Weyl Curvature,

Einstein Metrics, and

4-*Dimensional Geometry*

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Weyl tensor = Riemann curvature mod Ricci.

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$W^{a}{}_{bcd}$ unchanged if $g \rightsquigarrow \hat{g} = u^2 g$.

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Proposition. *Assume* $n \geq 4$. *Then* (M^n, g) *locally conformally flat* $\iff W \equiv 0$.

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- Do there exist minimizers?

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Warning: Proofs also require control of $\int s^2 d\mu$!

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Λ^+ self-dual 2-forms.

Λ^- anti-self-dual 2-forms.

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However, these are not independent!

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Euler characteristic

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(\frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu$$

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So $\int |W_+|^2 d\mu$ equivalent to Weyl functional.

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This is the pair of functionals we'll use henceforth.

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Reversing orientation $\rightsquigarrow$

self-duality $\longleftrightarrow$ anti-self-duality

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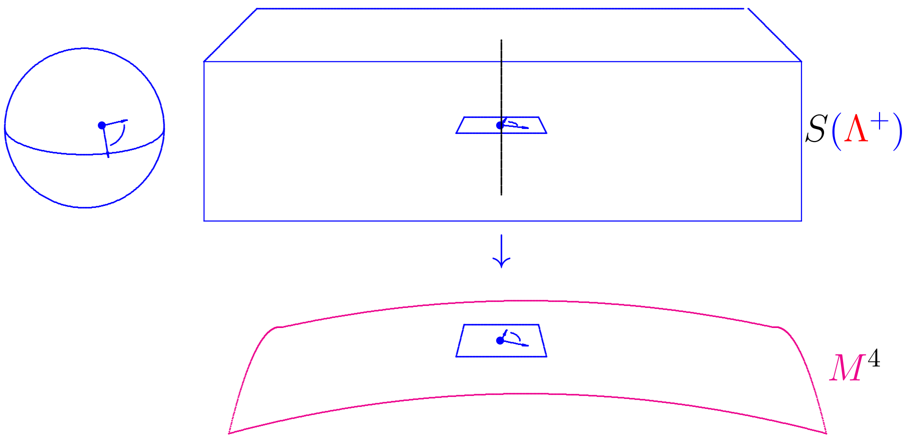

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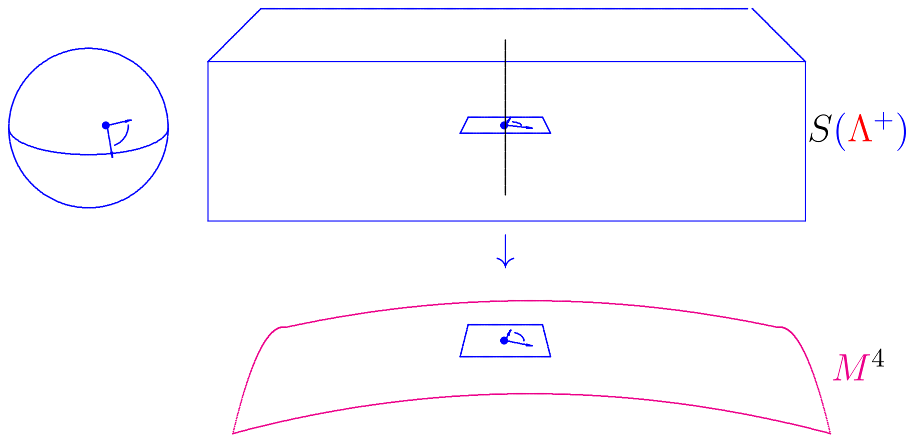
$Z = S(\Lambda^+)$, $J : TZ \to TZ$, $J^2 = -1$:



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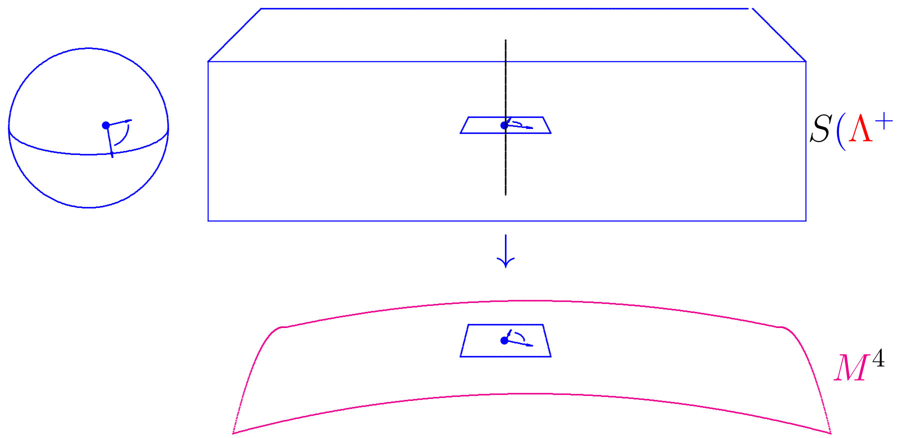

Theorem (Atiyah-Hitchin-Singer). (Z, J) *is a complex 3-manifold iff* $W_+ = 0$*.*

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Conformally flat! Developing map $\tilde{M} \to S^4$.

