On Conformally Kähler,

Einstein Manifolds

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$\Longleftrightarrow$ $\exists$ almost complex-structure J with $\nabla J = 0$ and $g(J\cdot, J\cdot) = g$.

$\Longleftrightarrow$ (M^4, J) is a complex surface and $\exists$ J-invariant closed 2-form ω such that $g = \omega(\cdot, J\cdot)$.

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Diffeotypes occurring in $\lambda > 0$ case:

$$\mathbb{CP}_2,\ S^2 \times S^2,\ \mathbb{CP}_2 \# \underbrace{\overline{\mathbb{CP}}_2 \# \cdots \# \overline{\mathbb{CP}}_2}_{3 \leq k \leq 8}.$$

Recall:

$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

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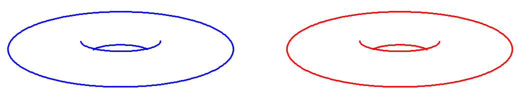
Connected sum $\#$:



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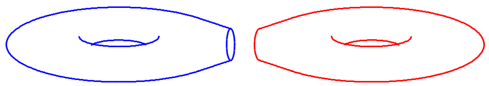

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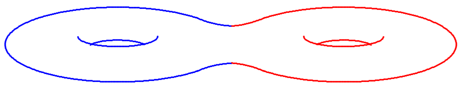

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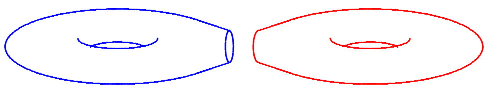

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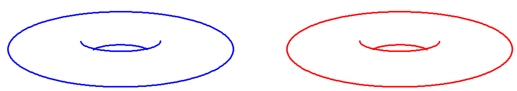

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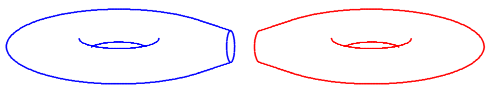

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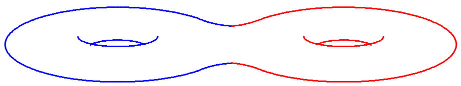

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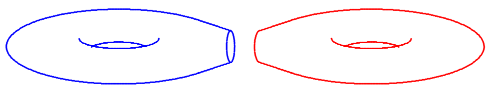

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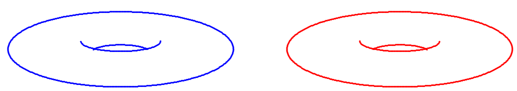

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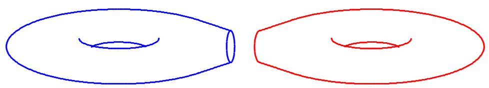

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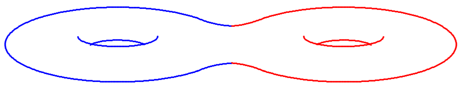

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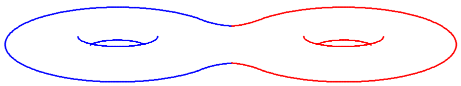

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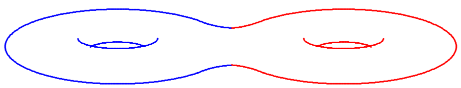

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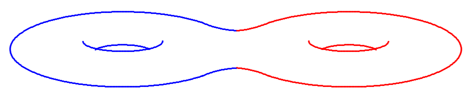

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(Matsushima/Lichnerowicz theorem)

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Proposition (L '96). *Let* (M^4, J) *be a compact complex surface, and suppose that* h *is an Einstein metric on* M *which is Hermitian with respect to* J*:*

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- h *has positive Ricci curvature;*
- g *is an extremal Kähler metric;*
- g *has scalar curvature* $s > 0$; *and*
- *after normalization,* $h = s^{-2} g$.

Theorem B. *A compact complex surface* (M^4, J) *admits an Einstein metric* h *which is Hermitian with respect to* J $\Longleftrightarrow$

$$c_1(M^4, J) = \kappa[\omega]$$

$\exists$ *Kähler class* $[\omega]$ *and* $\kappa \in \mathbb{R}$.

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Remark In K-E case, may take $[\omega]$ to be Kähler class of h. But in non-K-E case, $[\omega]$ is definitely not the Kähler class of conformally related Kähler metric g!

Theorem C. *Suppose that M is a smooth compact oriented 4-manifold which admits an integrable complex structure J. Then M admits an Einstein metric h with $\lambda > 0$*

$$\Longleftrightarrow \quad M \approx \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}_2}, & 0 \leq k \leq 8, \\ \quad or & \\ S^2 \times S^2 & \end{cases}$$

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Seiberg-Witten invariant must vanish.

Theorem D. *Suppose that* M *is a smooth compact oriented* 4*-manifold which admits a symplectic structure* ω*. Then* M *admits an Einstein metric* h *with* $\lambda > 0$

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X.X. Chen: always minimizers.

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Chen-Tian: unique modulo bihomorphisms.

Explicit lower bound:

Any Kähler (M^4, g, J) satisfies

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with $=$ $\Longleftrightarrow$ g extremal, where

$$\mathcal{A}([\omega]) := \frac{(c_1 \cdot [\omega])^2}{[\omega]^2} + \frac{1}{32\pi^2}\|\mathcal{F}_{[\omega]}\|^2$$

where $\mathcal{F}$ is Futaki invariant. Can compute $\mathcal{F}$ using any metric in Kähler class.

