

Einstein Constants

and

Differential Topology

Claude LeBrun

Stony Brook University

Mathematics Colloquium,
King's College London,
October 9, 2025.

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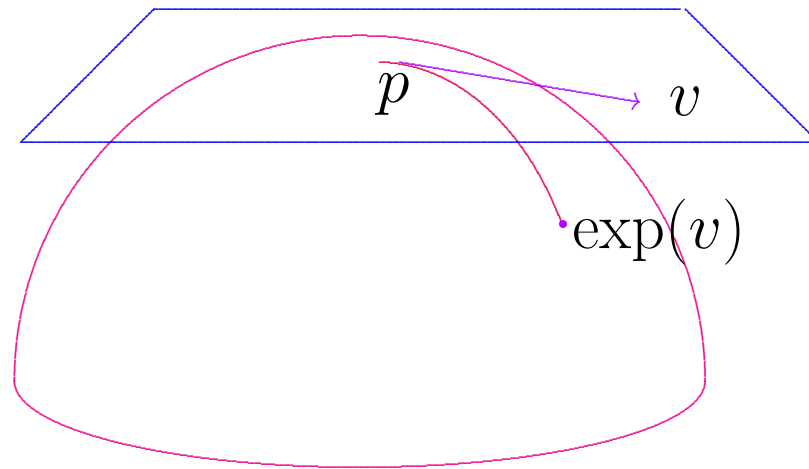
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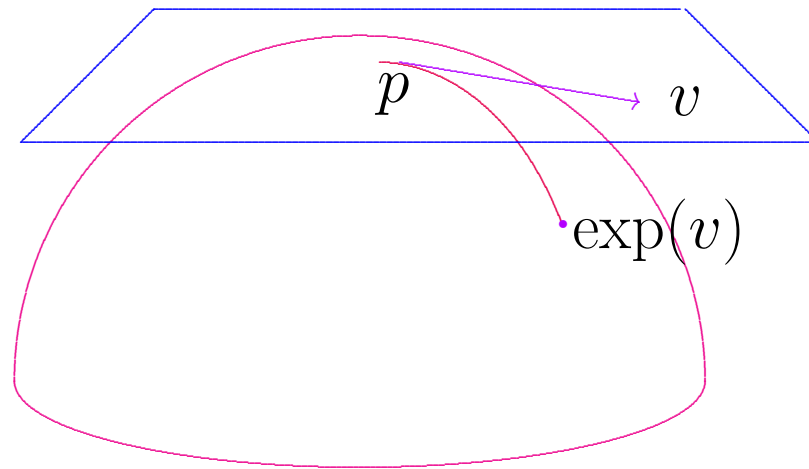
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Now choosing $T_p M \xrightarrow{\cong} \mathbb{R}^n$ via some orthonormal basis gives us special coordinates on M .

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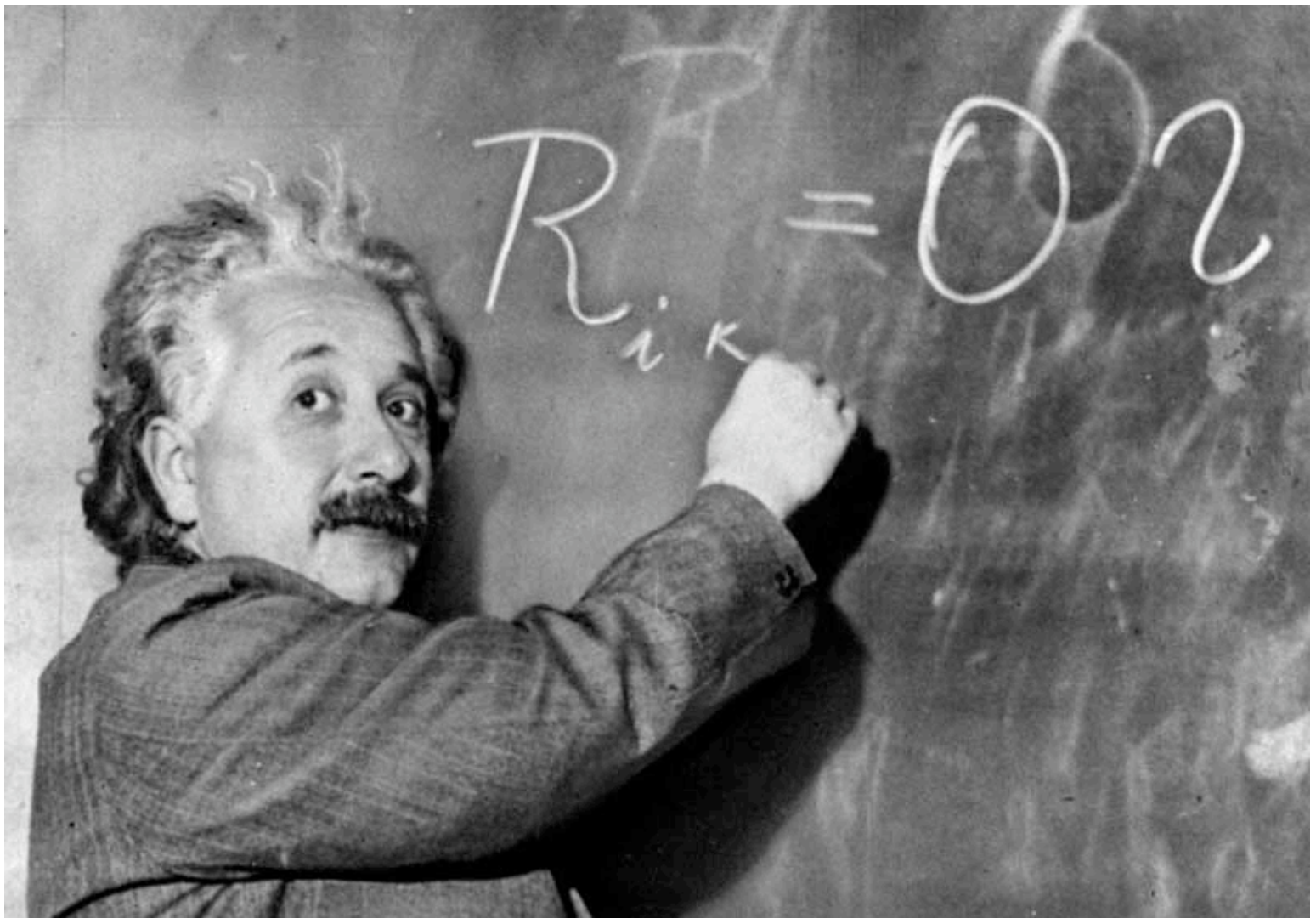
“...the greatest blunder of my life!”

— A. Einstein, to G. Gamow

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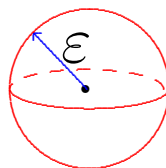
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$$\frac{\text{vol}_g(B_\varepsilon(p))}{c_n \varepsilon^n} = 1 - s \frac{\varepsilon^2}{6(n+2)} + O(\varepsilon^4)$$



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$$\Delta x^j = 0 \implies r_{jk} = \frac{1}{2} \Delta g_{jk} + \text{lots.}$$

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Ergebnisse der Mathematik und ihrer Grenzgebiete

3. Folge · Band 10

A Series of Modern Surveys in Mathematics

Arthur L. Besse

Einstein Manifolds



Springer-Verlag Berlin Heidelberg GmbH



Marcel Berger



Besse en Chandesse



L'Auvergne

Acknowledgements

Pour rassembler les éléments un peu disparates qui constituent ce livre, j'ai dû faire appel à de nombreux amis, heureusement bien plus savants que moi. Ce sont, entre autres, Geneviève Averous, Lionel Bérard-Bergery, Marcel Berger, Jean-Pierre Bourguignon, Andrei Derdzinski, Dennis M. DeTurck, Paul Gauduchon, Nigel J. Hitchin, Josette Houillot, Hermann Karcher, Jerry L. Kazdan, Norihito Koiso, Jacques Lafontaine, Pierre Pansu, Albert Polombo, John A. Thorpe, Liane Valère.

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Enfin, qu'il me soit permis de saluer ici mon prédécesseur et homonyme Jean Besse, de Zürich, qui s'est illustré dans la théorie des fonctions d'une variable complexe (voir par exemple [Bse]).

Vôtre,

A stylized, handwritten signature in black ink. It features a large, loopy 'A' shape on the left, followed by a series of overlapping loops and a horizontal stroke that ends in a small, pointed flourish on the right.