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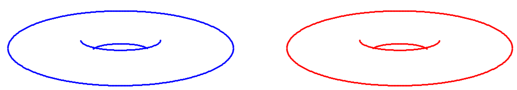

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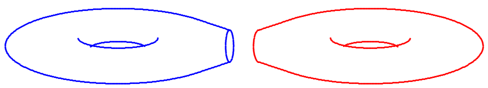

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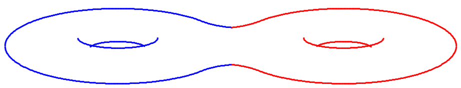

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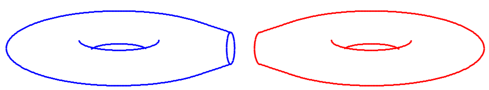

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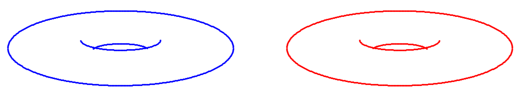

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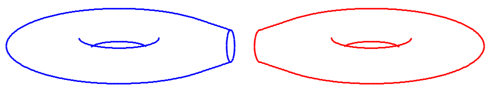

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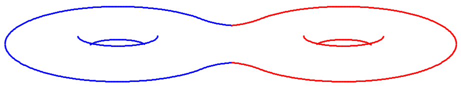

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Violate Hitchin-Thorpe inequality $2\chi + 3\tau \geq 0$.

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- What about $2 \leq m \leq 9$?
- What is $\inf \mathscr{W}$ when $m \leq 9$?
- Do minimizers exist?

Conjecture. $\mathbb{CP}_2 \# m\overline{\mathbb{CP}_2}$ *does not admit an ASD metric if* $m \leq 9$.

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for these manifolds.

Conjecture. $\mathbb{CP}_2 \# m \overline{\mathbb{CP}_2}$ *does not admit an ASD metric if* $m \leq 9$. *Moreover, for* $m \neq 1, 2$,

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To rephrase:

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"Anorexic Sequences": $\int |W_+|^2 d\mu \to 0$, etc.

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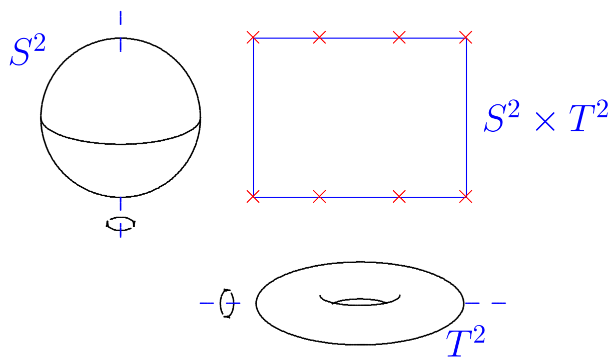

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Only two metrics arise in non-Kähler case!

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Strictly four-dimensional phenomenon.

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But independent for general Riemannian metrics.

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Natural Question. *When does Einstein metric* g *on* 4*-manifold* M *minimize one or both of these functionals?*

Theorem (L '95).

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Key idea due to Witten '94.

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Proof depends on Seiberg-Witten equations

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Proof depends on Seiberg-Witten equations

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Non-linear version of Dirac equation,

Theorem (L '95). *If smooth compact M^4 admits Kähler-Einstein metric g with $\lambda \leq 0$, then g is absolute minimizer of $\int_M s^2 d\mu$ among all Riemannian metrics on M. Moreover, any metric $\tilde{g}$ on M satisfies*

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only defined in dimension 4.

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Einstein metrics with $\lambda > 0$ never minimize $\int_M s^2 d\mu$!

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$Y([\tilde{g}]) > 0 \Longleftrightarrow \exists\ s > 0$ metrics in $[\tilde{g}]$.

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Doesn't answer our question, but suggestive!

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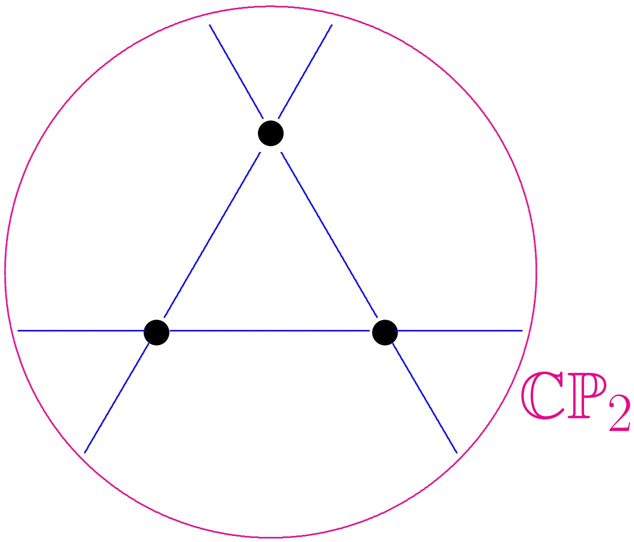
Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



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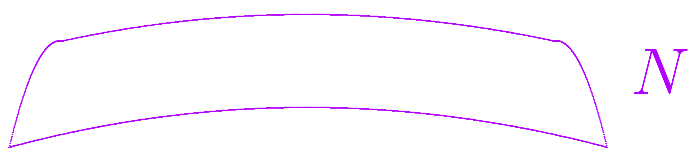

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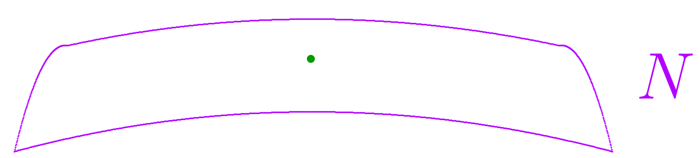

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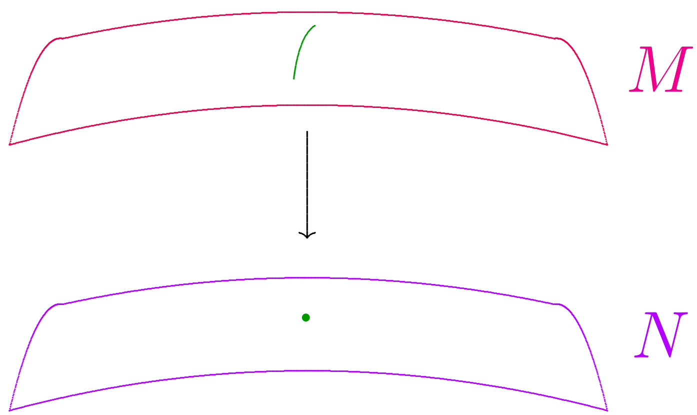