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Normalization chosen so that always have

$$\mathcal{A}([\omega]) \geq {c_1}^2.$$

Special character of dimension 4:

On oriented (M^4, g),

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

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$$\star : \Lambda^2 \to \Lambda^2,$$

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Λ^+ self-dual 2-forms.
Λ^- anti-self-dual 2-forms.

Riemann curvature of g

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where

$$\begin{aligned} s &= \text{scalar curvature} \\ \mathring{r} &= \text{trace-free Ricci curvature} \\ W_+ &= \text{self-dual Weyl curvature} \\ W_- &= \text{anti-self-dual Weyl curvature} \end{aligned}$$

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Conformally invariant Riemannian functional:

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$$\nabla^a B_{ab} = 0$$

4-dimensional Gauss-Bonnet formula

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and corresponds to harmonic primitive $(1,1)$-form

$$\psi := B(J\cdot, \cdot) = \frac{1}{12}\left[s\rho + 2i\partial\bar{\partial}s\right]_0$$

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So the critical points of restriction of $\mathcal{W}$ to $\{$Kähler metrics$\}$ also have $B = 0$!

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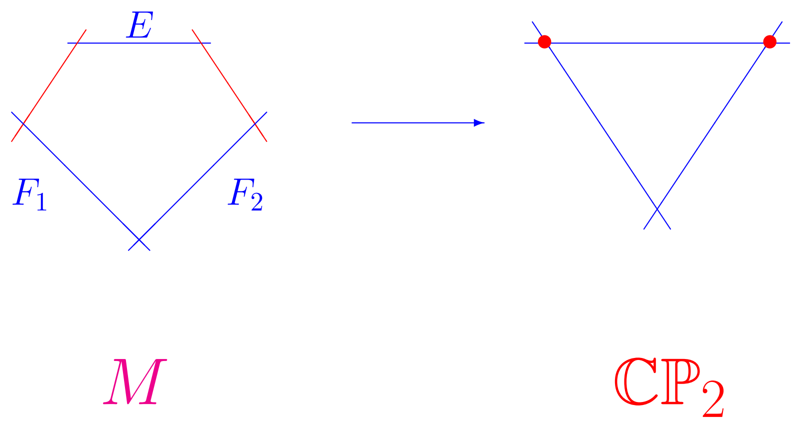
Warning. h undefined where $s = 0$!

Proposition. *Let* (M^4, J) *be a compact complex surface, and let* $\mathcal{KC} \subset H^2(M, \mathbb{R})$ *be its Kähler cone. If* $[\omega]$ *is a critical point of*

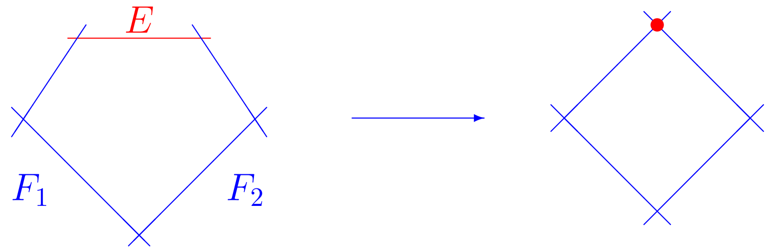
$$\mathcal{A} : \mathcal{KC} \to \mathbb{R}$$

and if $[\omega]$ *is represented by an extremal Kähler metric* g*, then* g *is Bach-flat. Moreover, if* g *has* $s > 0$*, then* $h = s^{-2}g$ *is an Einstein metric on* M*.*

Two-Point Blow-up of $\mathbb{CP}_2$:

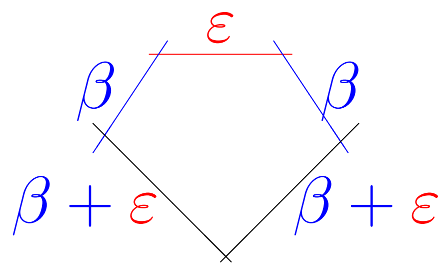


= One-Point Blow-up of $\mathbb{CP}_1 \times \mathbb{CP}_1$:



M $\qquad\qquad$ $\mathbb{CP}_1 \times \mathbb{CP}_1$

Bilaterally Symmetric Kähler Classes:



$$[\omega]_{\beta,\varepsilon} = (\beta + \varepsilon)(F_1 + F_2) - \varepsilon E$$

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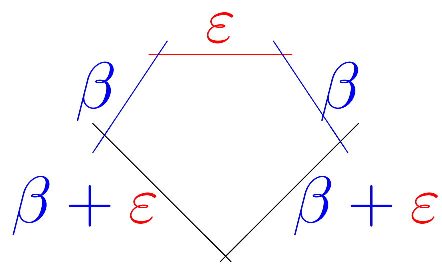
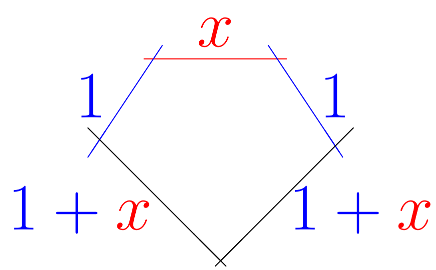


$$[\omega]_{\beta,\varepsilon} = (\beta + \varepsilon)(F_1 + F_2) - \varepsilon E$$

