Einstein Manifolds,

Weyl Curvature, &

Conformally Kähler Geometry

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“Differentialgeometrie im Großen,”
Mathematisches Forschungsinstitut Oberwolfach,
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"…the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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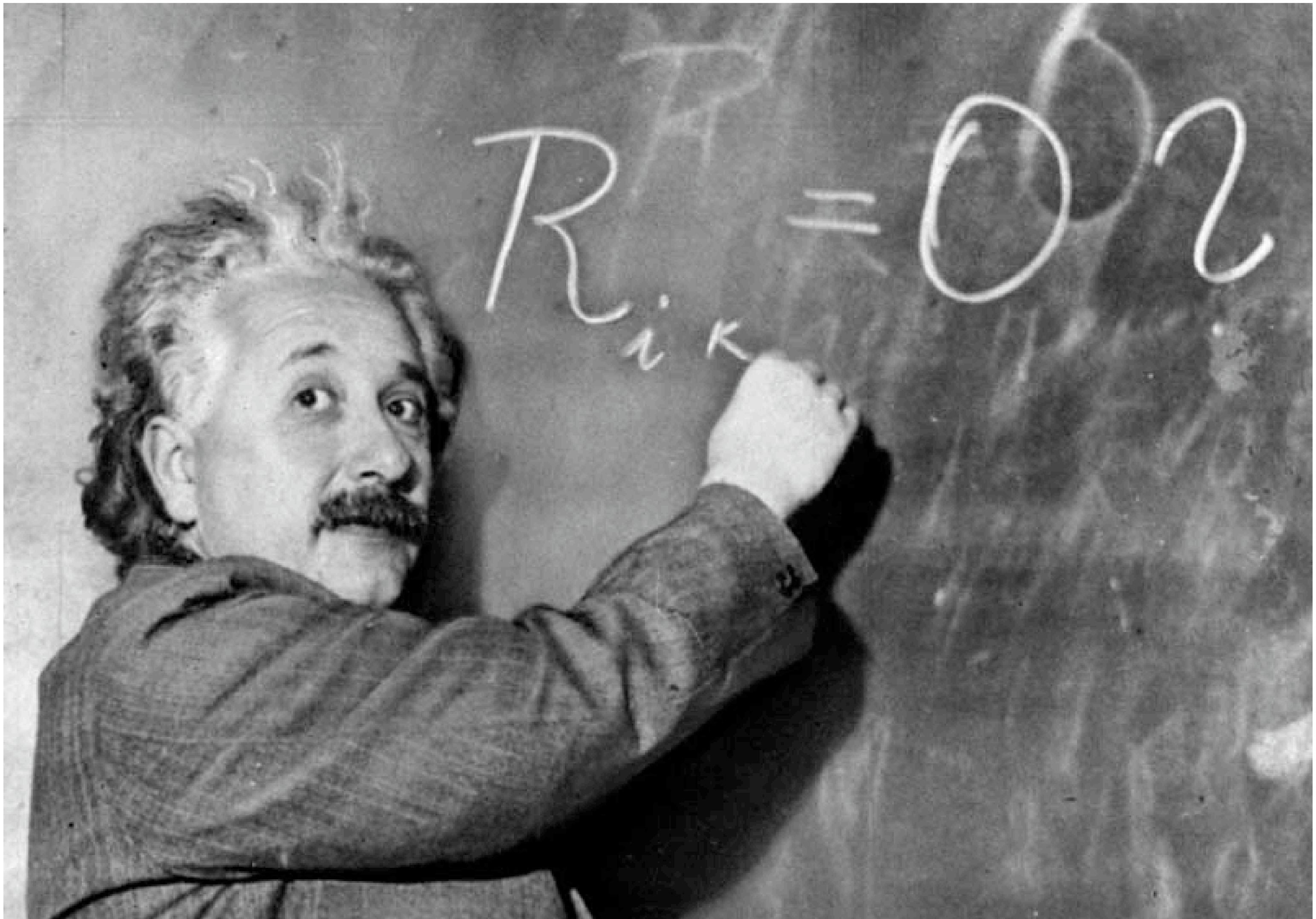

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Has same sign as the *scalar curvature*

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Surprisingly much!

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There is no higher-dimensional version of this story!

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Only depends on the conformal class

$$[g] := \{u^2 g \mid u : M \to \mathbb{R}^+\}.$$

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Reversing orientation interchanges $\Lambda^+ \leftrightsquigarrow \Lambda^-$.

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The numbers

$$b_\pm(M) = \dim \mathcal{H}^\pm_g$$

are independent of g, and so are invariants of M.

$b_{\pm}(M)$?

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Diagonalize:

$$\begin{bmatrix} +1 & & & & & \\ & \ddots & & & & \\ & & +1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{bmatrix}.$$

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$$b_2(M) = b_+(M) + b_-(M)$$

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$$\tau(M) = b_+(M) - b_-(M)$$

"Signature" of M.

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Example. For any symplectic (M^4, ω),
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This is one key ingredient in the proof of the following result about Einstein 4-manifold with $\lambda > 0$.