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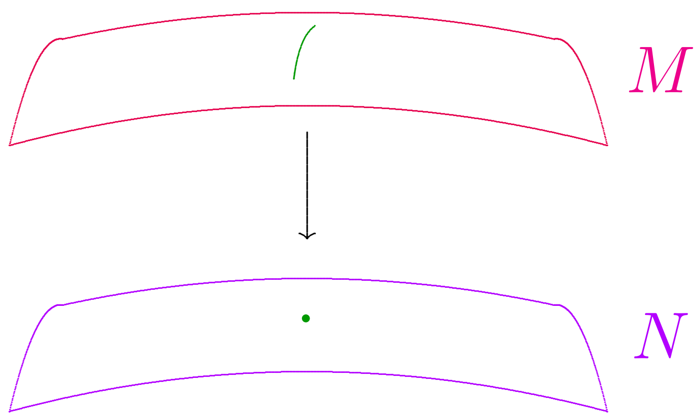
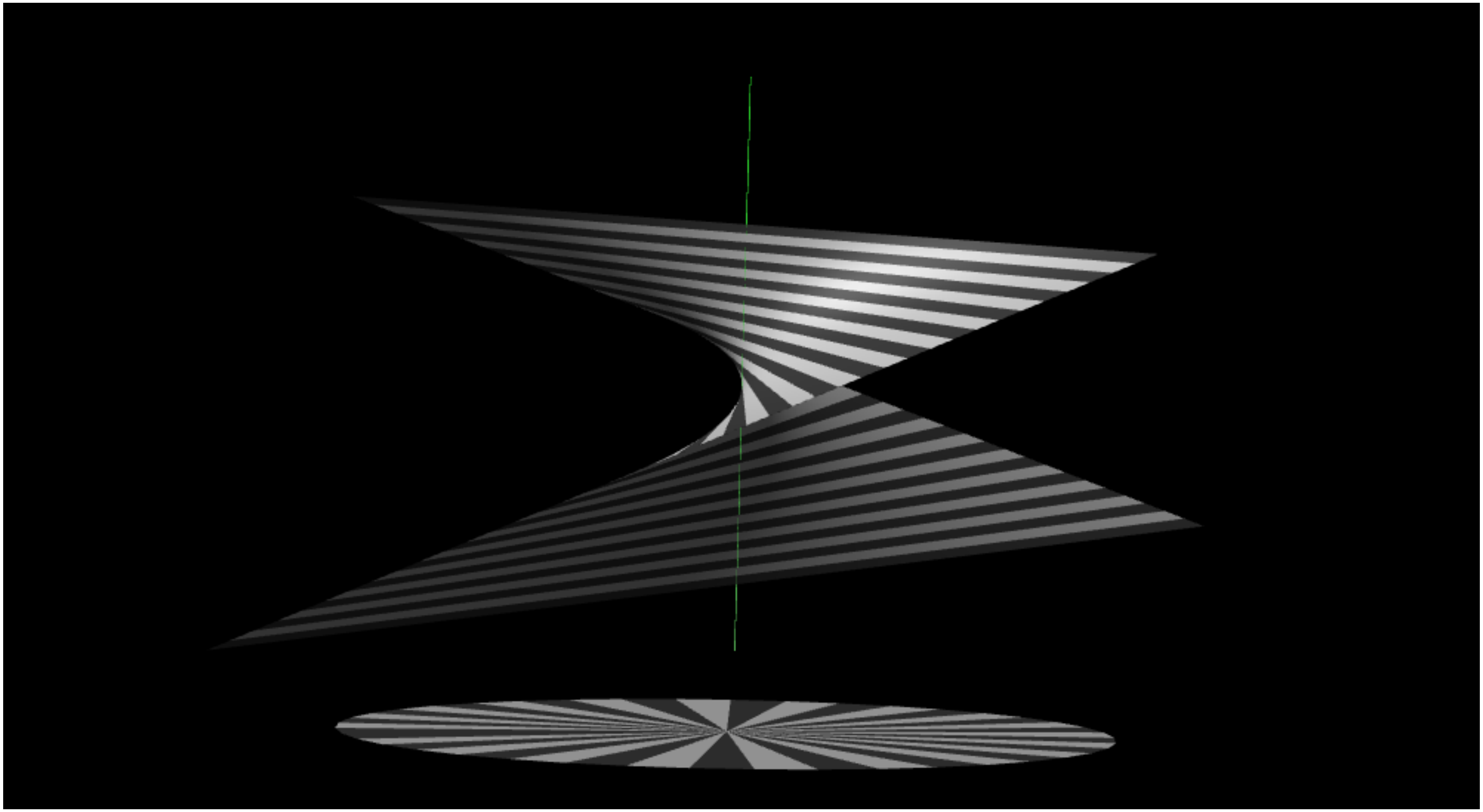
If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.





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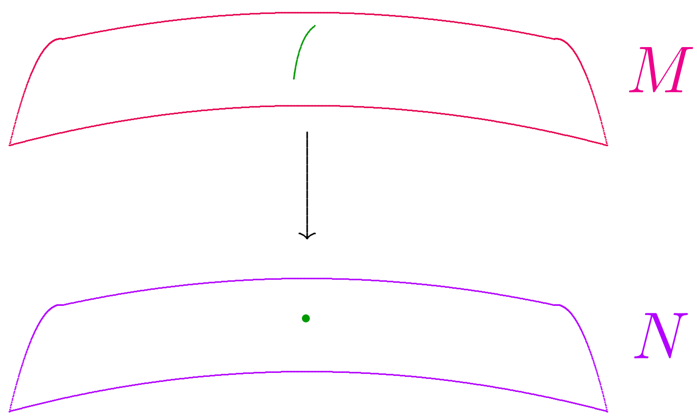