These are fixed points of involution of $\mathcal{KC}$

$$F_1 \longleftrightarrow F_2$$

Scale invariance reduces problem to



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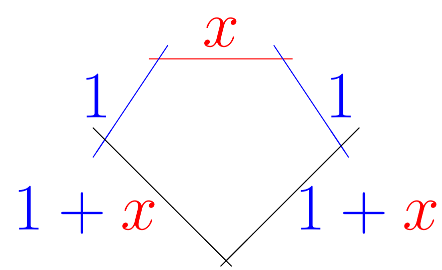

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and

$$f(x) = \mathcal{A}([\omega]_x)$$

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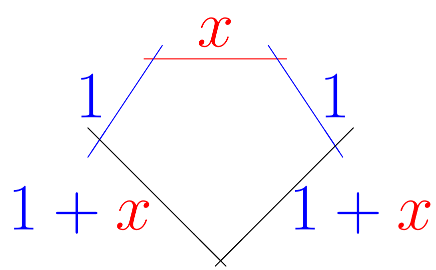

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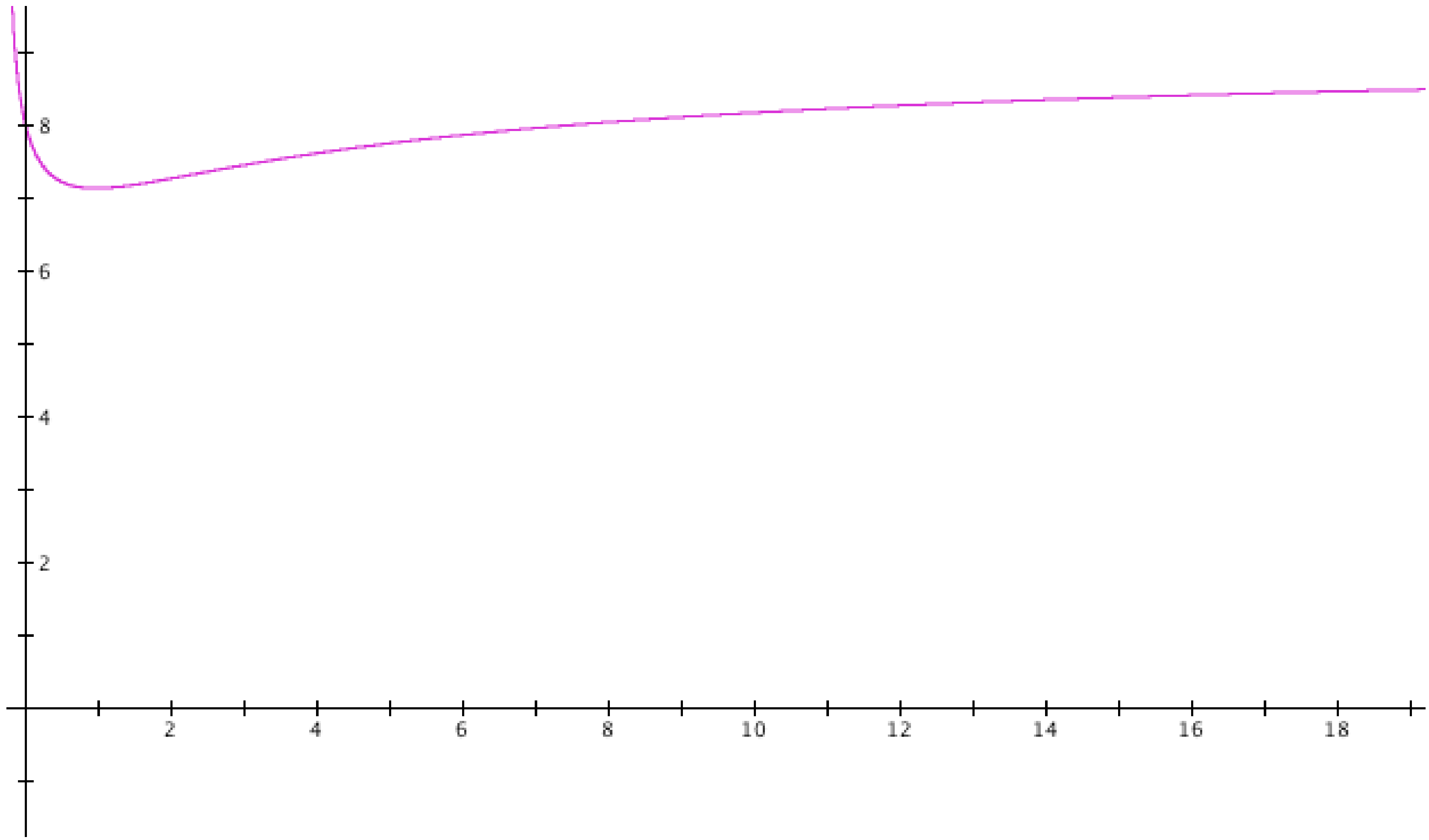
NEED TO SHOW: $\exists x_0 > 0$ with

$$f'(x_0) = 0$$

such that $[\omega]_{x_0}$ represented by extremal Kähler metric g with $s > 0$.

$$f(x) = 9\left(\frac{32 + 176x + 318x^2 + 280x^3 + 132x^4 + 32x^5 + 3x^6}{36 + 216x + 414x^2 + 360x^3 + 162x^4 + 36x^5 + 3x^6}\right)$$