Arthur Besse

Le Faux, le 15 septembre 1986

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But it turns out that the answer is actually **Yes!**

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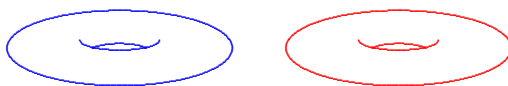
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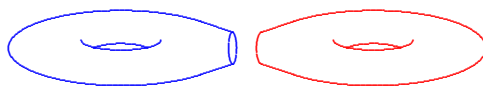
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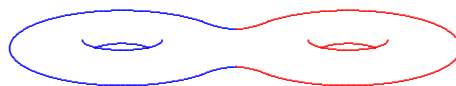
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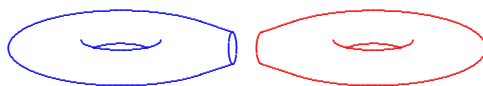
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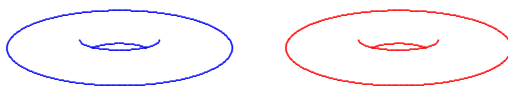
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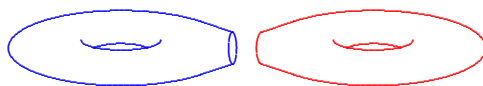
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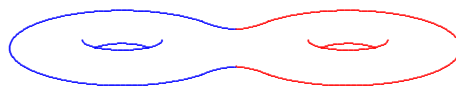
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of two Einstein manifolds is Einstein

$\iff$ they have the same Einstein constant λ .

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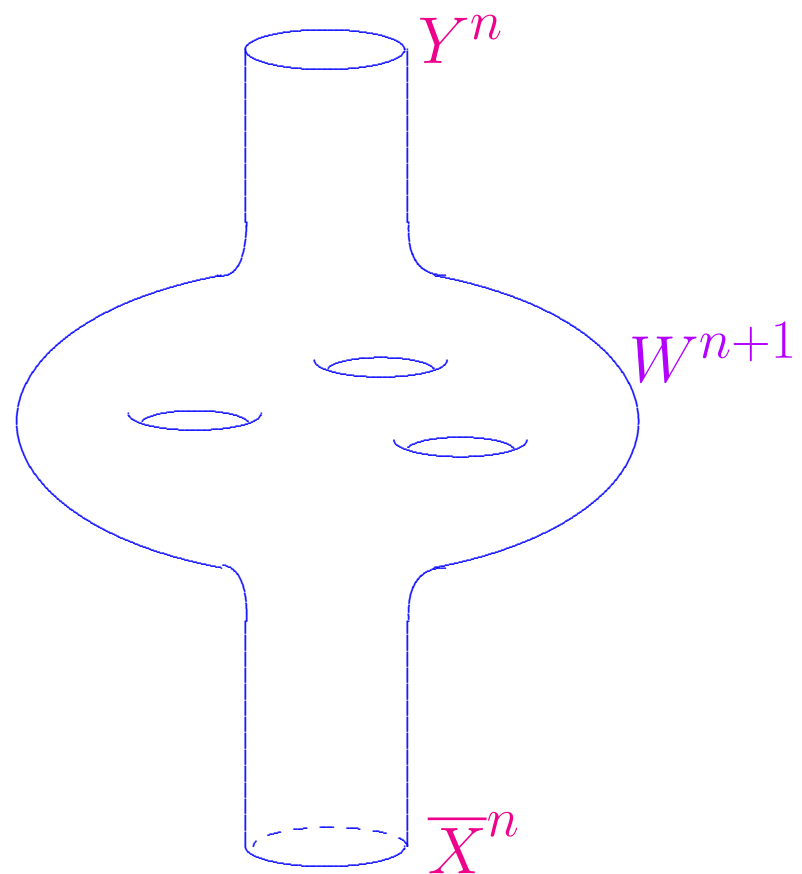
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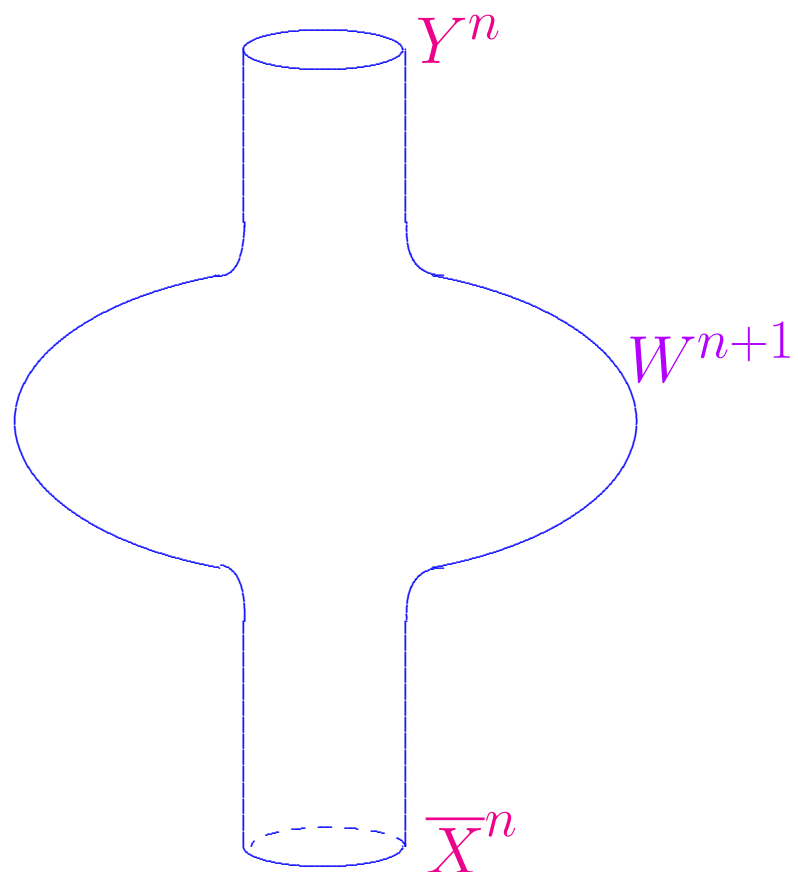
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- Smale's h -cobordism theorem.