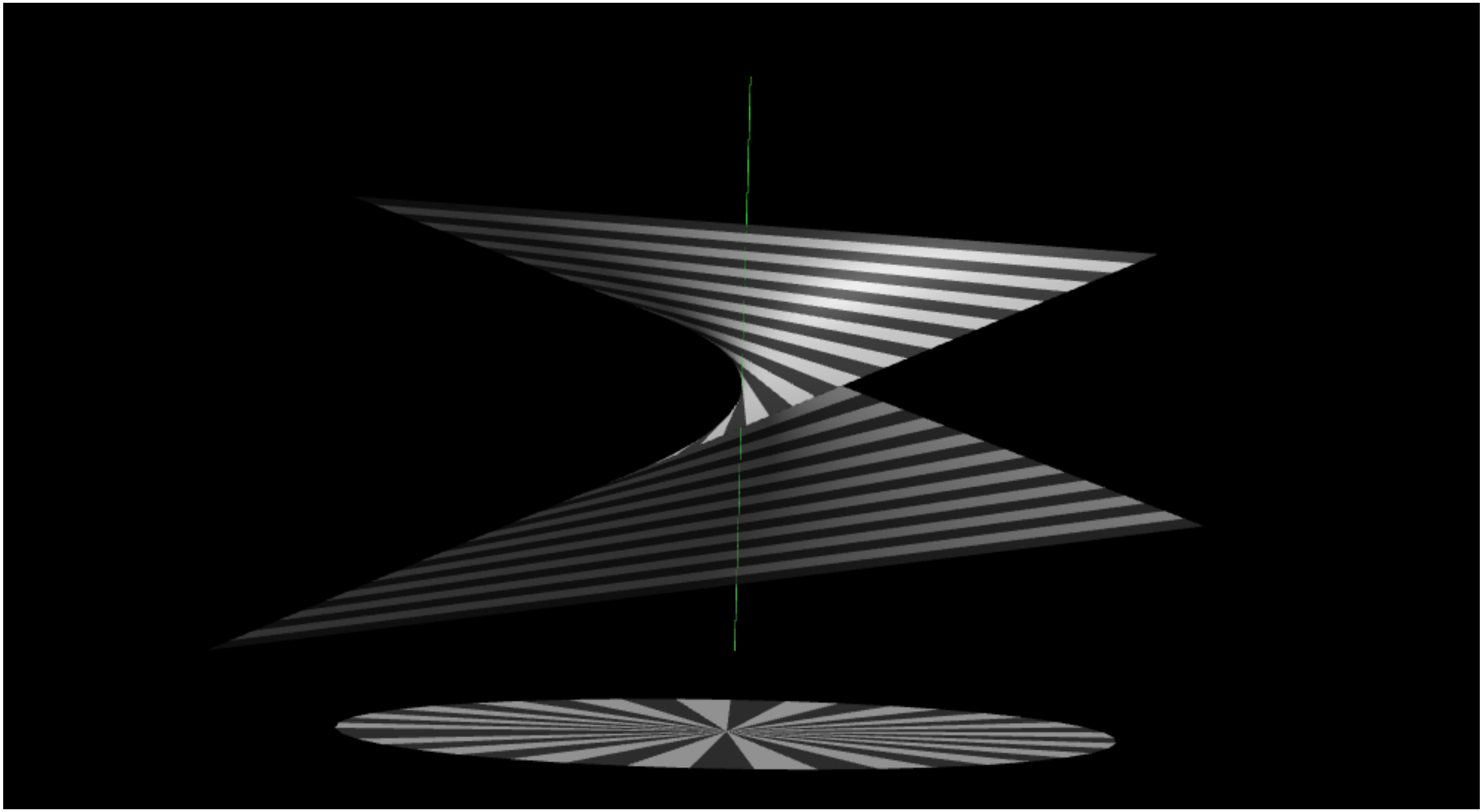

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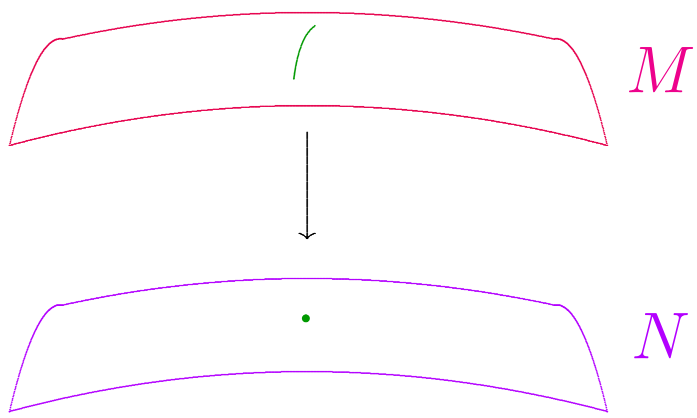