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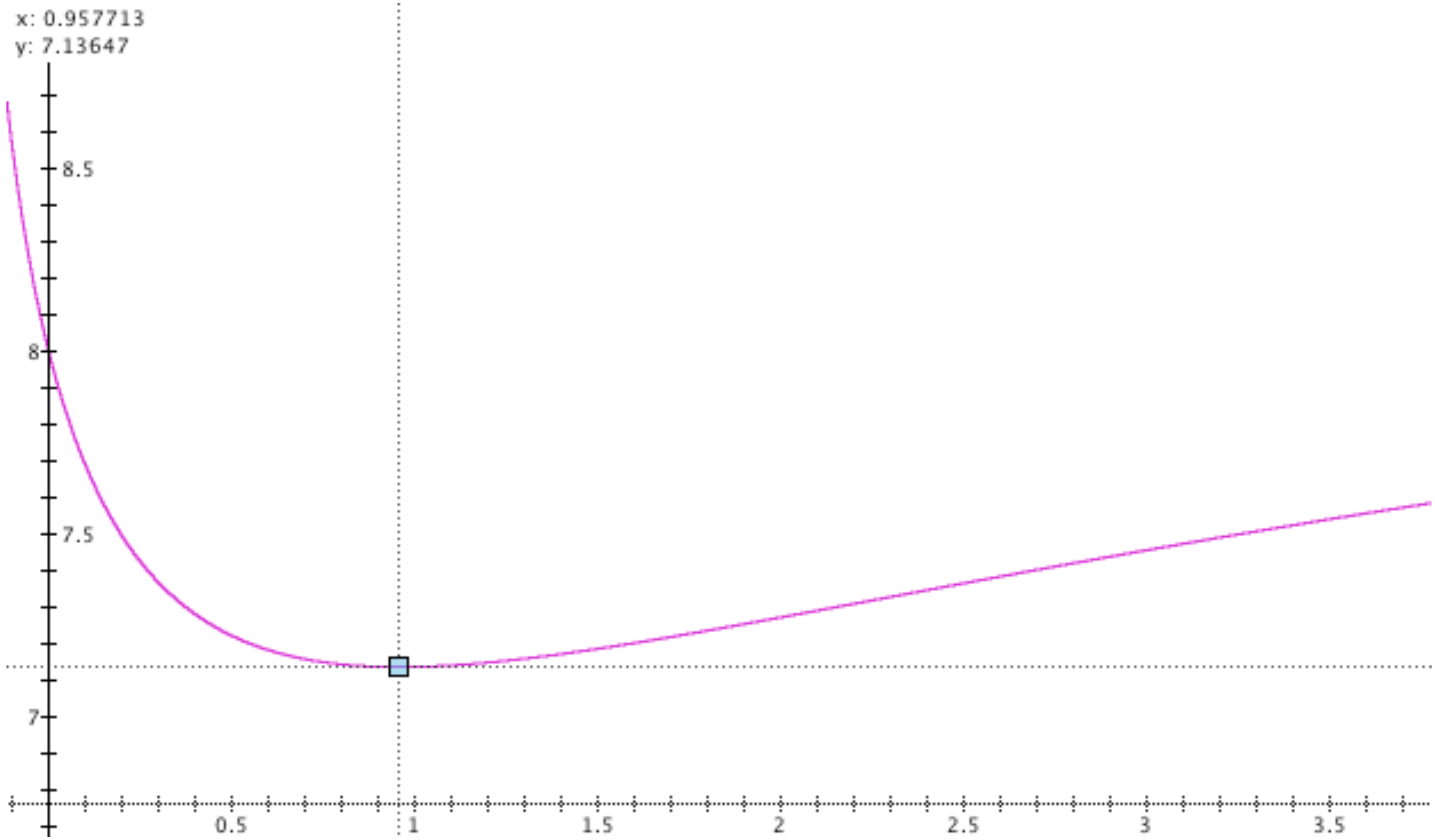

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$$f(0)=8, \quad f'(0)<0, \qquad \lim_{x\to\infty} f(x) = 9.$$

Define x_0 to be smallest $x > 0$ in $(f')^{-1}(0)$.
Then $f(x) < 8$ on $(0, x_0]$.

