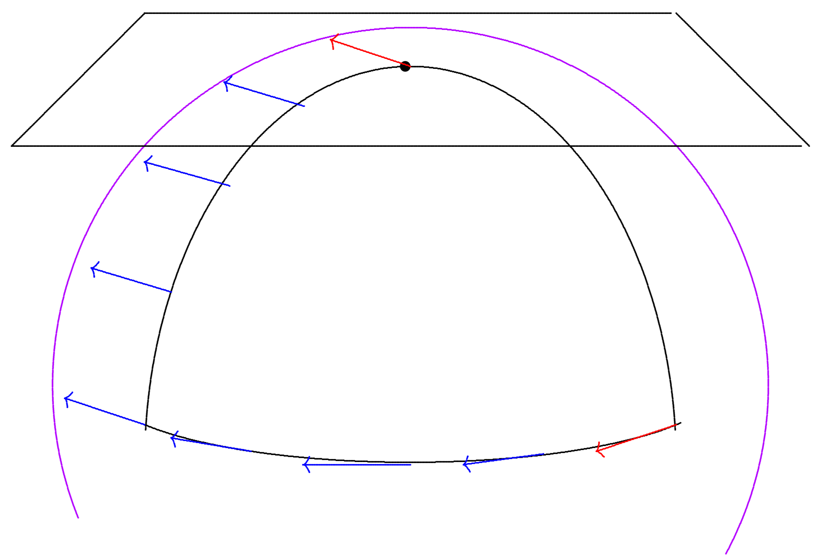

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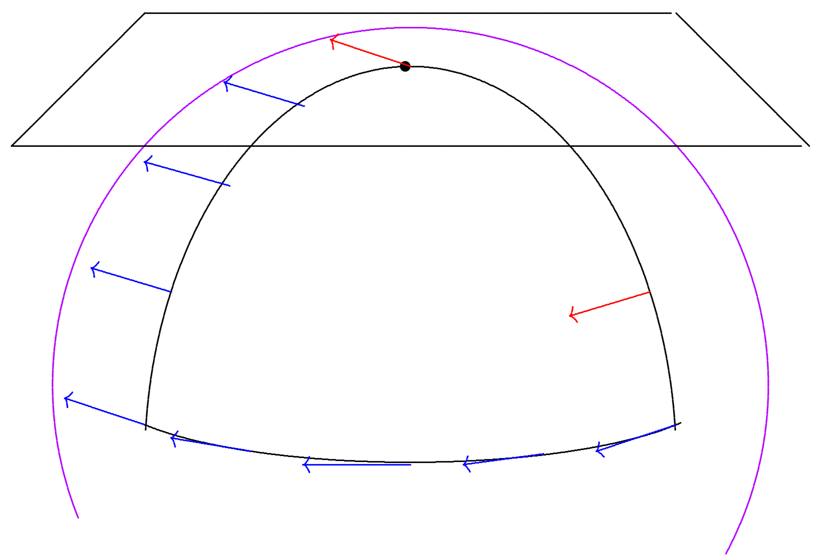

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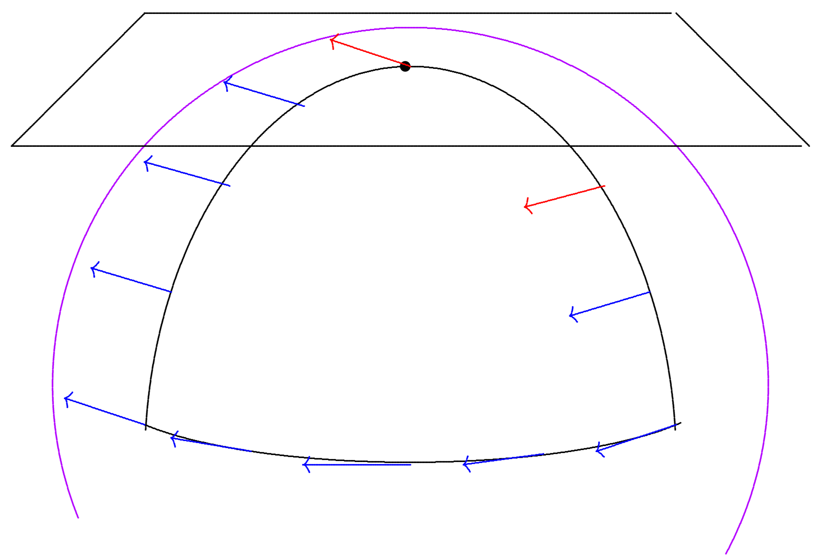