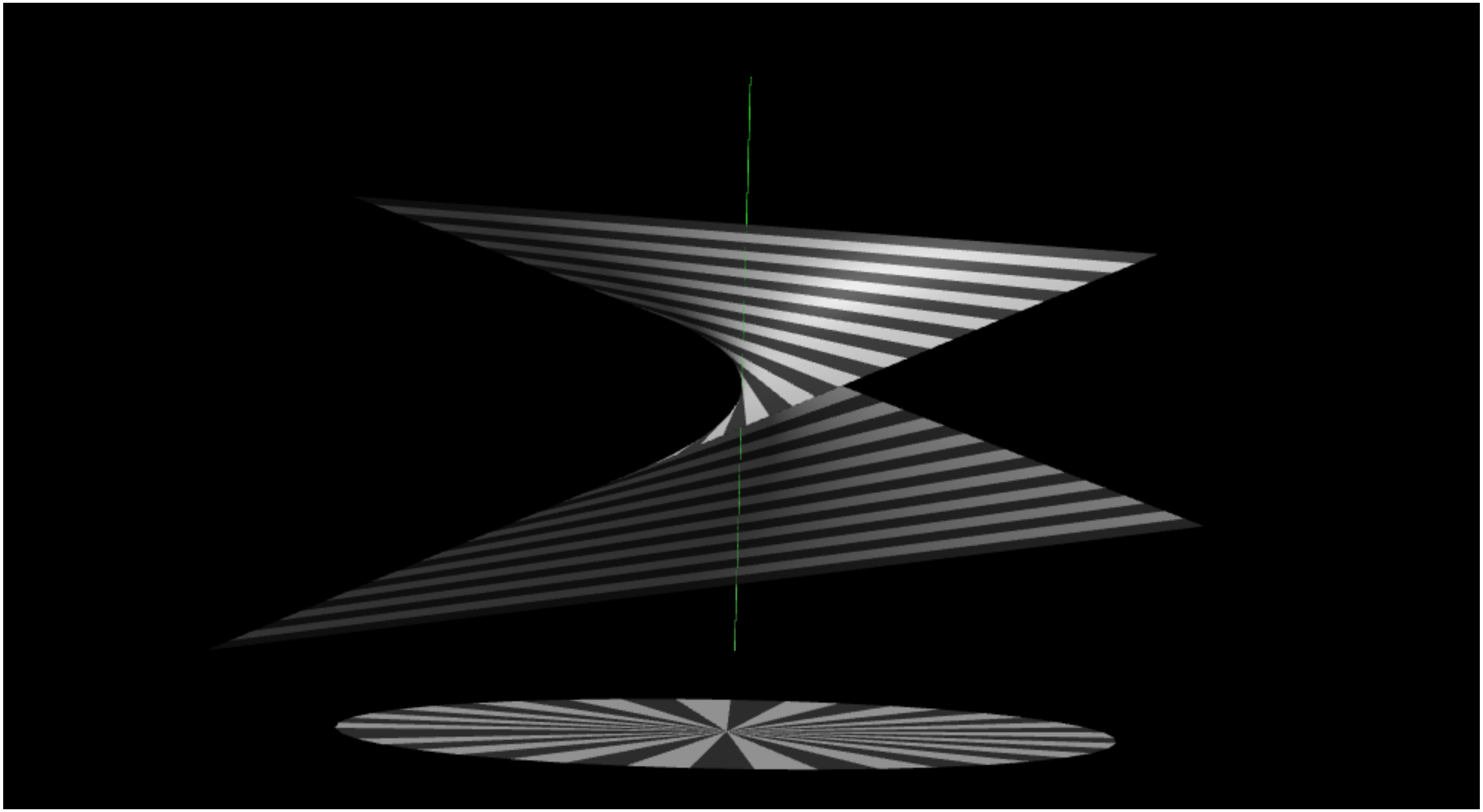

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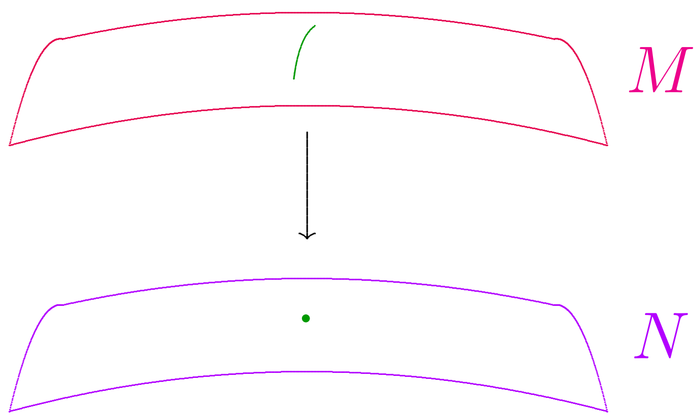

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- holds in Kähler case;
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This follows from a stronger inequality…