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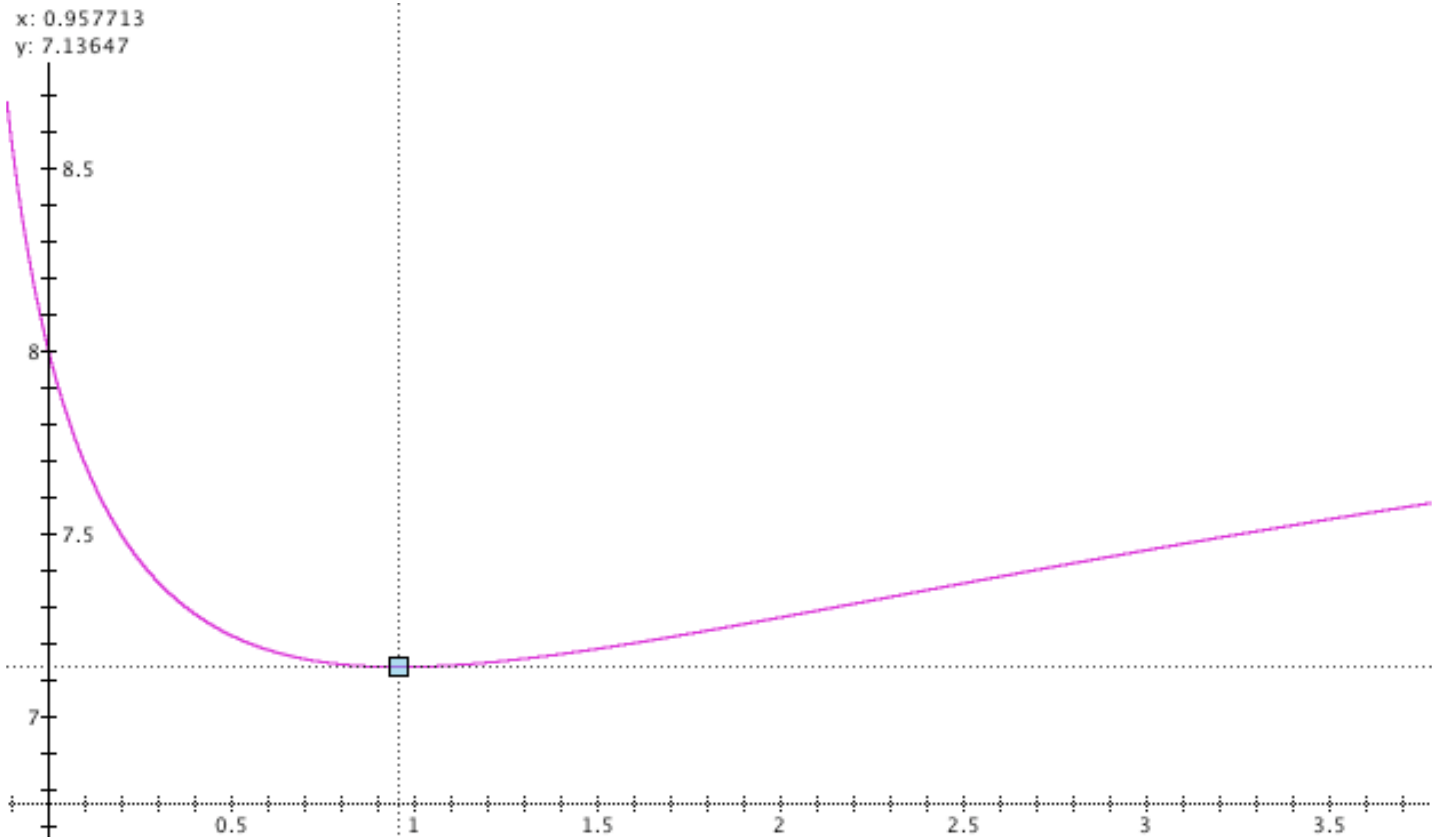

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Gluing theorem: attach small Burns metric to product $S^2 \times S^2$, perturb.

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Proposition (LeBrun-Simanca). *Set of $x > 0$ s.t. $[\omega]_x$ contains extremal Kähler metric is open.*

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So closed is the difficult issue!

Definition. *On a compact complex surface* (M, J), *the controlled cone is the set of Kähler classes* $[\omega]$ *which satisfy*

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Related to (positive) Yamabe constant:

$$Y_{[g]} = \inf_{u \not\equiv 0} \frac{\int (6|\nabla u|^2 + s_g u^2)\, d\mu_g}{\left(\int u^4 d\mu_g\right)^{1/2}}.$$

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Theorem (Chen-Weber). *Let g_i be an arbitrary sequence of unit-volume extremal Kähler metrics on M^4 with uniformly bounded energies $\mathcal{A}$ and Sobolev constants C_S. Then $\exists$ subsequence which Gromov-Hausdorff converges to an extremal Kähler metric on a compact complex 2-orbifold.*

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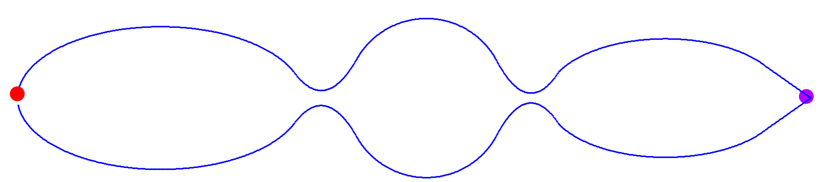

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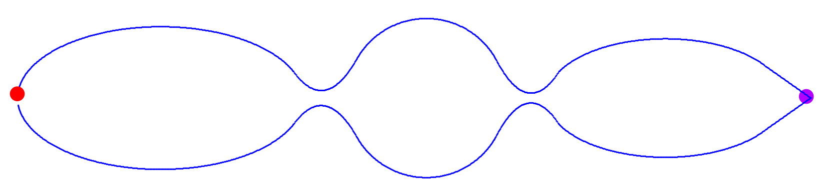

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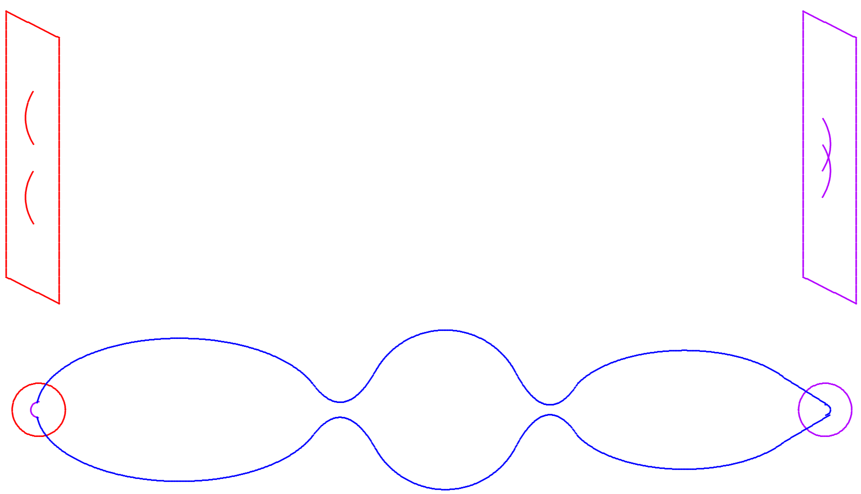

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Goal: rule out deepest bubbles.