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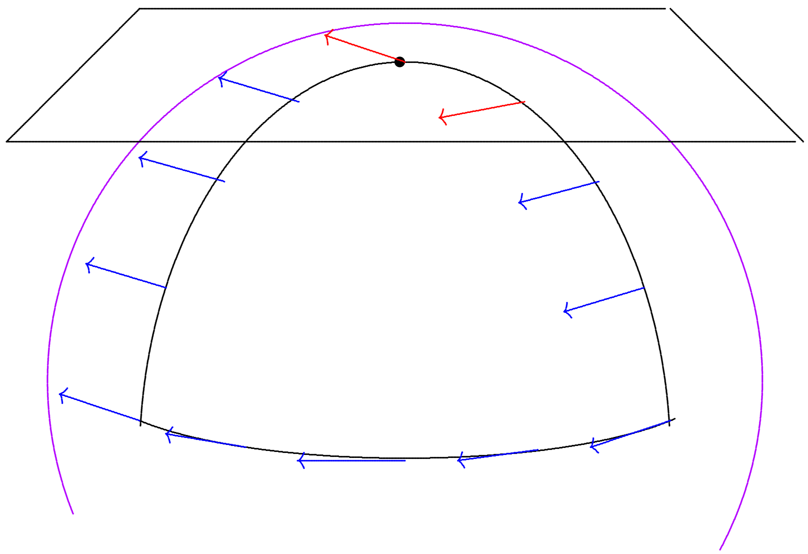

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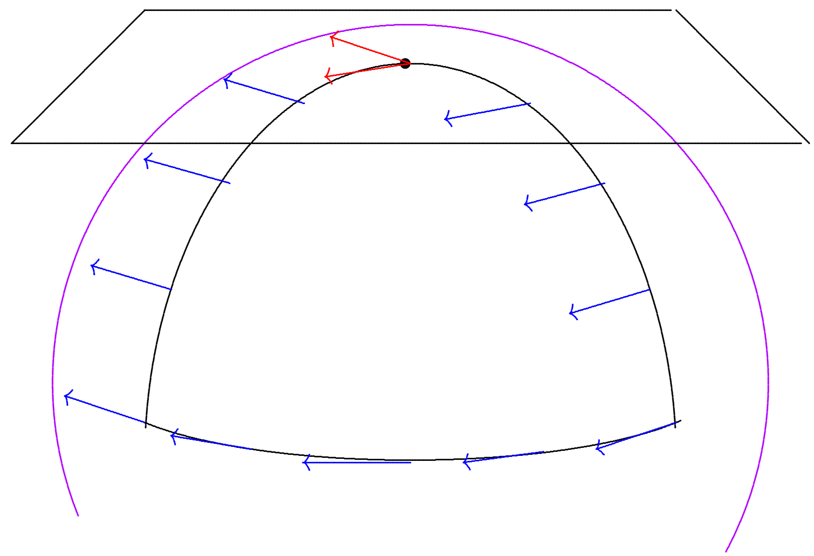