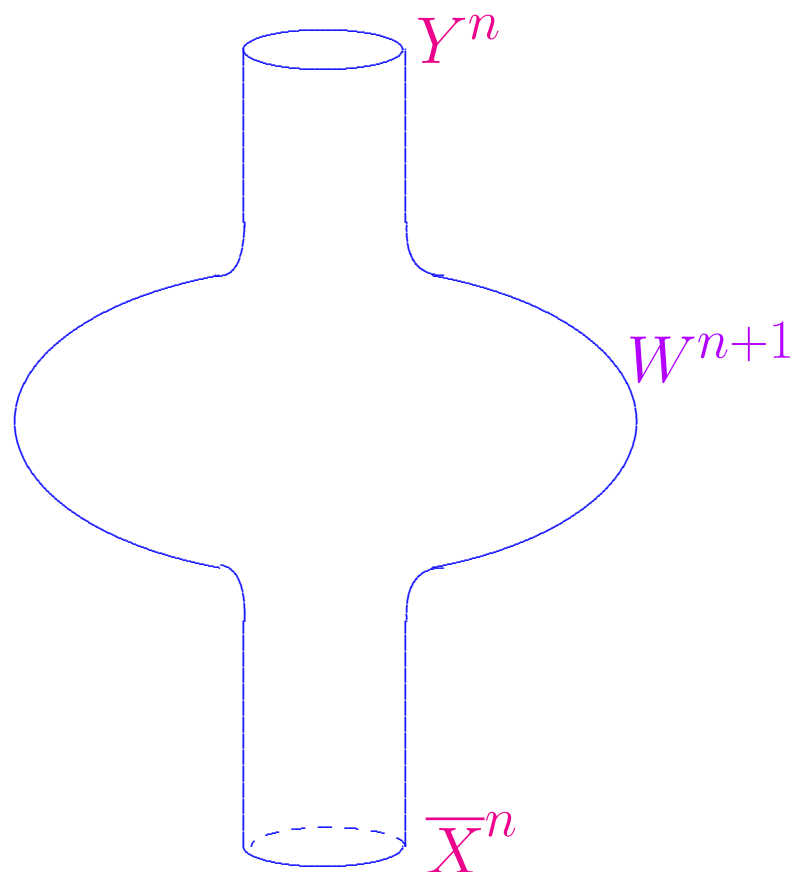


Cobordism

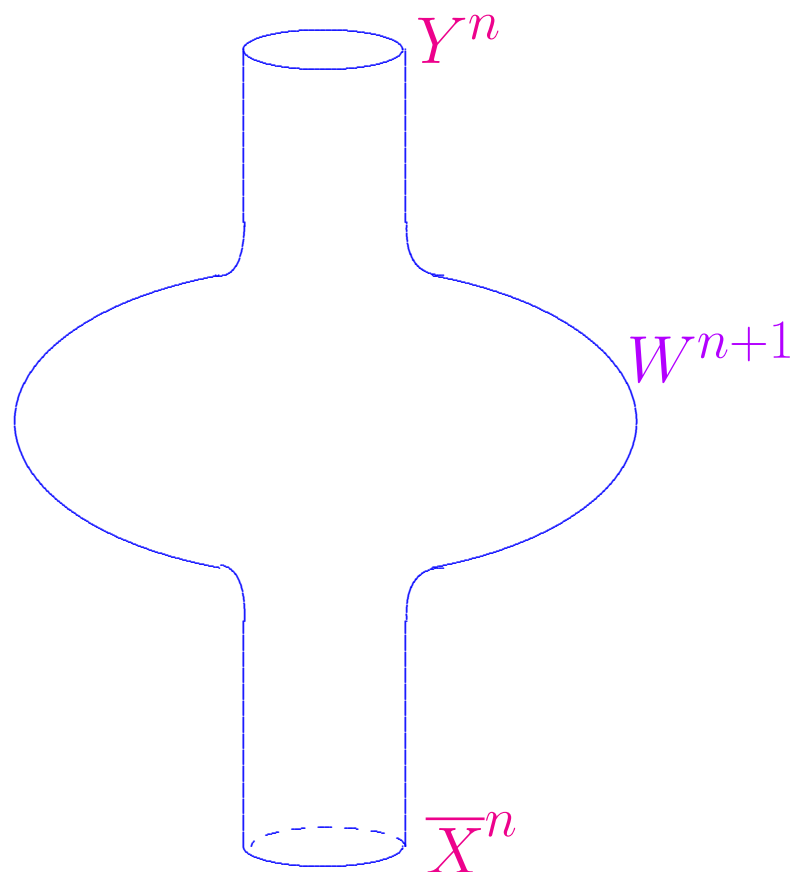


h -Cobordism

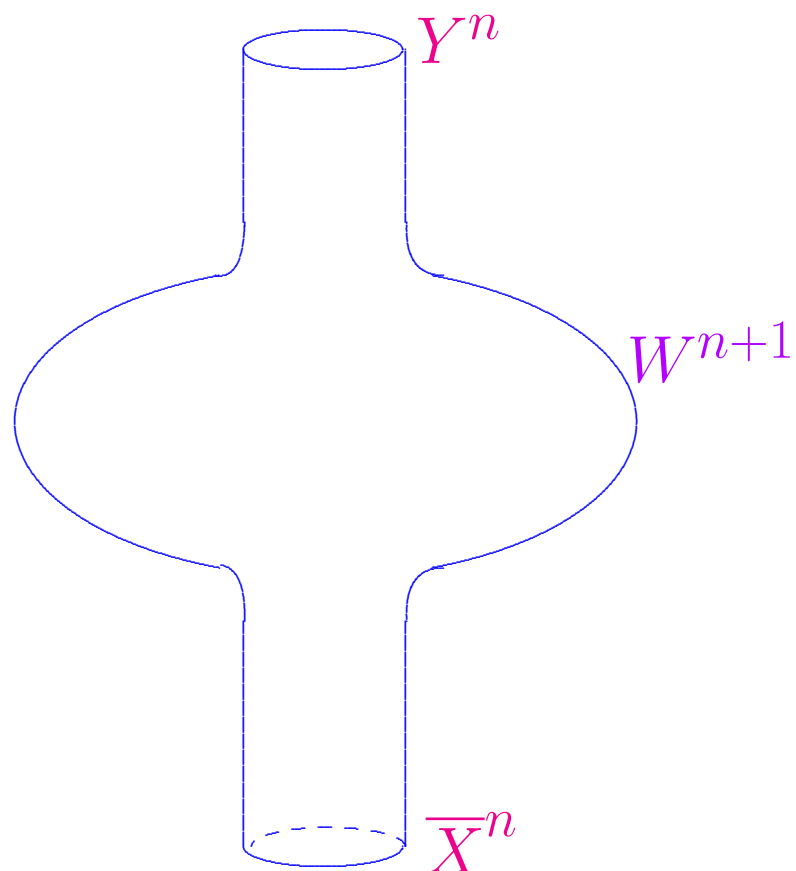
if $X \hookrightarrow W$, $Y \hookrightarrow W$ both homotopy equivalences



Smale: Suppose that X^n is h -cobordant to Y^n . If $\pi_1 = 0$ and $n > 4$, then X is diffeomorphic to Y .

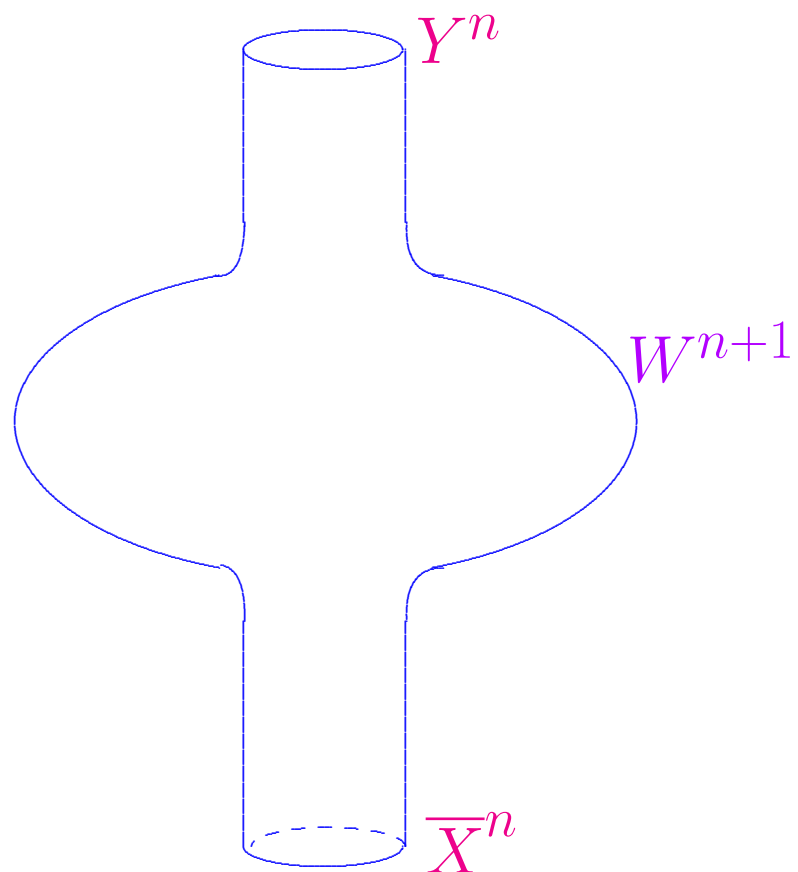


Wall: Suppose that X^4 homotopy equivalent to Y^4 .
 If $\pi_1 = 0$, then X is h -cobordant to Y .

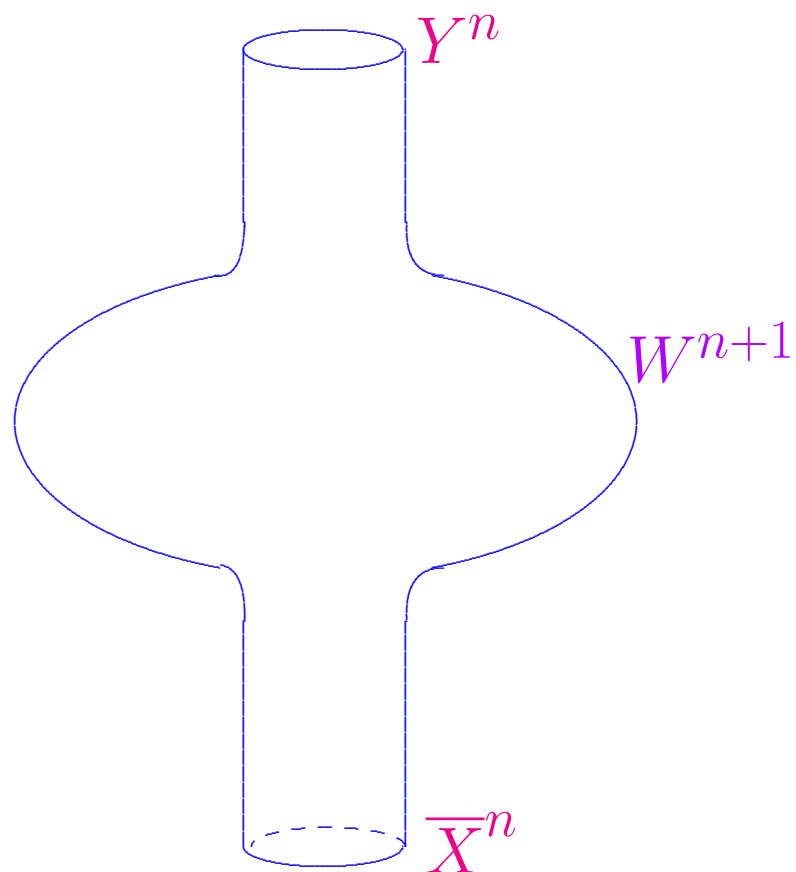


Wall: Suppose that X^4 homotopy equivalent to Y^4 .
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But Smale doesn't apply when $n = 4$!