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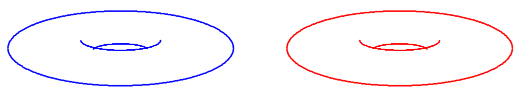

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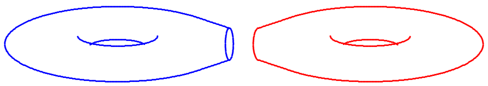

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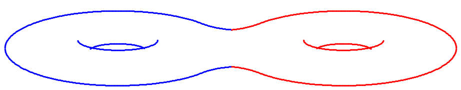

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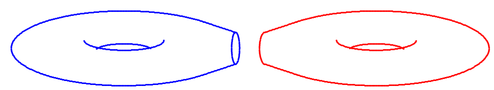

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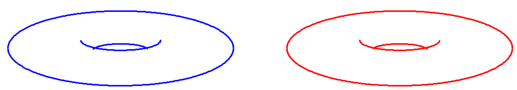

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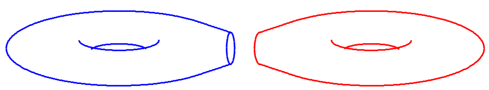

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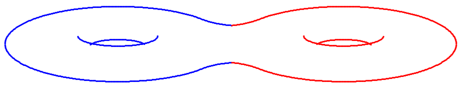

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Up to diffeomorphism, although exotic differentiable structure do exist on most of these manifolds!

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Allowed diffeotypes: exactly the Del Pezzo surfaces.

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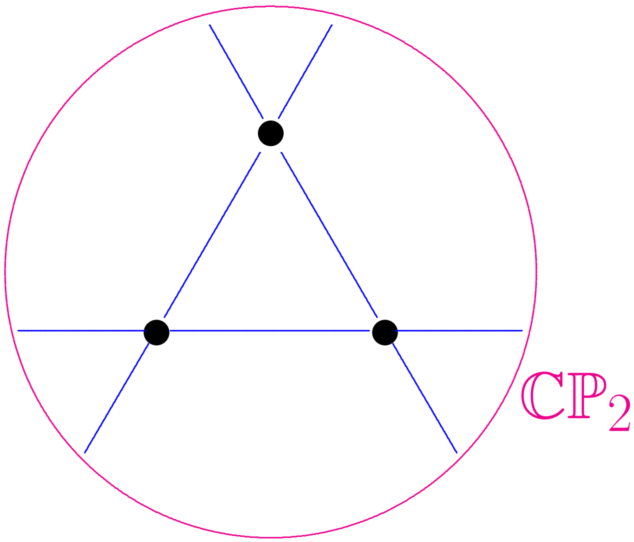

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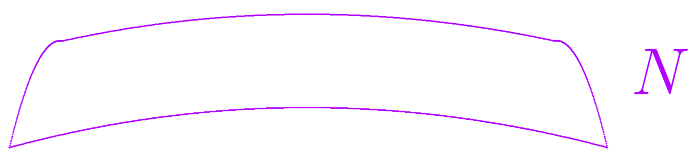

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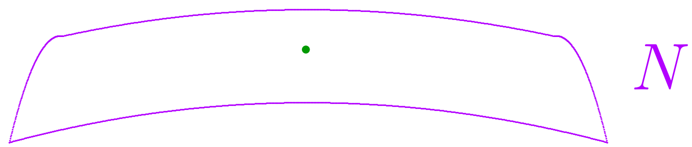
If N is a complex surface, may replace $p \in N$



Blowing up:

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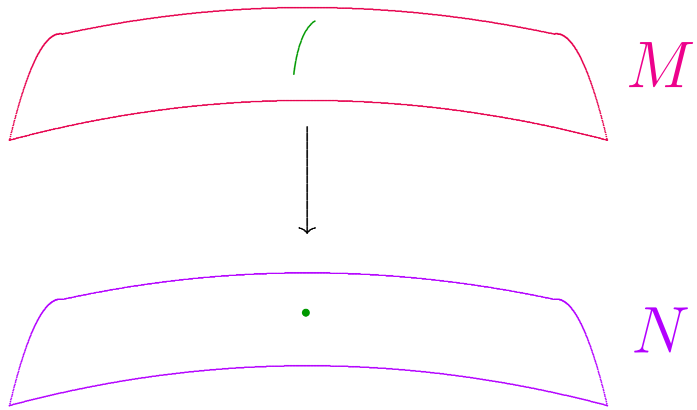

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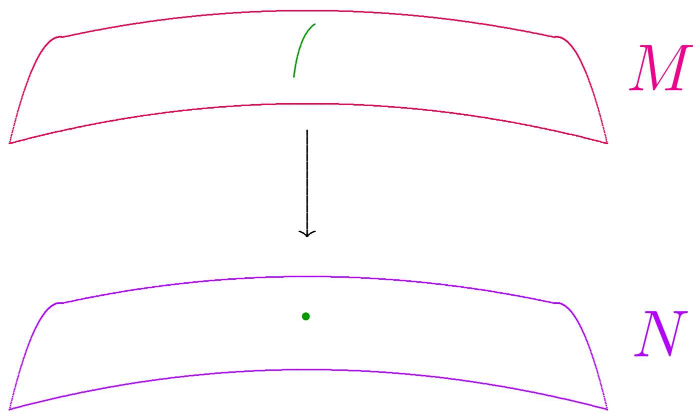
$$M \approx N \# \overline{\mathbb{CP}}_2$$