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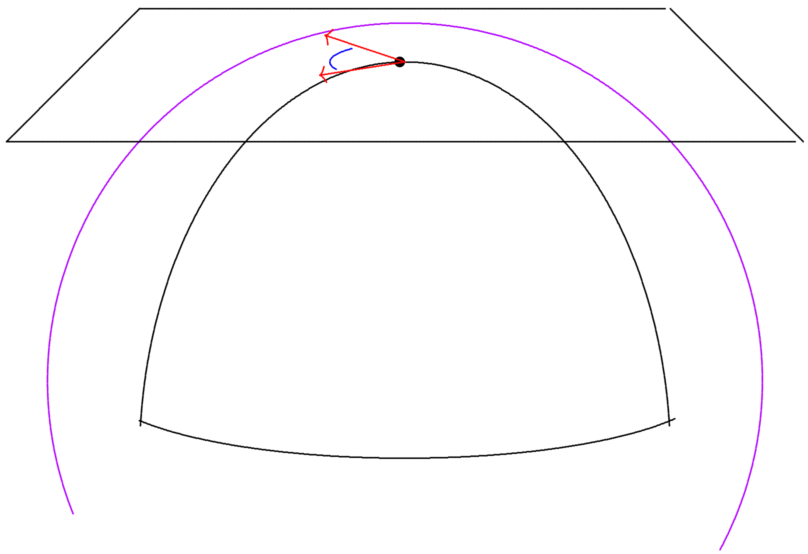
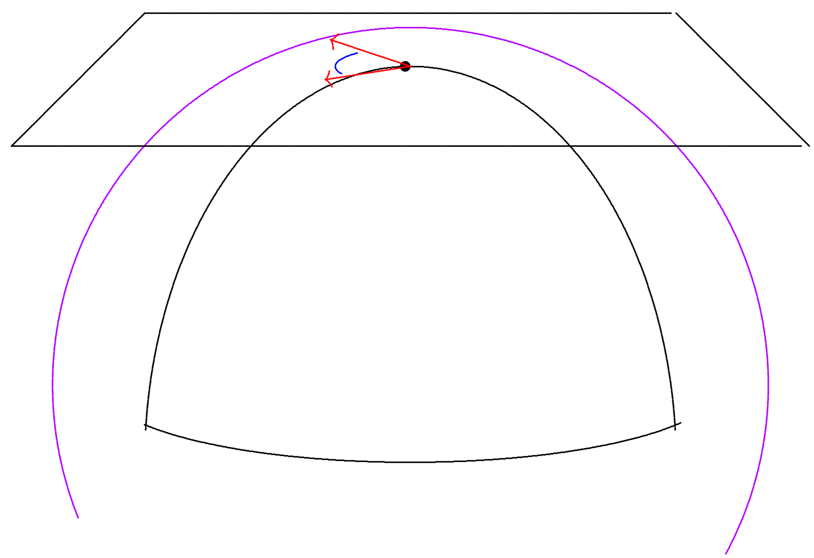


(M^n, g): holonomy $\subset \mathbf{O}(n)$



Kähler metrics:

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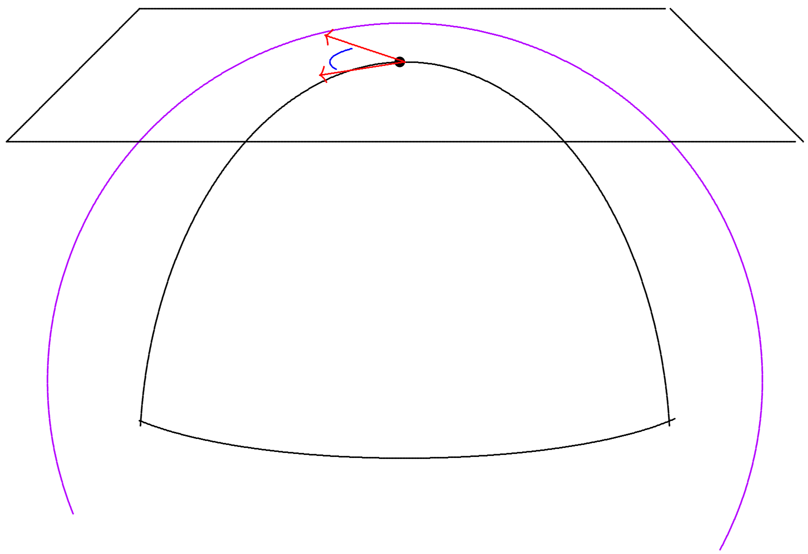