Lemma. *If* $\exists$ *deepest bubble* (X, g_∞), *then toric, ALE, scalar-flat Kähler, with*

$$\int_X |\mathring{r}|^2 d\mu_{g_\infty} \leq \limsup_{i\to\infty} \int_M |\mathring{r}|^2 d\mu_{g_i}$$

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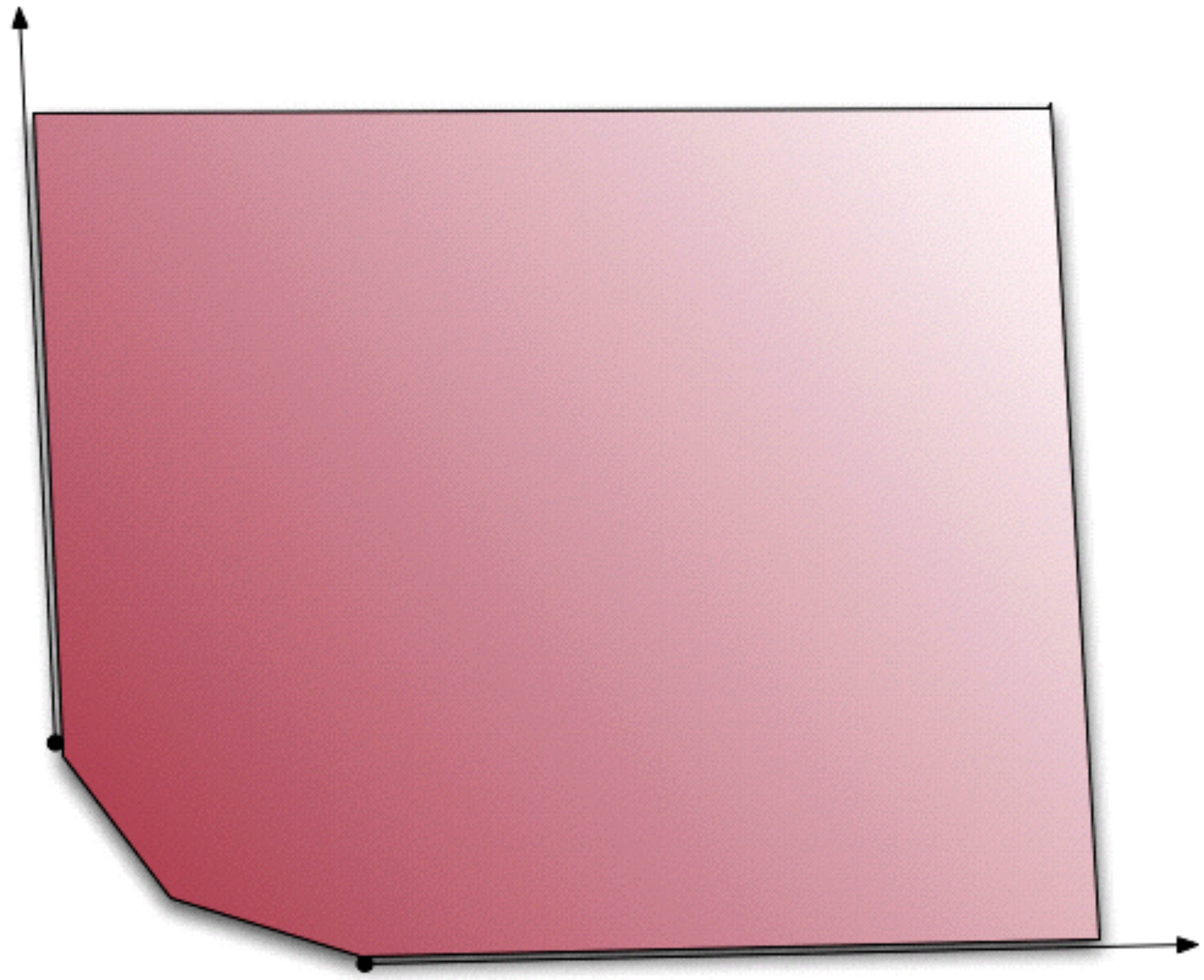