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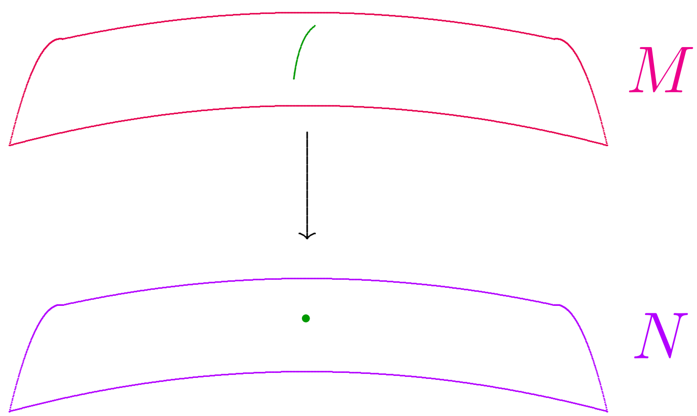
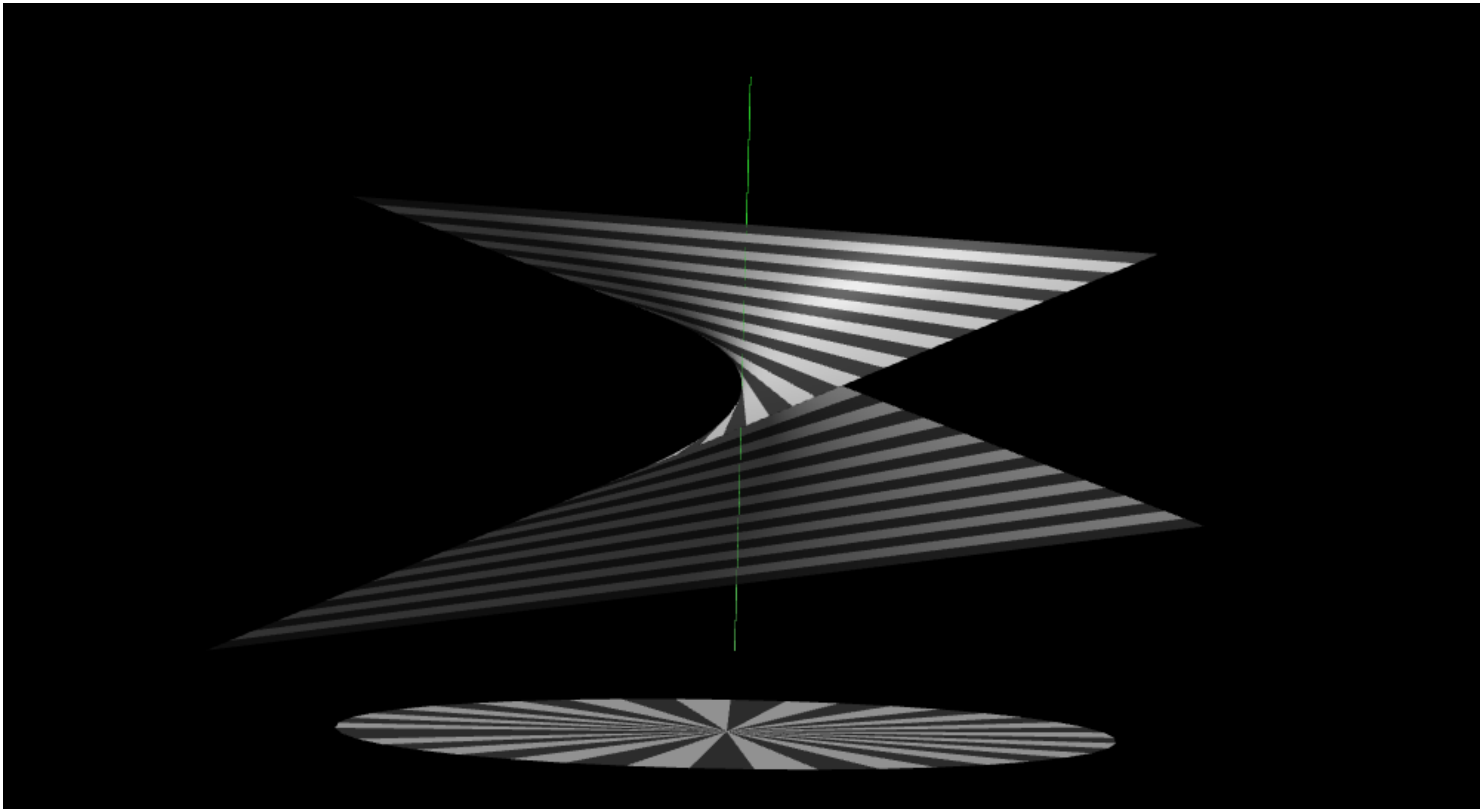
in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.