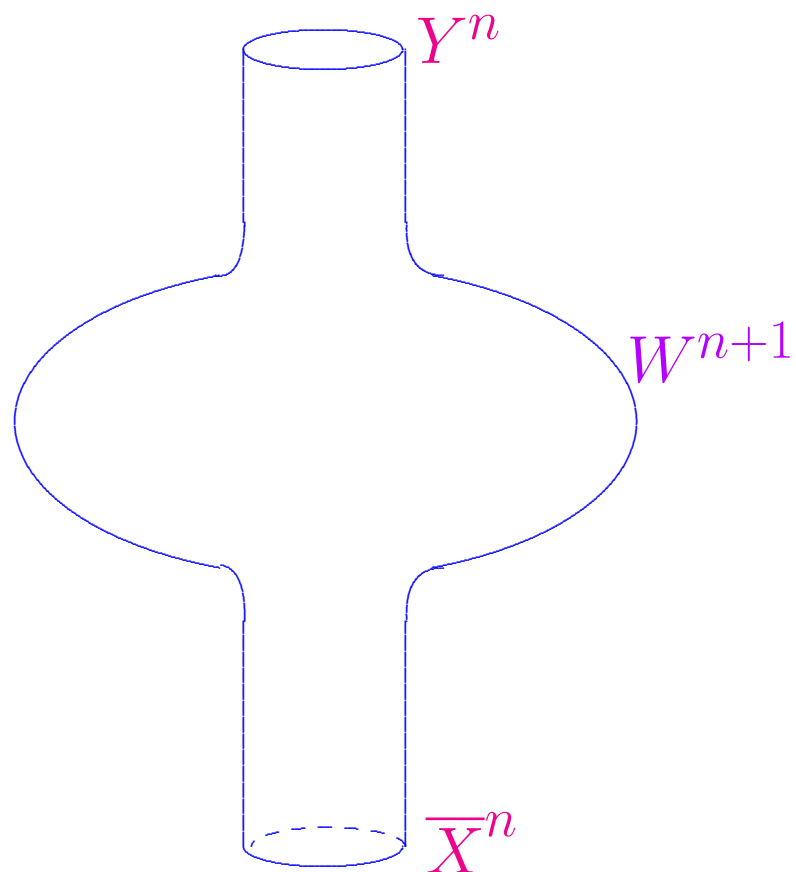


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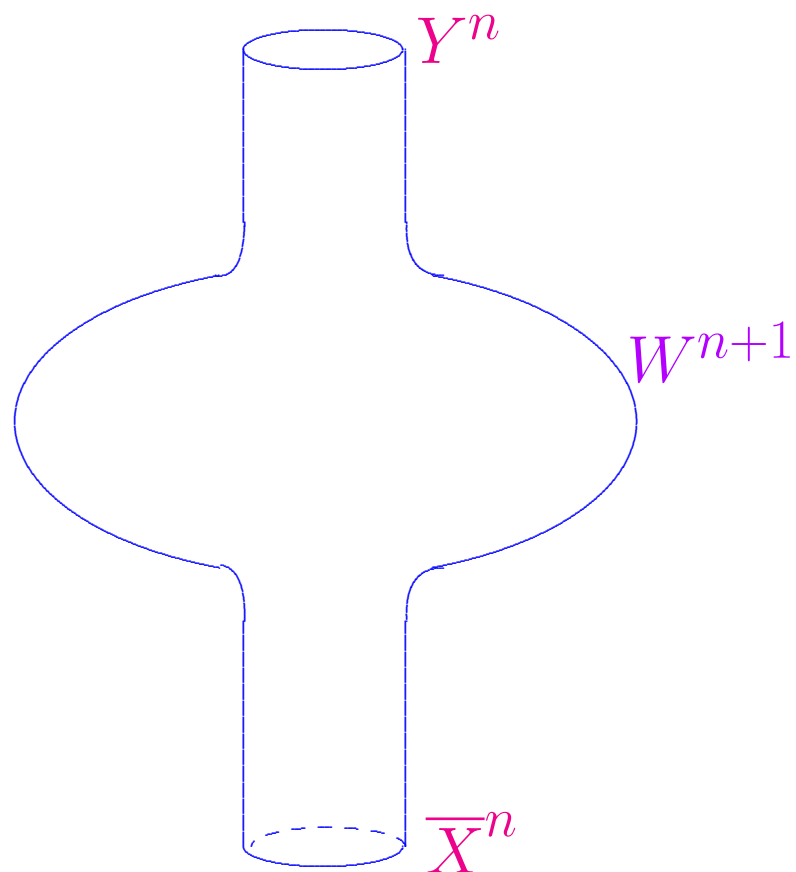
Lemma. *If X^4 and Y^4 are homotopy equivalent and simply connected, then $X \times X$ is actually diffeomorphic to $Y \times Y$.*



Indeed, if W is h -cobordism $X \sim Y$, then

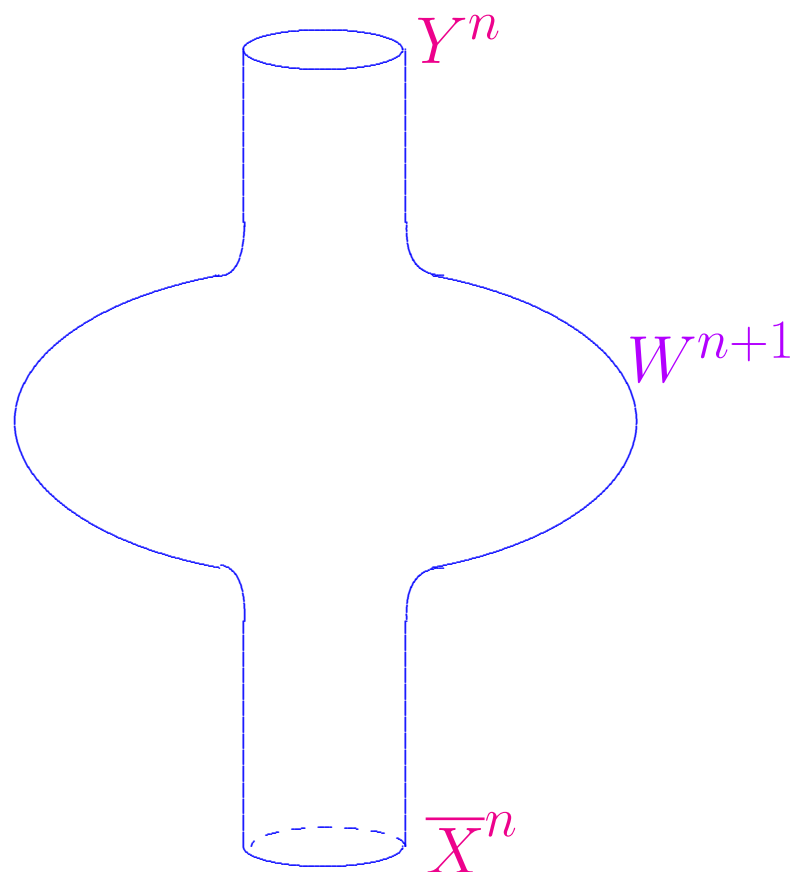
$$(X \times W) \cup_{X \times Y} (W \times Y)$$

is an h -cobordism $(X \times X) \sim (Y \times Y)$.