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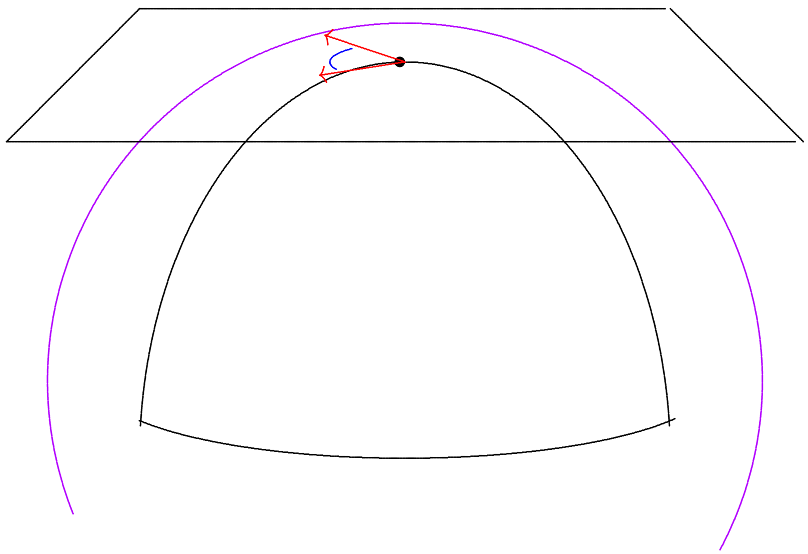

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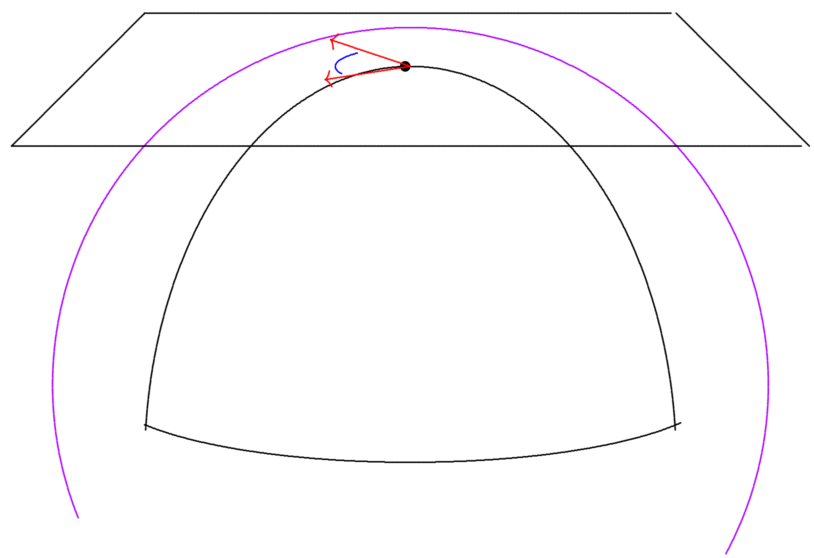

$$\mathbf{U}(m) := \mathbf{O}(2m) \cap \mathbf{GL}(m, \mathbb{C})$$

Theorem (Catanese-L 1997). *For each* $k \geq 2$*, there is a closed simply-connected* $4k$*-manifold* M *that admits both* $\lambda > 0$ *and* $\lambda < 0$ *Einstein metrics.*

Barlow 1985: Found compact complex surface (X^4, J) homotopy equivalent to $\mathbb{CP}_2 \# 8\overline{\mathbb{CP}_2}$ with $c_1 \leq 0$.

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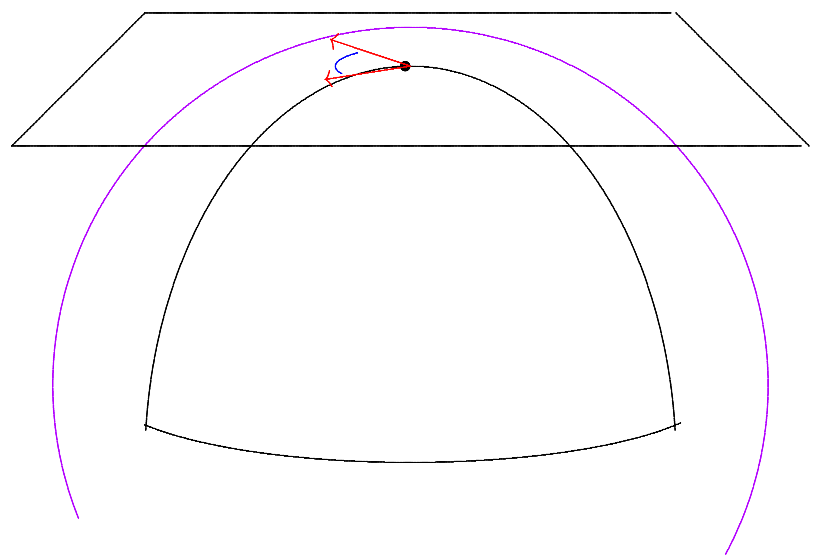
Kähler, but for wildly different complex structures.