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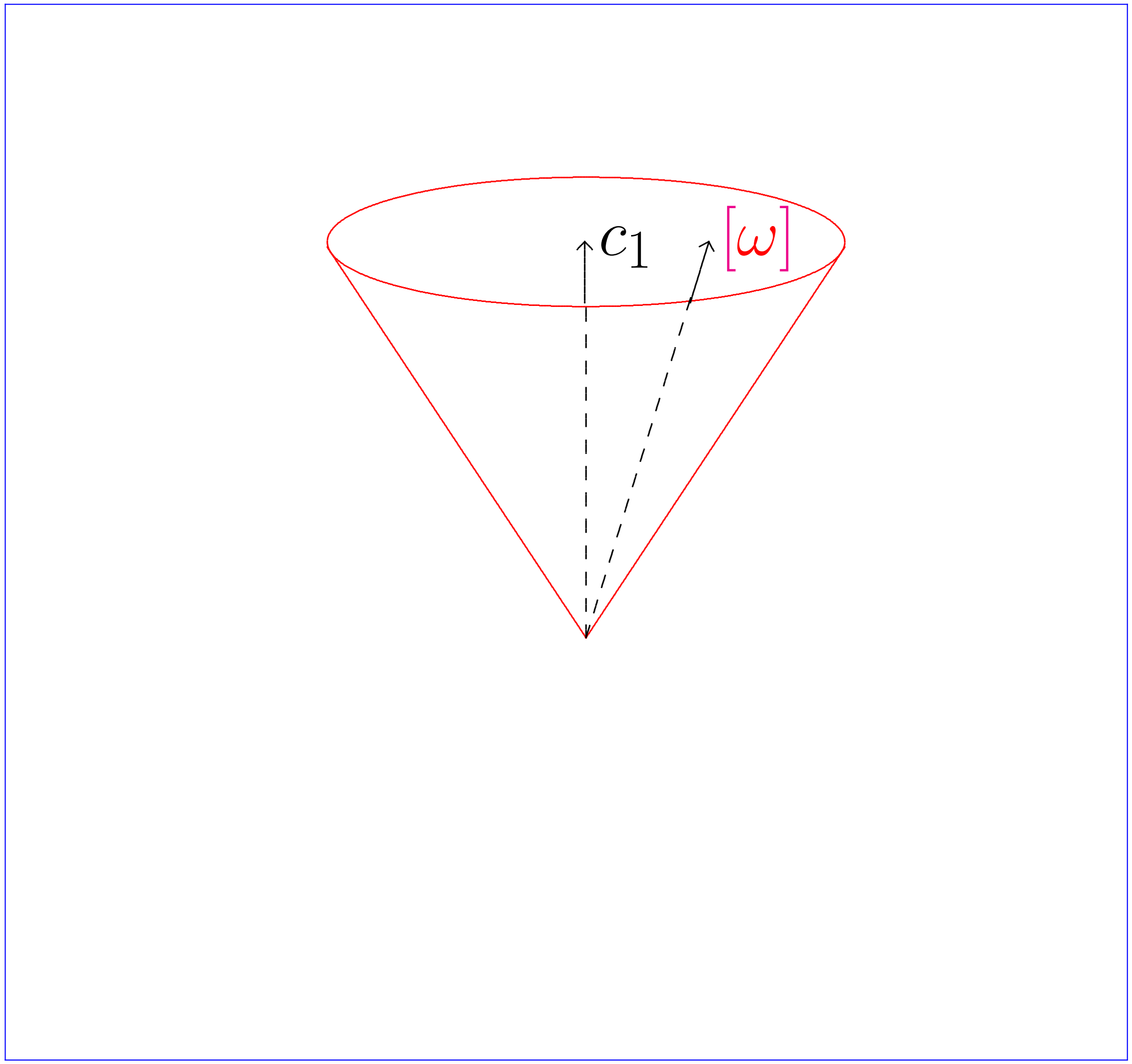
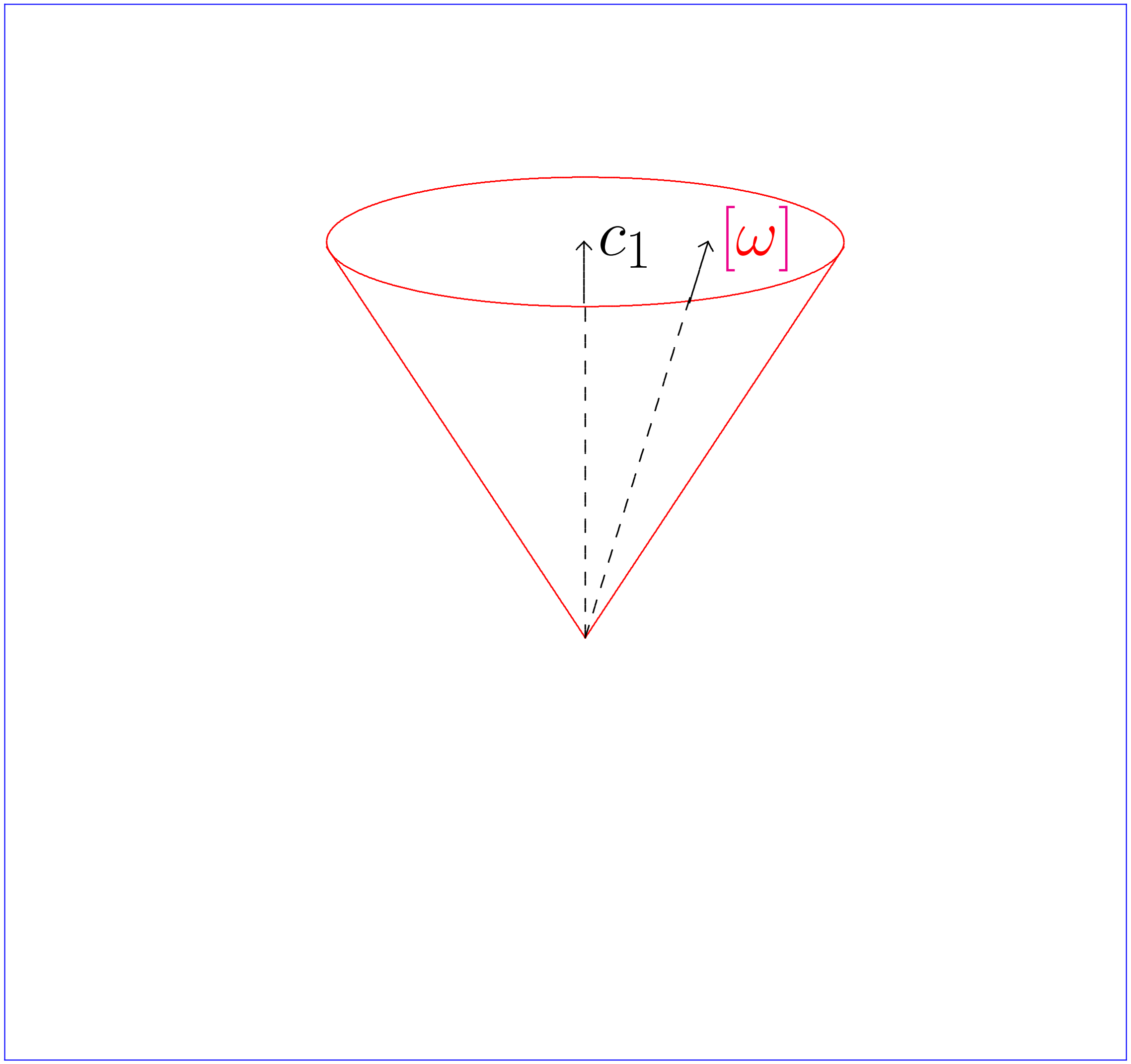


$H^2(M, \mathbb{R})$

M Del Pezzo



$$c_1 \cdot [\omega] \geq \sqrt{{c_1}^2 \, [\omega]^2}$$

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$\exists$ connection on $L = \Lambda^{0,2}$ such that