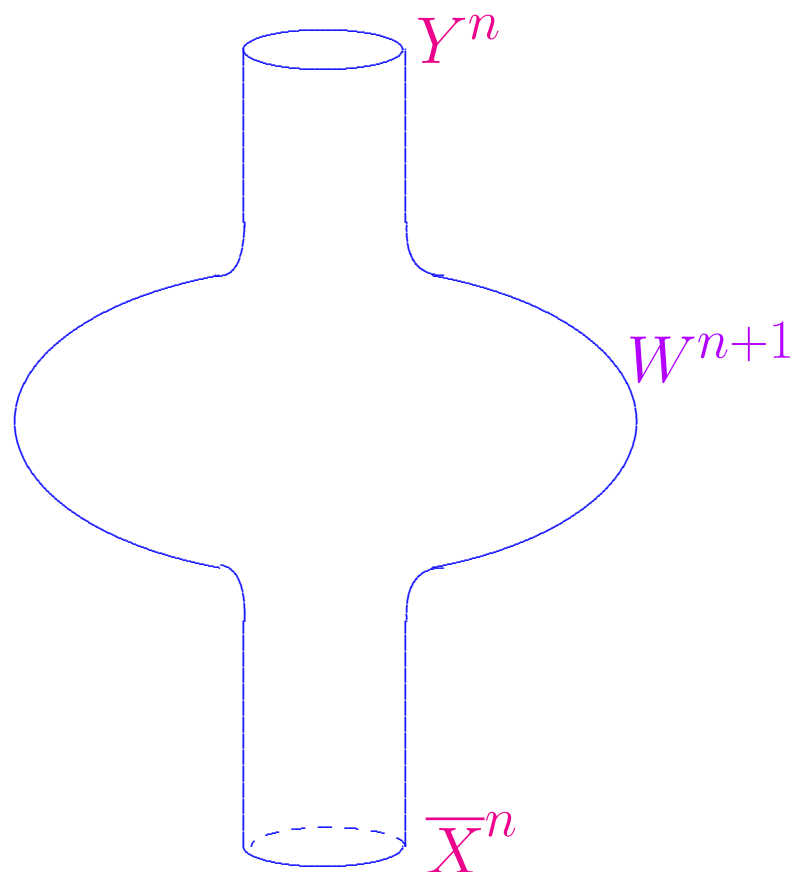
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Lemma. *If X^4 and Y^4 are simply connected, non-spin, with $\chi(X) = \chi(Y)$, $\tau(X) = \tau(Y)$, then*

$$\underbrace{X \times \cdots \times X}_k \approx_{\text{diff}} \underbrace{Y \times \cdots \times Y}_k \quad \forall k \geq 2.$$

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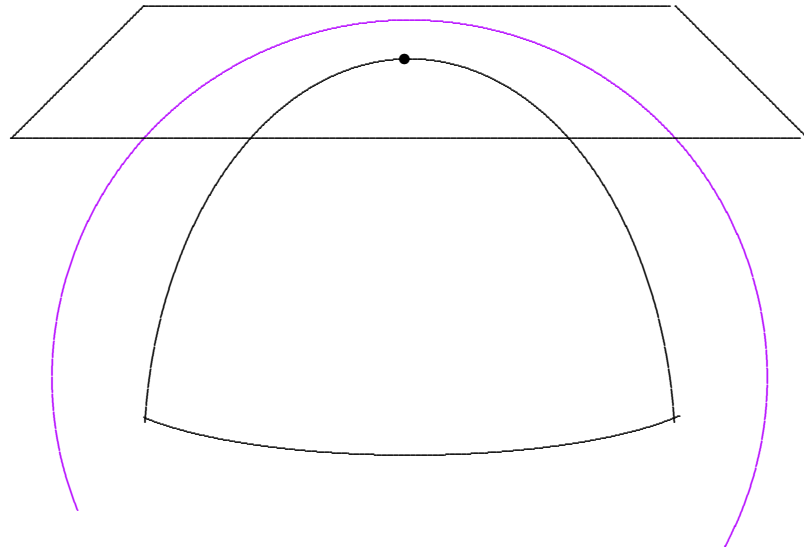
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holonomy