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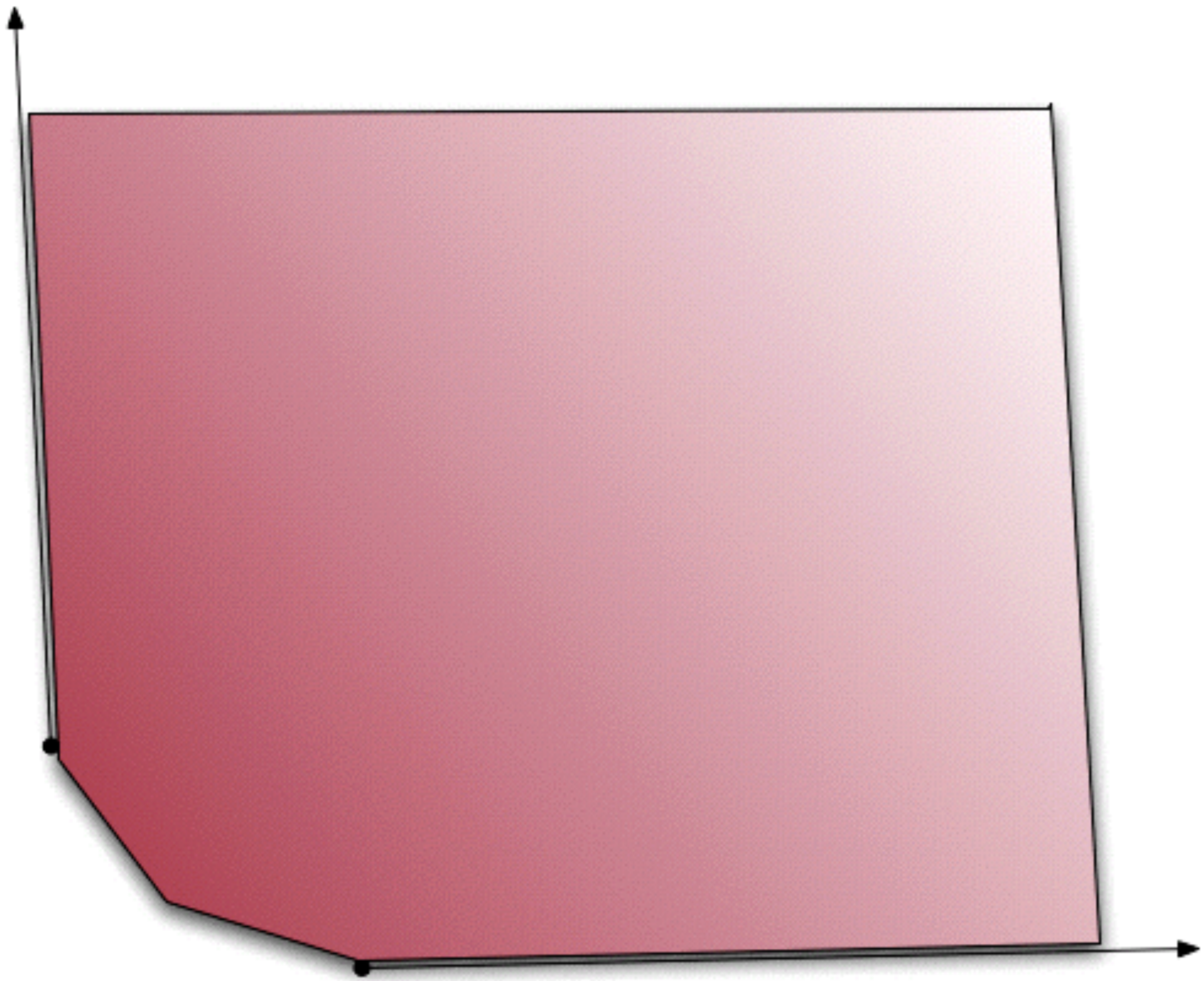

Calderbank-Singer

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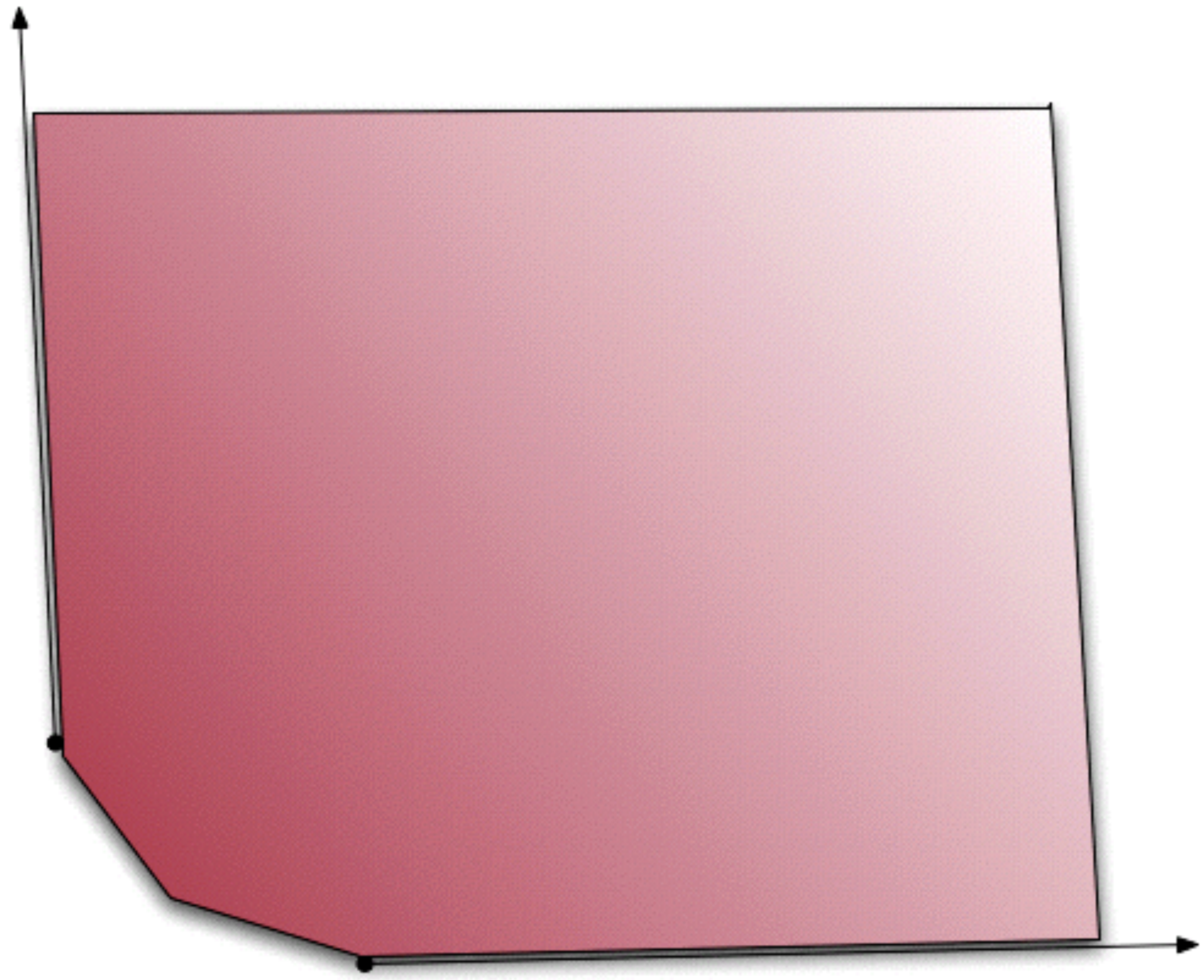