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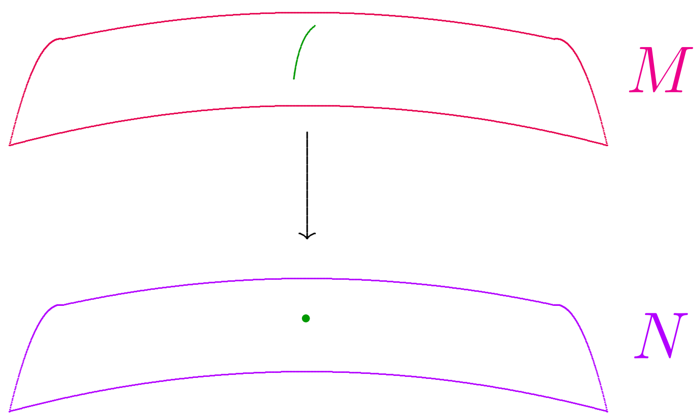

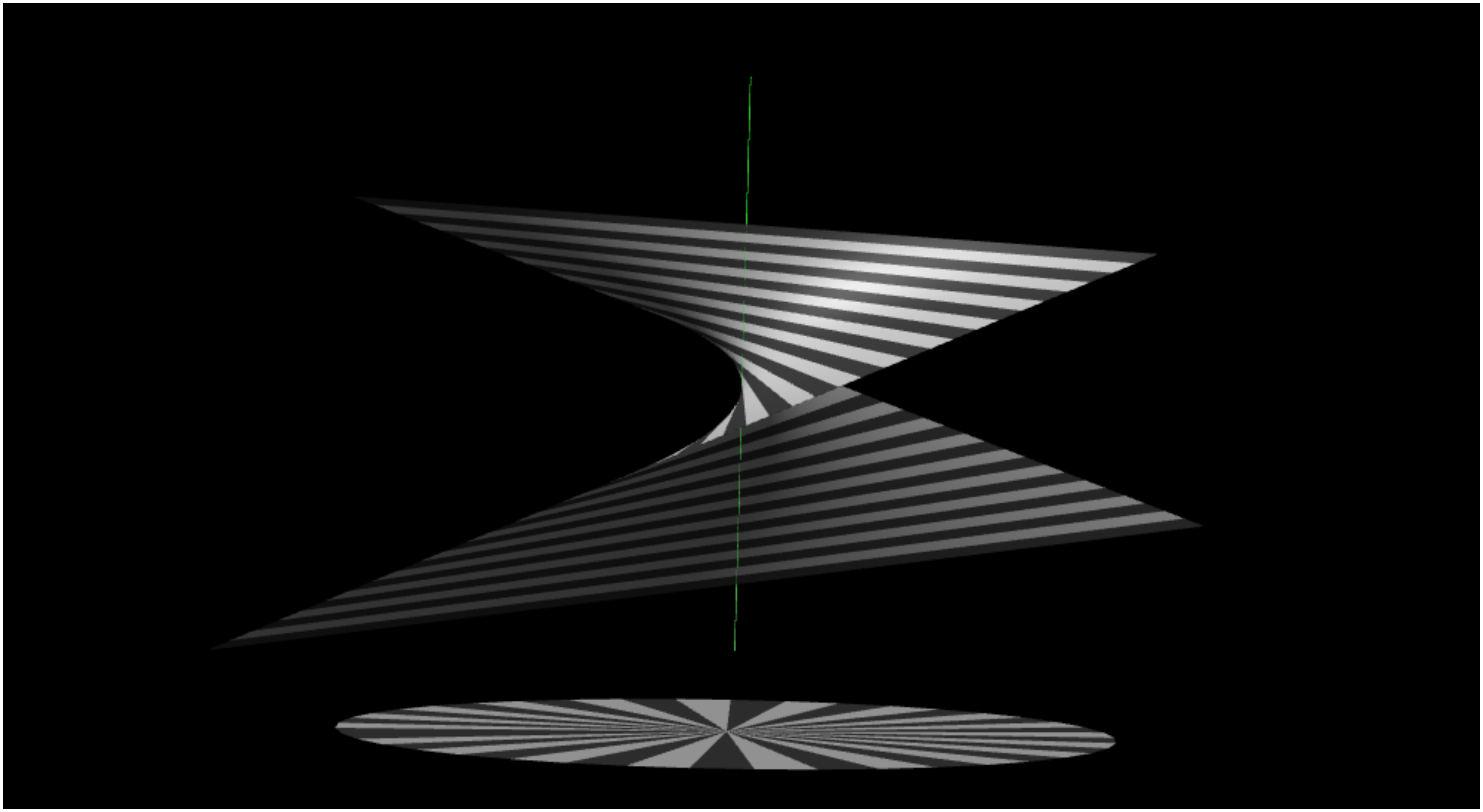

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