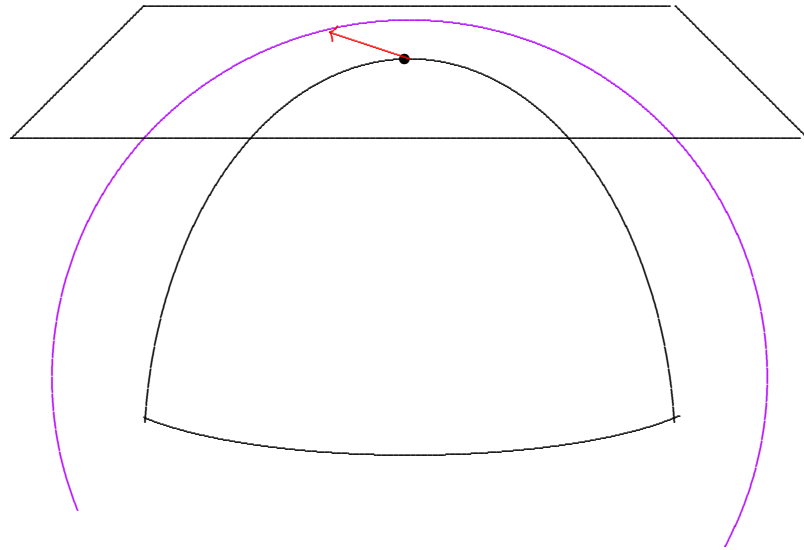
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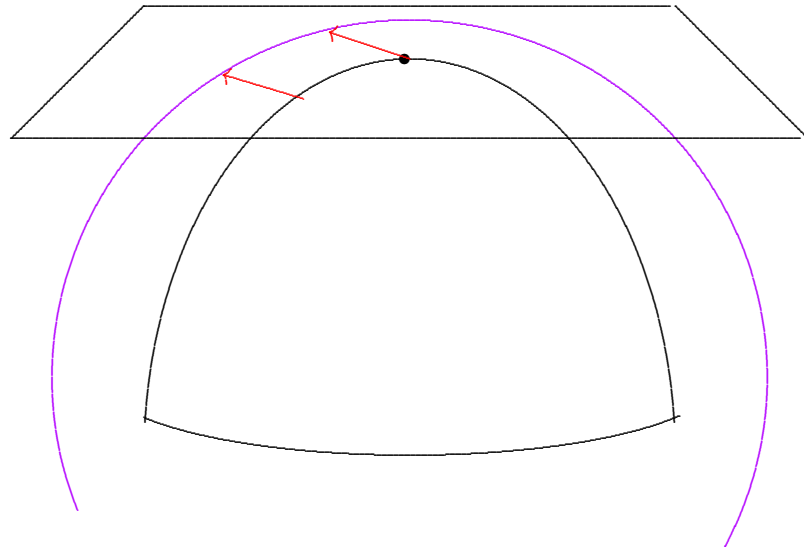
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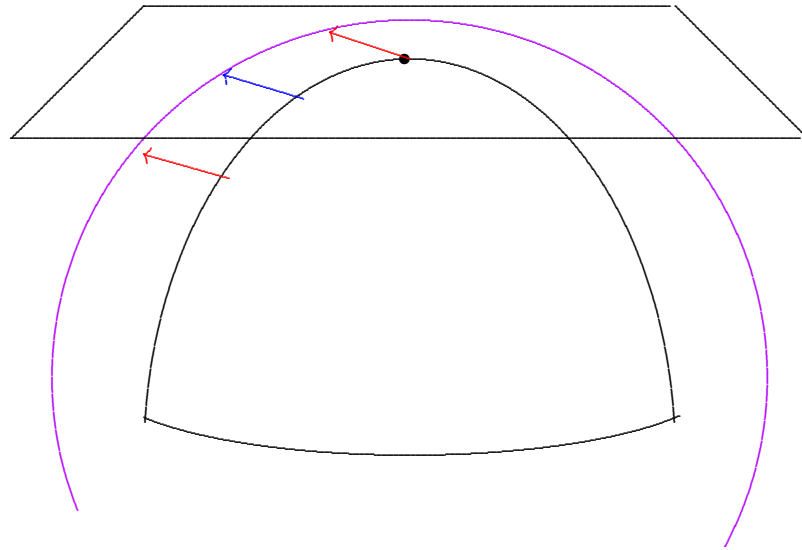
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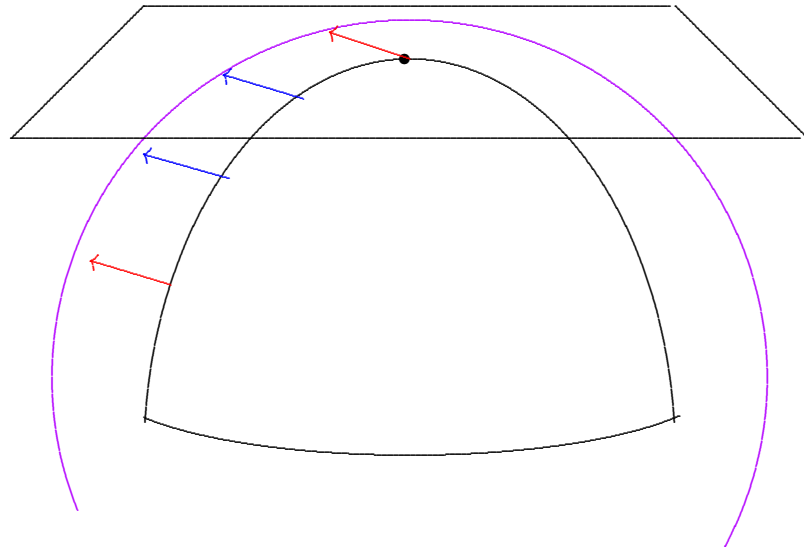
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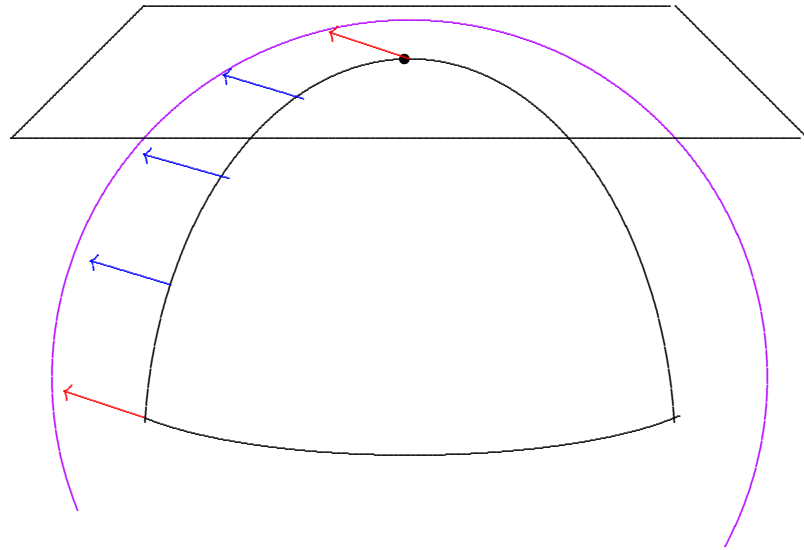
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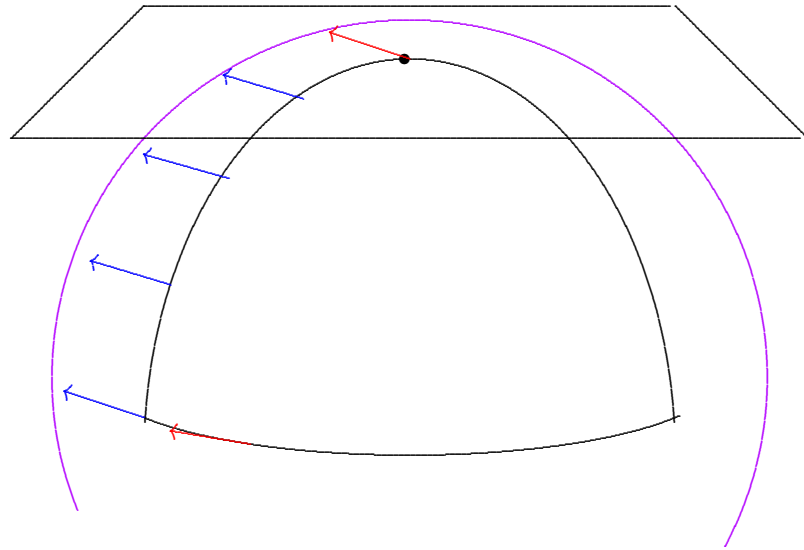
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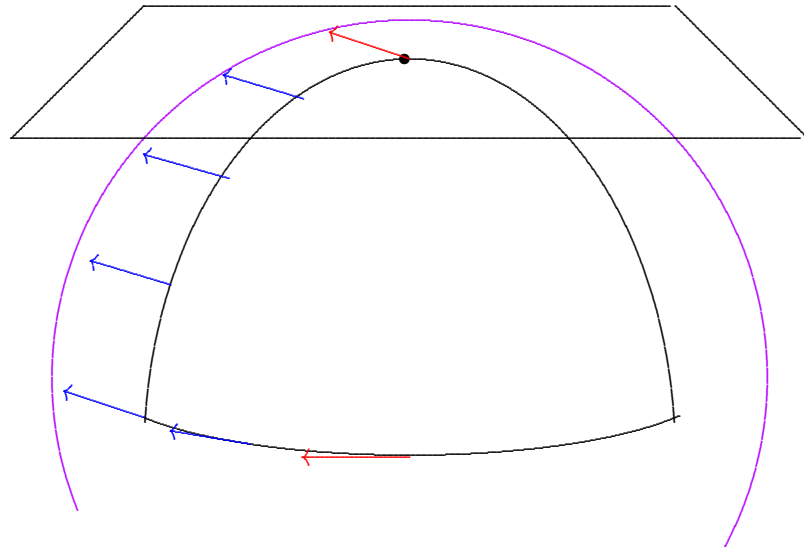
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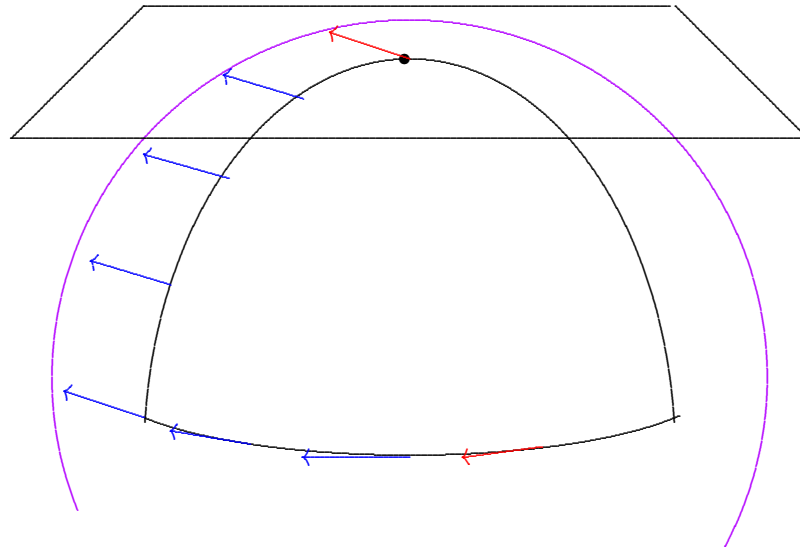