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$\exists$ connection on $L = \Lambda^{0,2}$ such that

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Theorem B. *Let M be the underlying 4-manifold of a del Pezzo surface. Then any conformal class $[g]$ of symplectic type on M satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}\frac{(c_1 \cdot [\omega])^2}{[\omega]^2},$$

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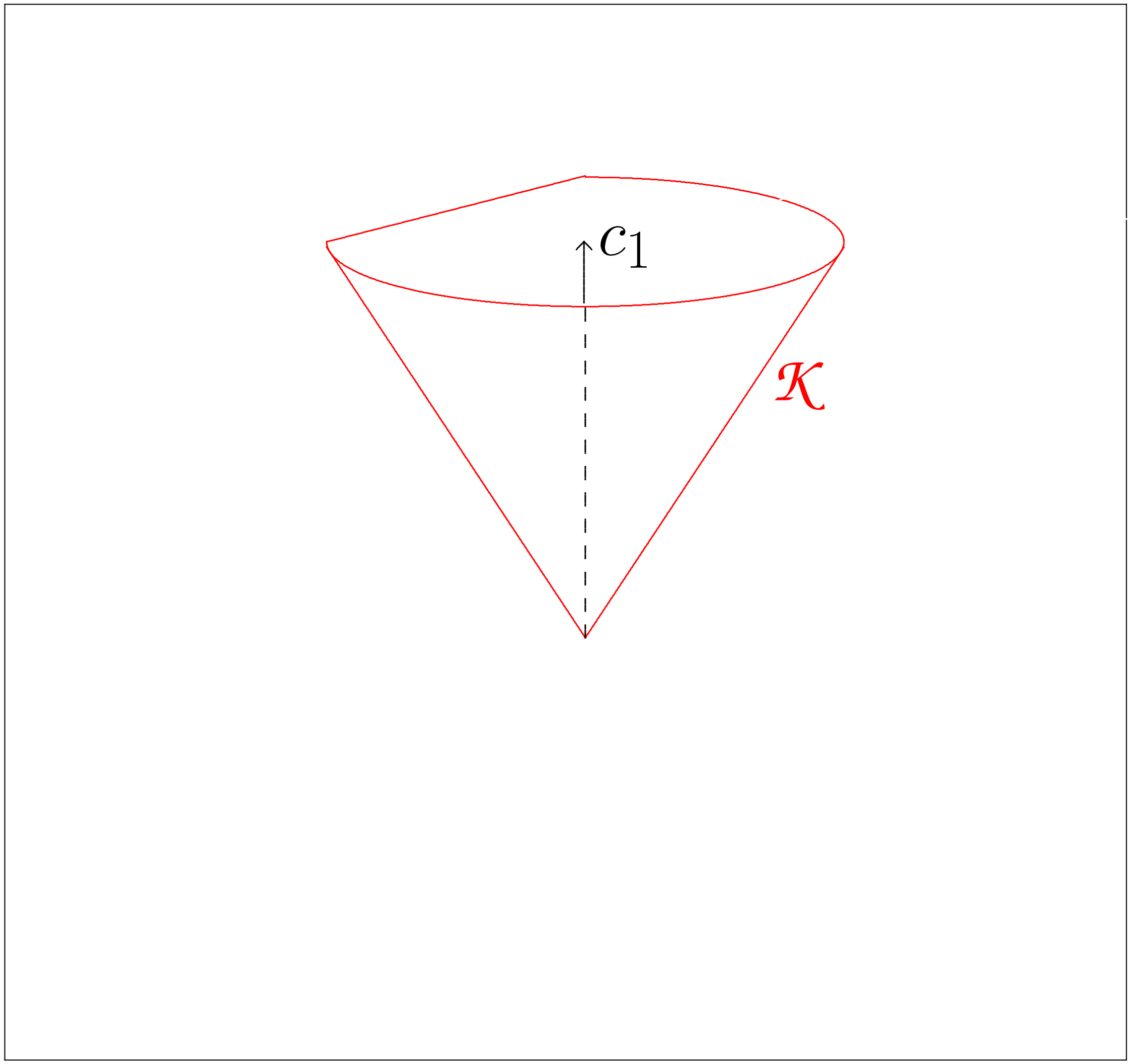