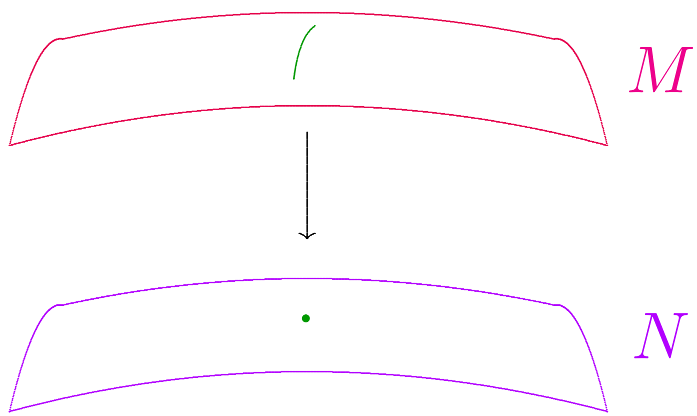

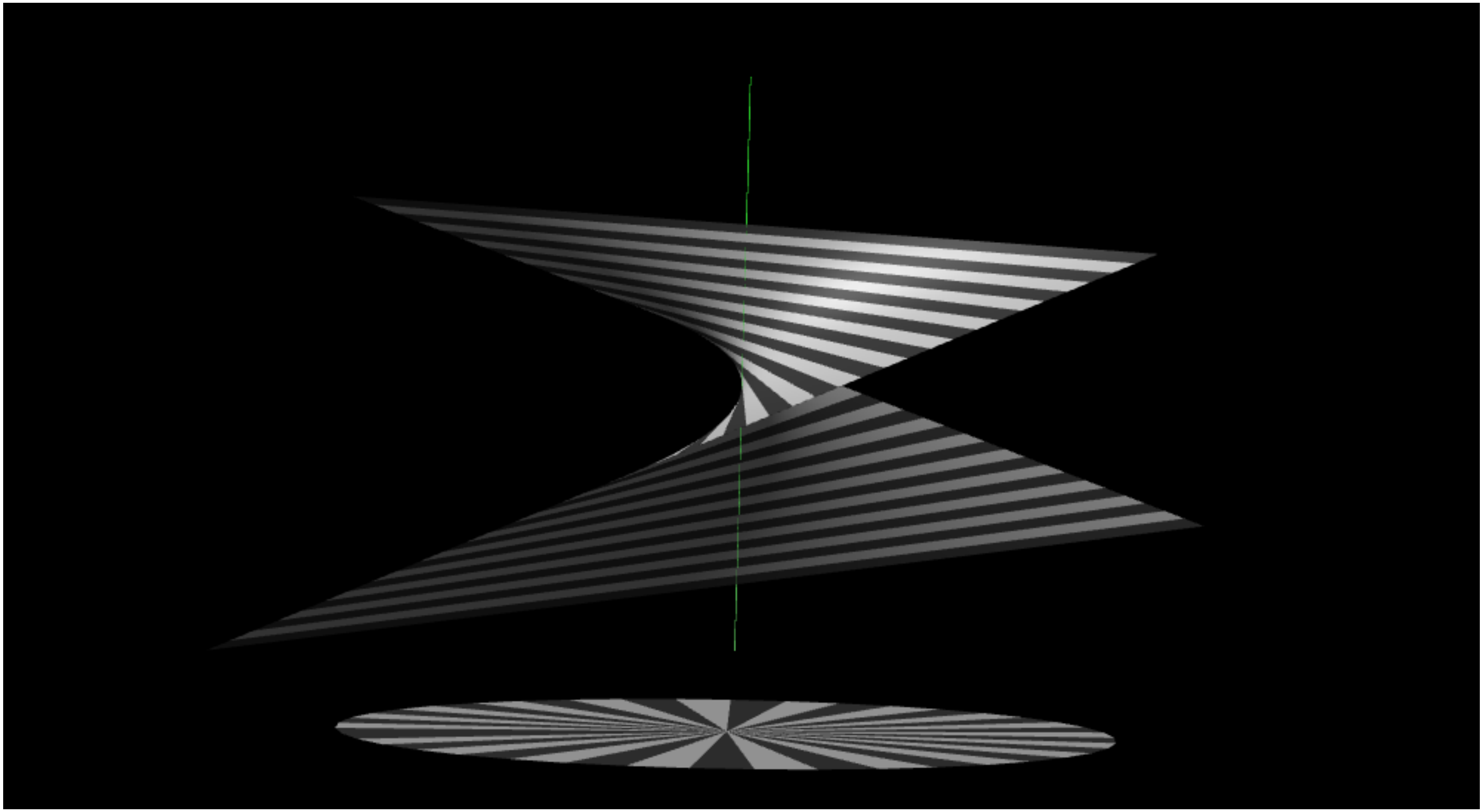

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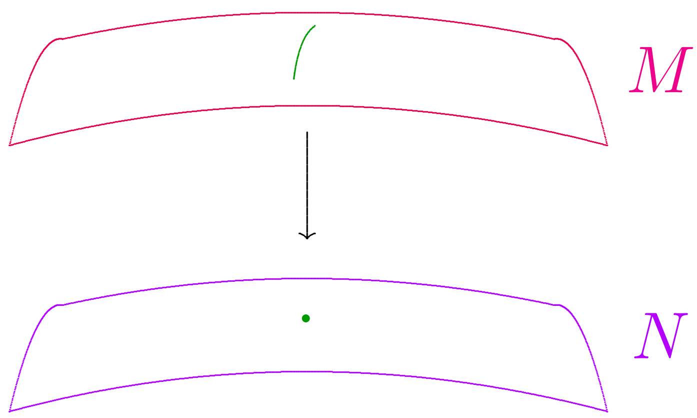