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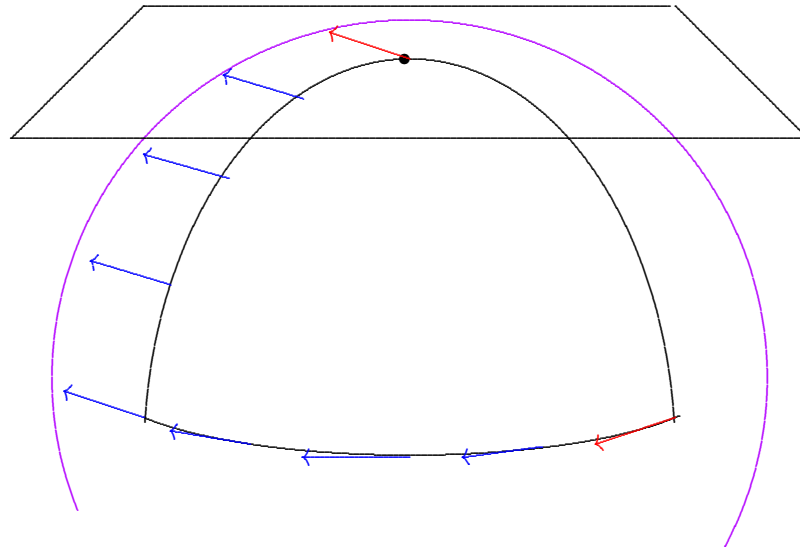
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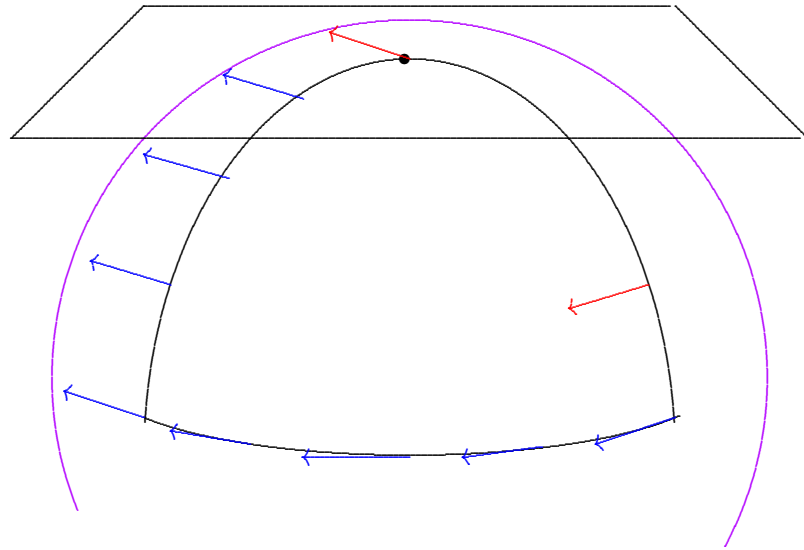
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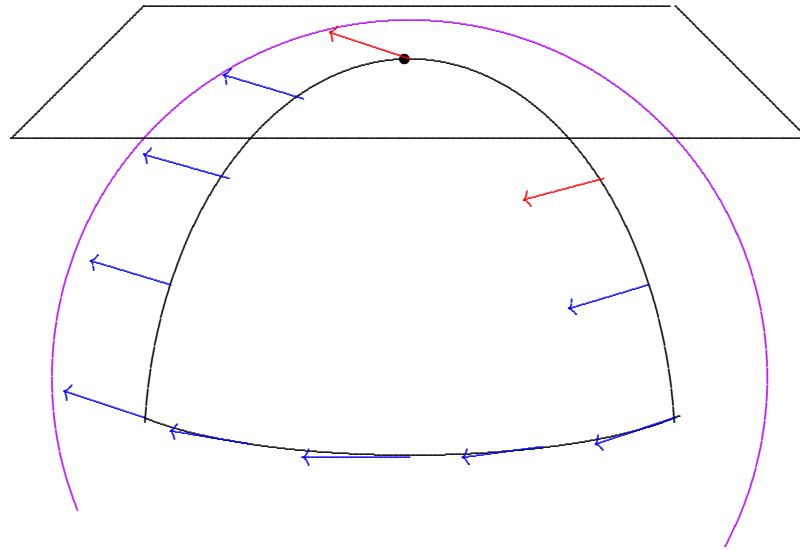
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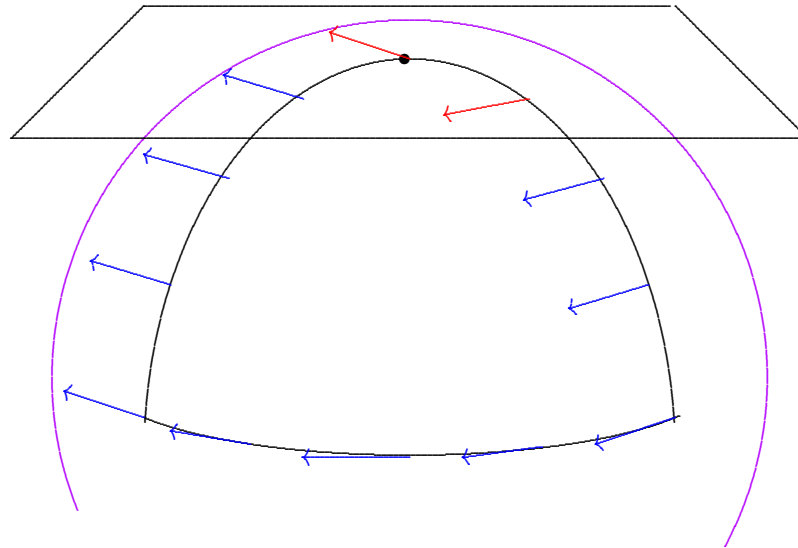
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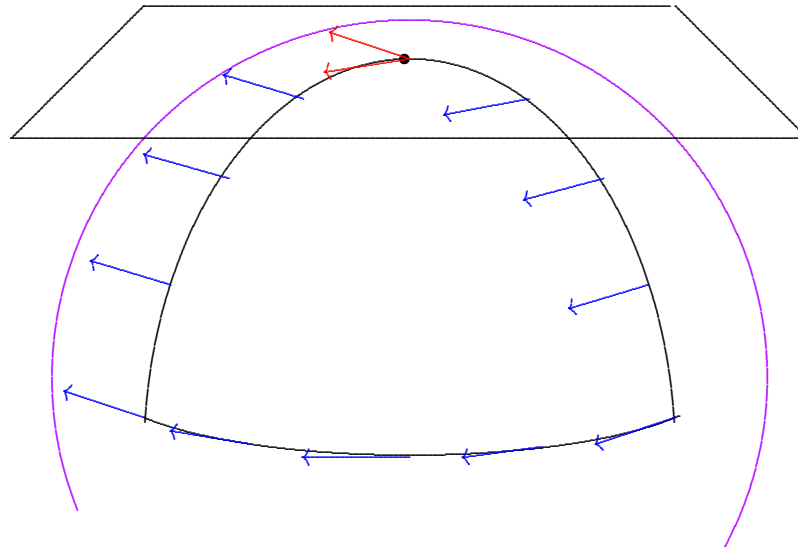
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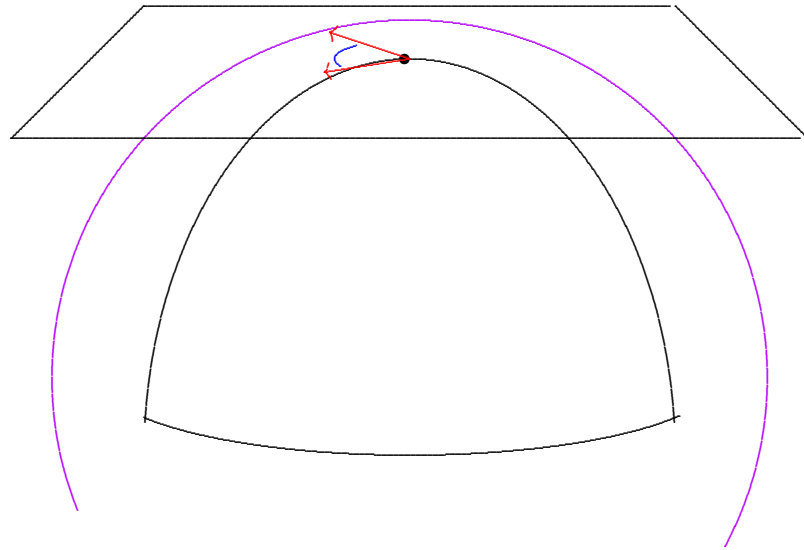
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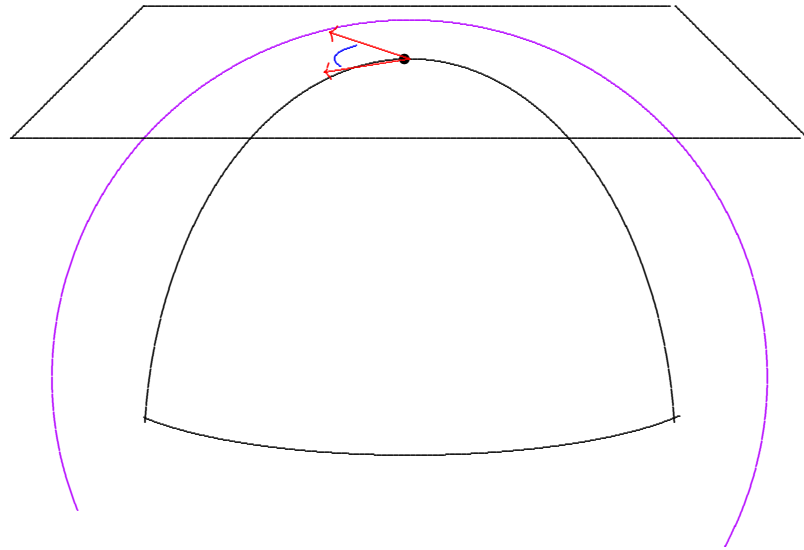
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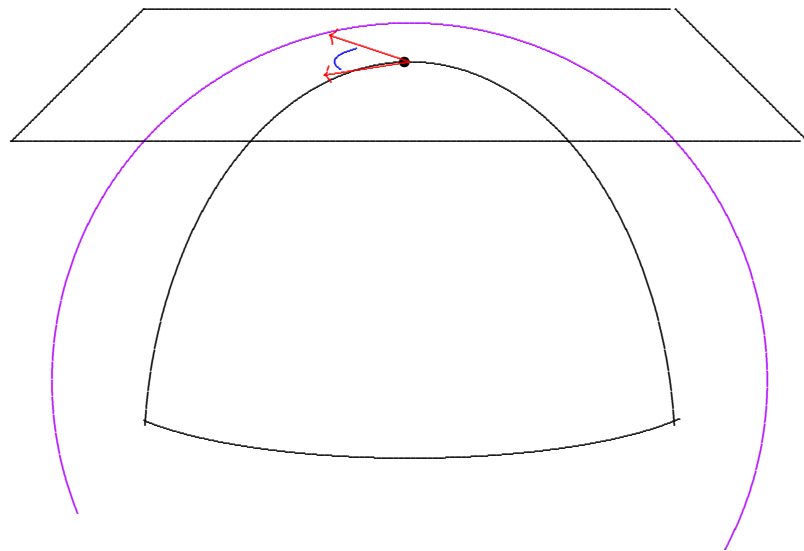
holonomy $\subset \mathbf{O}(n)$