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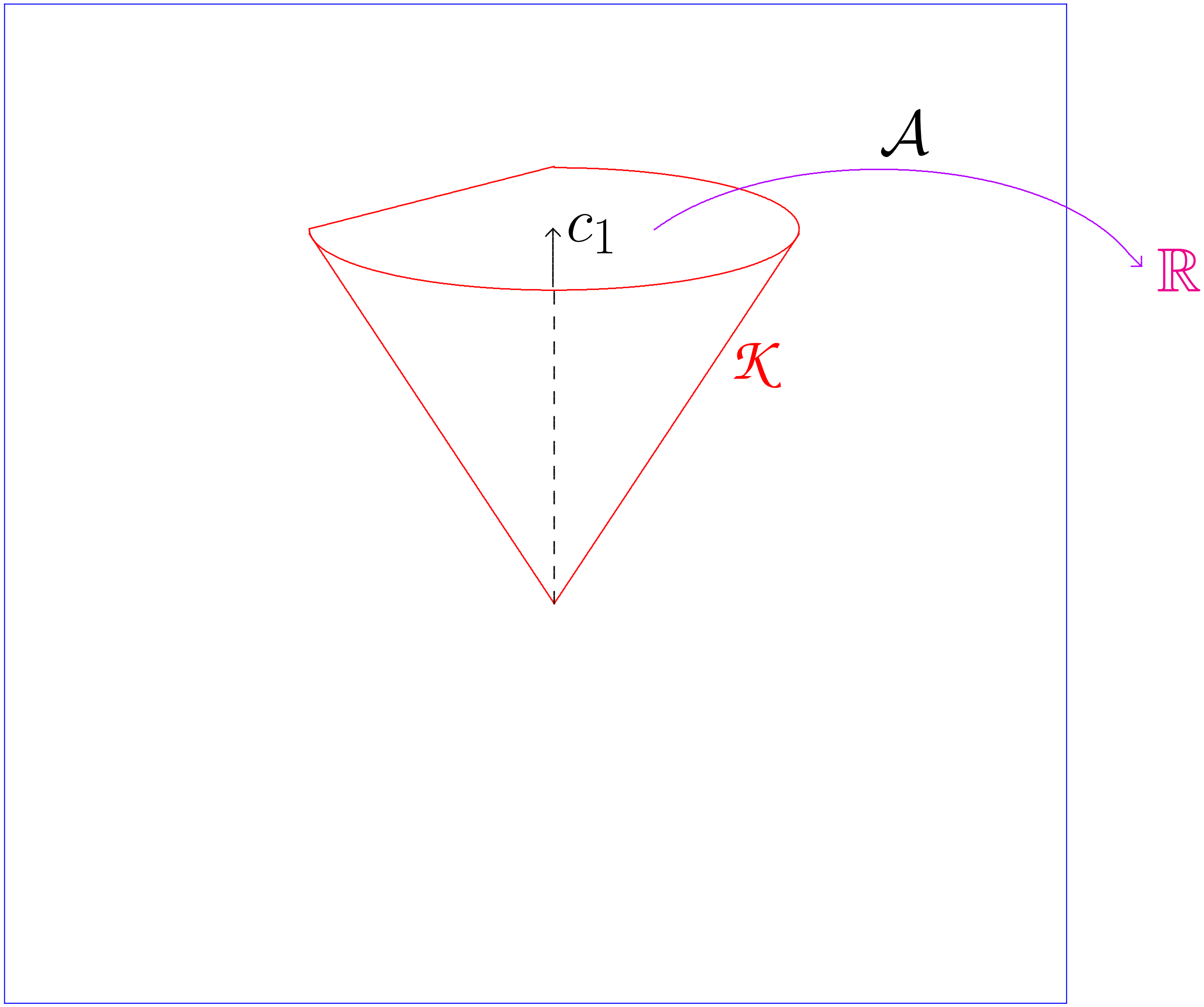

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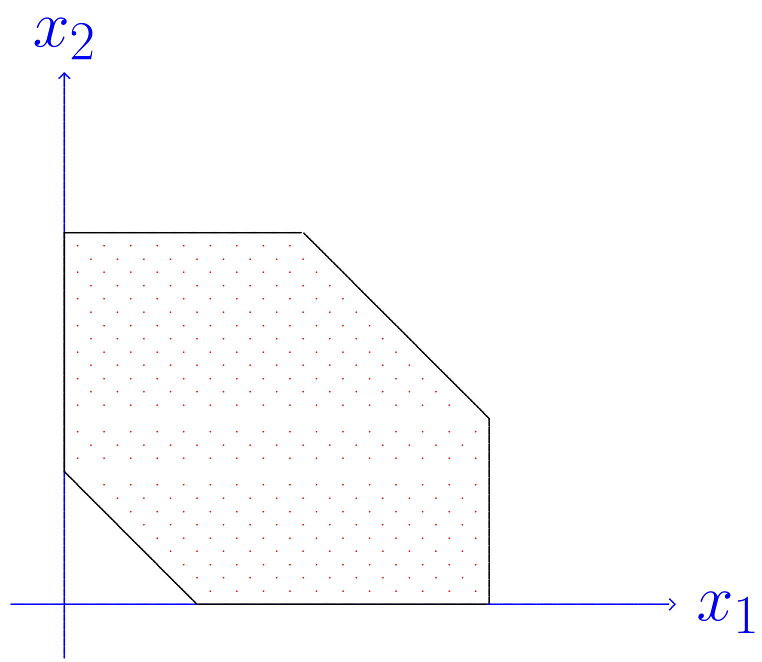

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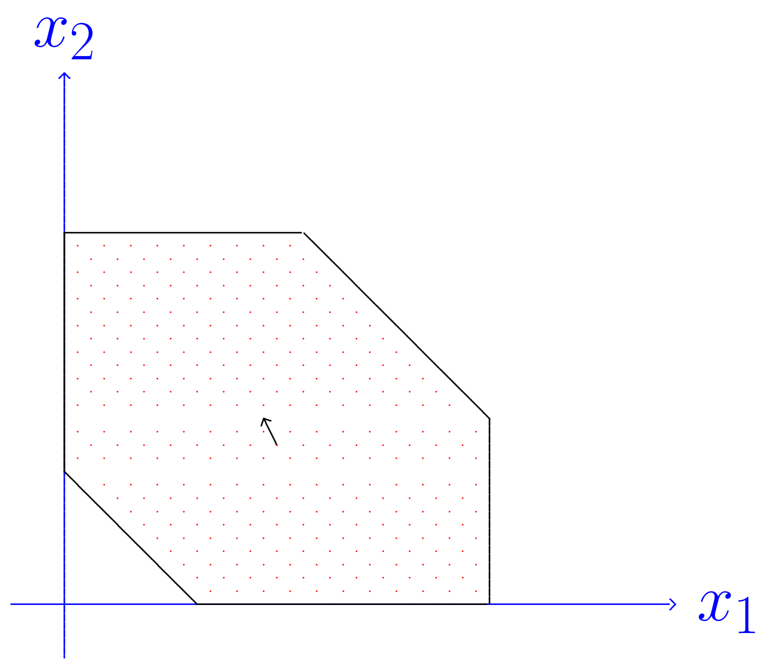

$$\mathcal{A}([\omega]) = \frac{|\partial P|^2}{2}\left(\frac{1}{|P|} + \vec{\mathfrak{D}} \cdot \Pi^{-1}\vec{\mathfrak{D}}\right)$$

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Non-Kähler cases: eliminate toric condition?

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$$M \approx \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{cases}$$