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

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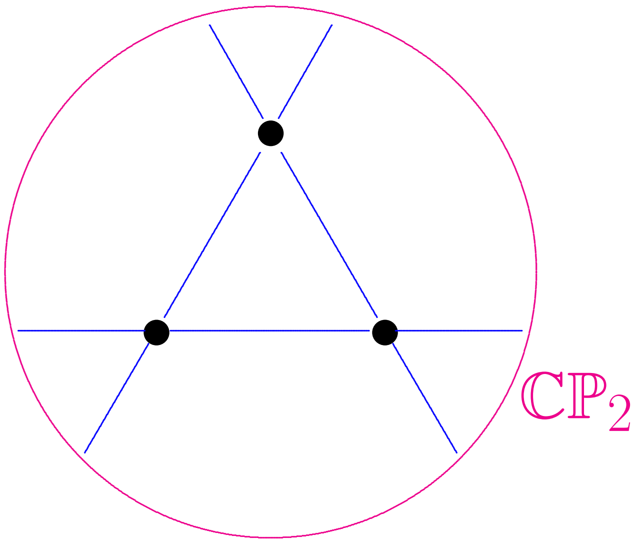

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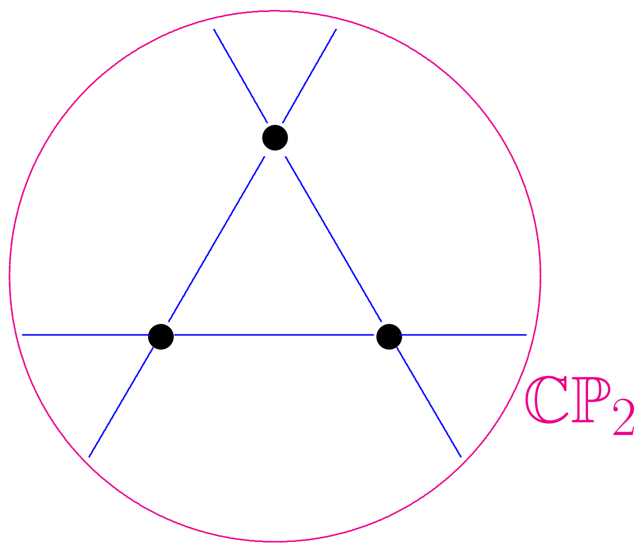

No 3 on a line,

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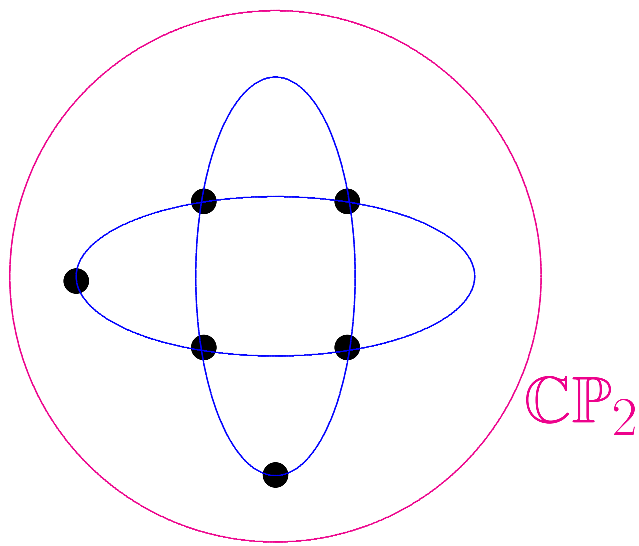

No 3 on a line, no 6 on conic,

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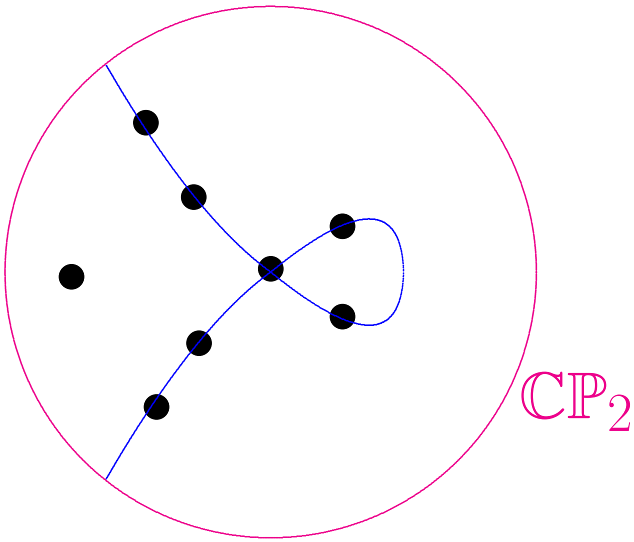

No 3 on a line, no 6 on conic, no 8 on nodal cubic.