Kähler metrics:

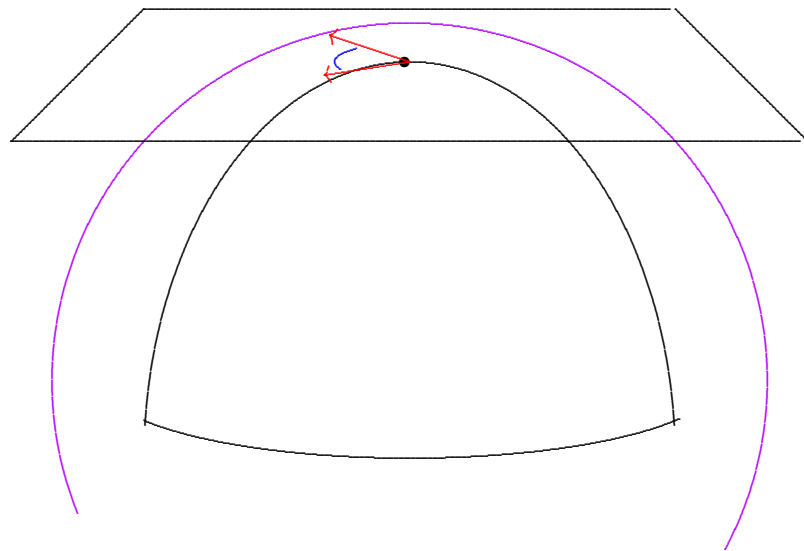
(M^{2m}, g) :

holonomy



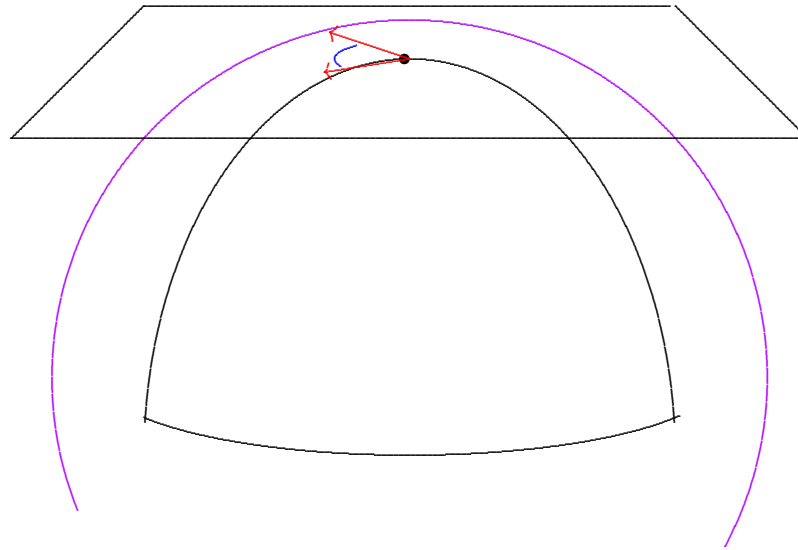
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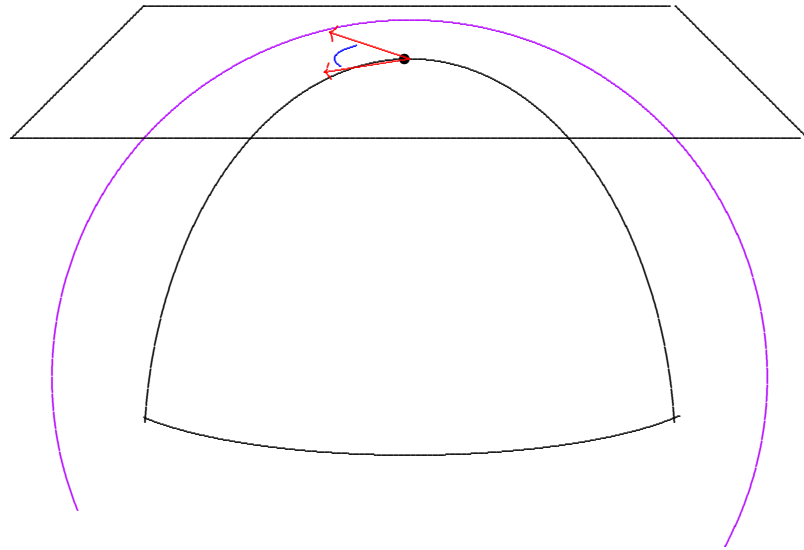
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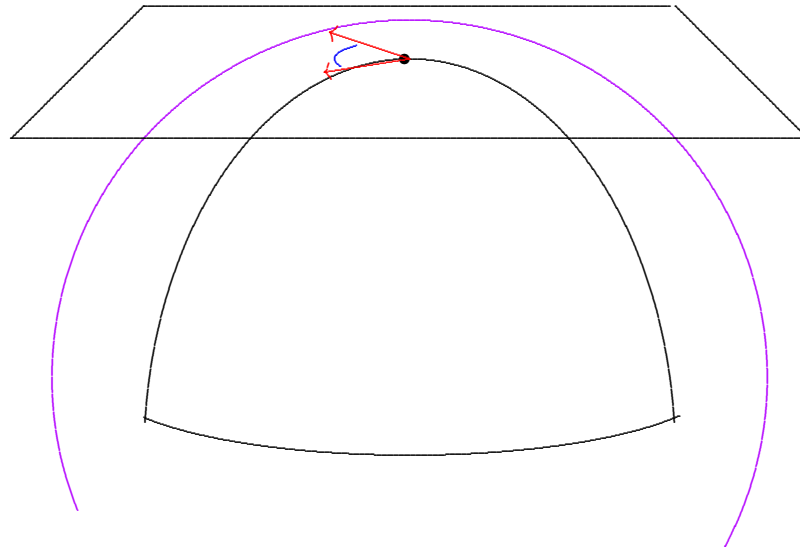
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What are the possible holonomy groups?

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$$\mathbf{Spin}(7) := \langle S^7 \subset \mathbb{O}^\times \rangle \subset \mathbf{SO}(8)$$

Theorem (Berger 1955...). *Let (M^n, g) be a complete, simply-connected Riemannian manifold. If (M, g) is neither a product of lower-dimensional manifolds nor a symmetric space, then its holonomy group appears on the following table:*

dimension n	$\text{Hol}(M^n, g)$	geometry	relevant field
n	$\mathbf{SO}(n)$	generic	$\mathbb{R}$
$2m$	$\mathbf{U}(m)$	Kähler	$\mathbb{C}$
$2m$	$\mathbf{SU}(m)$	Calabi-Yau	$\mathbb{C}$
$4k$	$\mathbf{Sp}(k)$	Hyper-Kähler	$\mathbb{H}$
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Simons, Calabi, Hitchin, Bryant, Salamon, Joyce...

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dimension n	$\text{Hol}(M^n, g)$	geometry	Einstein?
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The latter seems to be especially delicate!

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Can such metrics coexist?

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Theorem. *No smooth compact M^7*

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To make this plausible, will first illustrate assertion for prototypical examples due to S. Kobayashi '63.

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Replace Hard Lefschetz on $H^*(X, \mathbb{R})$ with transverse version on $H_B^*(M, \mathfrak{F})$ due to El Kacimi-Alaoui.

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$$\hat{A}(M) = \begin{cases} 2, & \text{if Hol} = \mathbf{SU}(2m) \text{ and } n = 2m, \\ k + 1, & \text{if Hol} = \mathbf{Sp}(k) \text{ and } n = 4k, \\ 1, & \text{if Hol} = \mathbf{Spin}(7) \text{ and } n = 8. \end{cases}$$

$\Rightarrow \nexists$ metrics with $s > 0$.

$\Rightarrow \nexists$ Einstein metrics with $\lambda > 0$.

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But no one has found Calabi-Yau partners for these Fano manifolds!

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