(M^n, g):

holonomy $\subset \mathbf{O}(n)$



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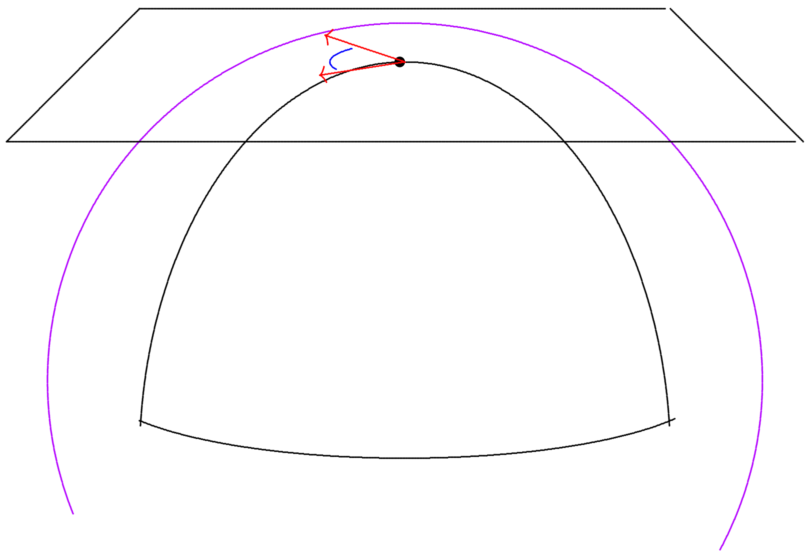

What are the possible holonomy groups?

Theorem (Berger 1955...). *Let* (M^n, g) *be a complete, simply-connected Riemannian manifold.*

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Simons, Aleksevskii, Calabi, Hitchin, Bryant, Joyce...

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Theorem (Berger 1955. . .). *Let* (M^n, g) *be a complete, simply-connected Riemannian manifold. If* (M, g) *is neither a product of lower-dimensional manifolds nor a symmetric space, then its holonomy group appears on the following table:*

dimension n	$\mathrm{Hol}(M^n, g)$	geometry	Ricci-flat?
n	$\mathbf{SO}(n)$	generic	?
$2m$	$\mathbf{U}(m)$	Kähler	✗
$2m$	$\mathbf{SU}(m)$	Calabi-Yau	✓
$4k$	$\mathbf{Sp}(k)$	Hyper-Kähler	✓
$4k$	$\mathbf{Sp}(k)\mathbf{Sp}(1)$	Quaternion-Kähler	✗
7	$\mathbf{G}_2$	Exceptional I	✓
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The latter seems to be especially delicate!

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Gromov-Lawson: $\widetilde{M}^7$ admits metrics with $s > 0$.

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Can such metrics coexist?

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Theorem (L '25)**.**

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To make this plausible, will first illustrate assertion for prototypical examples due to S. Kobayashi '63.

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Sub-complex of deRham complex:

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Replace Hard Lefschetz on $H^*(X, \mathbb{R})$ with transverse version on $H^*_B(M, \mathfrak{F})$ due to El Kacimi-Alaoui.

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CTC Wall: Set of invariants that determines when a given 6-manifold is diffeomorphic to one of these.

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But no one has yet discovered a Calabi-Yau parter for any of these Fano manifolds!

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Thanks again to the organizers and to the MFO for inviting me to participate in this wonderful workshop!