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Theorem. *Each del Pezzo* (M^4, J) *admits a* J*-compatible conformally Kähler, Einstein metric, and this metric is unique*

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Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is unique up to complex automorphisms and constant rescalings.*

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Conformally Kähler:

$$g = u^2 h$$

$\exists$ some Kähler metric h & some smooth function u.

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where Kähler metric h is extremal & $u = s_h^{-1}$.

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Existence: Page

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Existence: Page-Derdziński,

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Existence: Page-Derdziński, Siu,

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Existence: Page-Derdziński, Siu, Tian-Yau, Tian, Odaka-Spotti-Sun,

2016

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2008

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Understand all Einstein metrics on del Pezzos.

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Is Einstein moduli space connected?

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Exactly one connected component of moduli space!

Theorem A (L '15)**.**

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Corollary. *These known Einstein metrics on any del Pezzo* M^4

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Corollary. *These known Einstein metrics on any del Pezzo M^4 sweep out exactly one connected component of the Einstein moduli space $\mathscr{E}(M)$.*

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But $W^+(\omega, \omega) > 0$ is not purely local condition!

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Kähler $\Longrightarrow \Lambda^+ = \mathbb{R}\omega \oplus \Re e \Lambda^{2,0}$

$$W^+ = \text{trace-free part of } \begin{bmatrix} 0 & & \\ & 0 & \\ & & \frac{s}{4} \end{bmatrix}$$

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for these metrics & conformal rescalings:

$$g \rightsquigarrow h = u^2 g \quad \Longrightarrow \quad \det(W^+) \rightsquigarrow u^{-6} \det(W^+).$$

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method also proves more general results.

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L (2021b): related classification result.

Theorem B.

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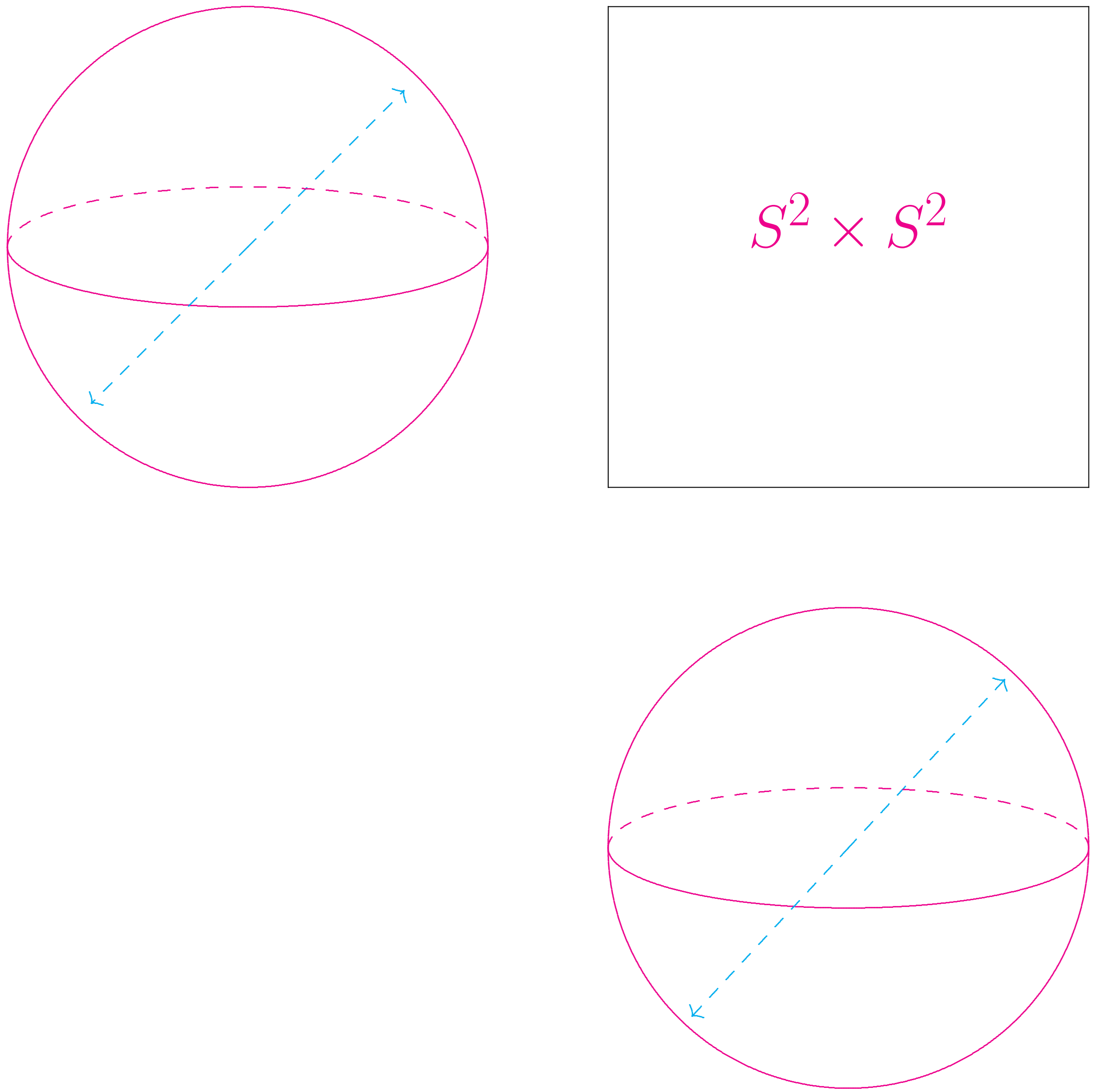
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Similar results govern moduli spaces in these cases.