$\sim$ Calderbank-Singer: topology, $\int \mathring{r}^2 d\mu$, $\int |W_-|^2 d\mu$.

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Exclude: $[\omega]$, areas of homology generators.

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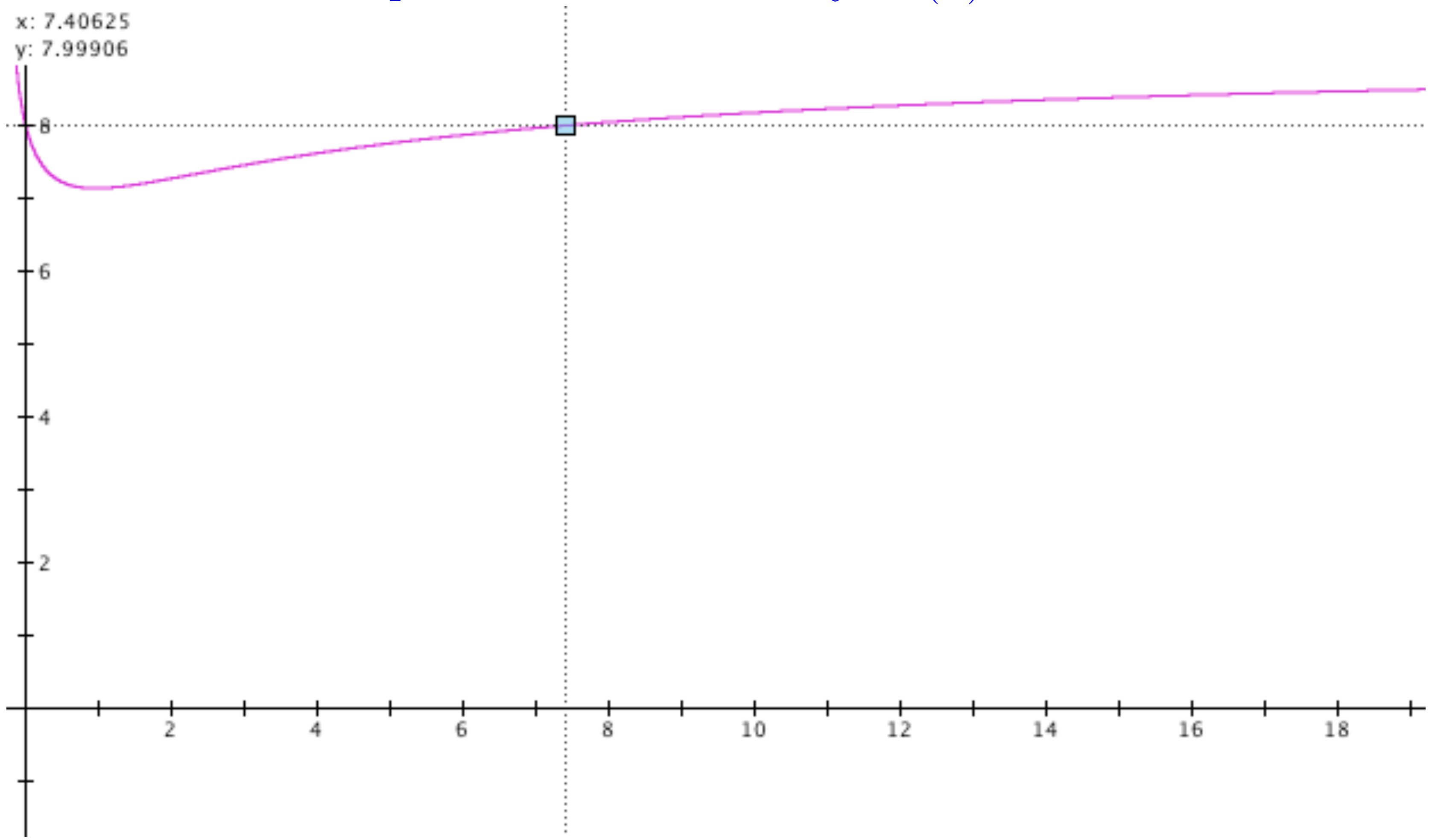

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Since $x_0 < L$, Theorem A follows.