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Weighted conformal invariance of $\delta W^+ = 0$.

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for $f W^+ \in \mathrm{End}(\Lambda^+)$.

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Now choose $\omega \in \Gamma\Lambda^+$ so that

$$W_h^+(\omega) = \alpha\, \omega, \quad |\omega|_g \equiv \sqrt{2},$$

after at worst passing to double cover $\hat{M} \to M$.

$$0 = \int_{\hat{M}} \Big[\langle W^+, \nabla^* \nabla (\omega \otimes \omega) \rangle + \frac{s}{2} W^+(\omega, \omega) - 6|W^+(\omega)|^2 + 2|W^+|^2|\omega|^2 \Big] f \, d\mu$$

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$$0 = \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) - 2W^+(\omega, \nabla^e \nabla_e \omega) + \frac{s}{2} W^+(\omega, \omega) - 6|W^+(\omega)|^2 + 2|W^+|^2|\omega|^2 \Big] f \, d\mu$$

$$0 = \int_M \Big[-2W^+(\nabla_e\omega, \nabla^e\omega) - 2\alpha\langle\omega, \nabla^e\nabla_e\omega\rangle$$
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because

$$W_h{}^+(\omega) = \alpha\omega$$

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$$0 \geq \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) + 2\alpha\langle \omega, \nabla^*\nabla\omega\rangle$$

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because

$$|W_h{}^+|^2 \geq \frac{3}{2}\alpha^2$$

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$$|\omega|^2_h = 2 \implies (\nabla_e \omega) \perp \omega$$

$$\det(W^+) > 0 \implies W^+ \sim \begin{bmatrix} + & & \\ & - & \\ & & - \end{bmatrix}$$

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$$|\omega|^2_h = 2 \quad \Longrightarrow \quad (\nabla_e \omega) \perp \omega$$

$$\det(W^+) > 0 \quad \Longrightarrow \quad W^+(\nabla_e \omega, \nabla^e \omega) \leq 0$$

$$0 \geq \int_M \Big[\quad 2\alpha\langle\omega, \nabla^*\nabla\omega\rangle$$

$$+\frac{s}{2}\alpha|\omega|^2 - 3\alpha^2|\omega|^2\Big] f \; d\mu$$

$$|\omega|^2_h = 2 \quad \Longrightarrow \quad (\nabla_e\omega) \perp \omega$$

$$\det(W^+) > 0 \quad \Longrightarrow \quad -W^+(\nabla_e\omega, \nabla^e\omega) \geq 0$$

$$0 \geq \int_M \left[2\alpha \langle \omega, \nabla^* \nabla \omega \rangle + \frac{s}{2} \alpha |\omega|^2 - 3\alpha^2 |\omega|^2 \right] f \, d\mu$$

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But

$$\alpha f \equiv 1$$

$$0 \geq \int_M \left[2\langle \omega, \nabla^* \nabla \omega \rangle + \frac{s}{2} |\omega|^2 - 3|\omega|^2 \alpha \right] d\mu$$

$$0 \geq \int_M \left[2\langle \omega, \nabla^* \nabla \omega \rangle - 3W^+(\omega, \omega) + \frac{s}{2}|\omega|^2 \right] d\mu$$

$$0 \geq \int_M \left[\frac{1}{2}|\nabla\omega|^2 + \frac{3}{2}\langle\omega, \left(\nabla^*\nabla - 2W^+ + \frac{s}{3}\right)\omega\rangle\right] d\mu$$

$$0 \geq \int_M \left[\frac{1}{2}|\nabla\omega|^2 + \frac{3}{2}\langle\omega, \quad (d+d^*)^2 \quad \omega\rangle\right] d\mu$$

Because

$$(d+d^*)^2 = \nabla^*\nabla - 2W^+ + \frac{s}{3}$$

on $\Gamma\Lambda^+$.

$$0 \geq \frac{1}{2}\int_M |\nabla\omega|^2 \, d\mu + 3\int_M |d\omega|^2 \, d\mu$$

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So $\nabla\omega \equiv 0$, and h is Kähler!

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Techniques used extend today's results.

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