Gravitational Instantons,

Weyl Curvature, &

Conformally Kähler Geometry

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Kansas Geometric Analysis Workshop
Wichita State University
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Joint work with

Joint work with

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Sorbonne Université

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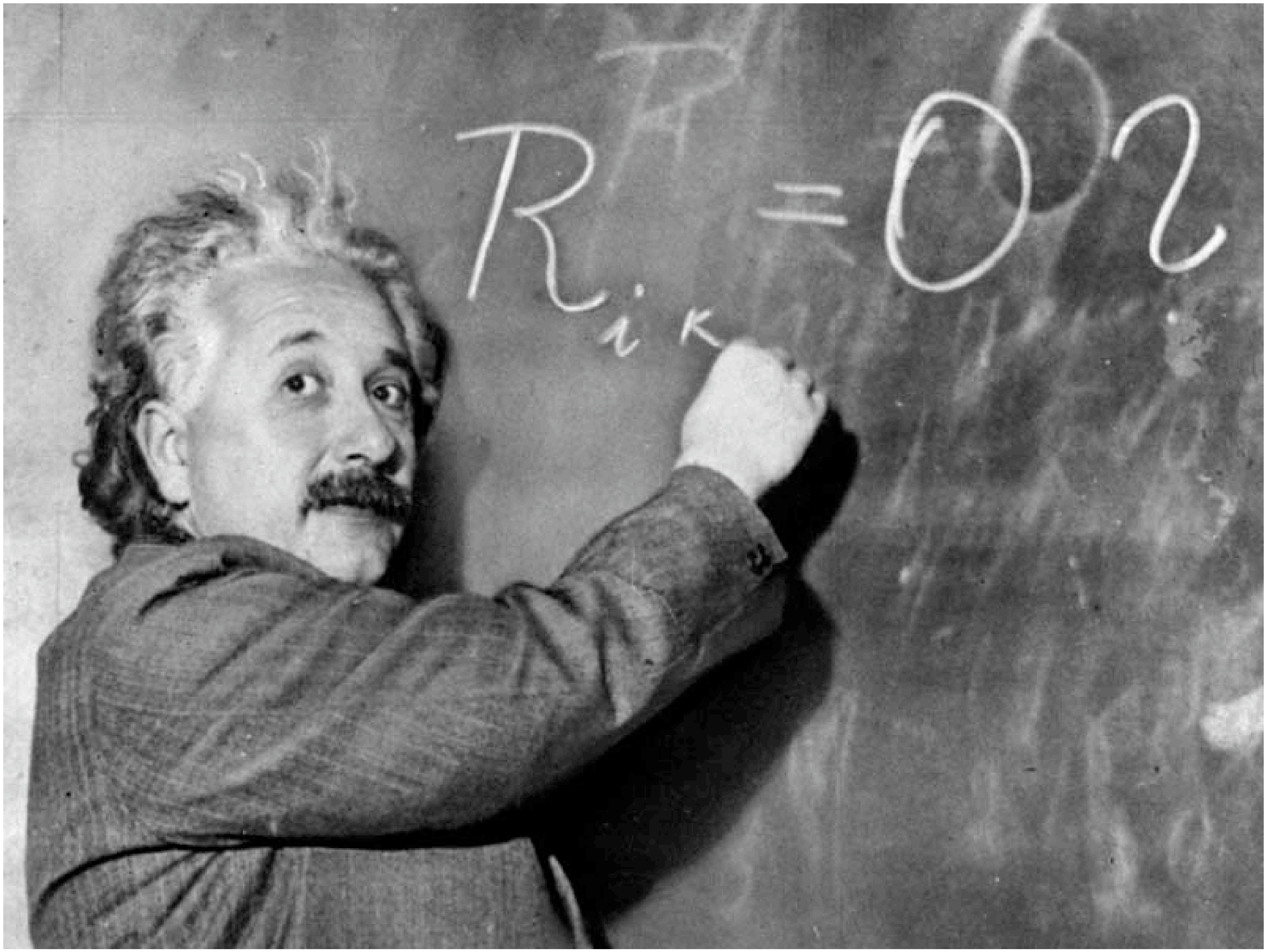

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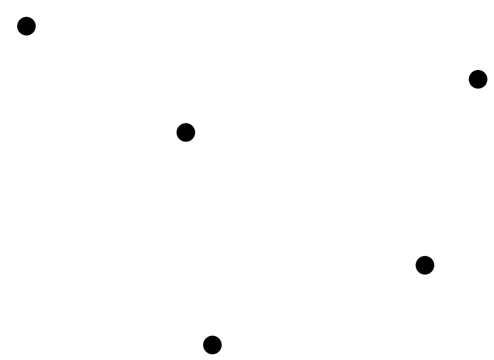

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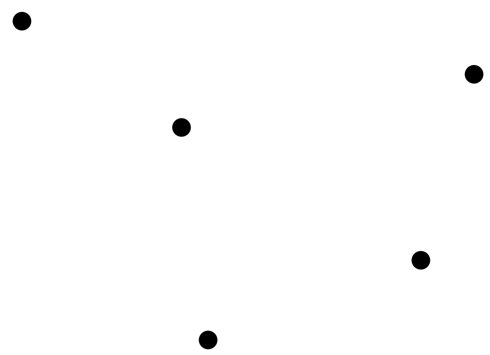

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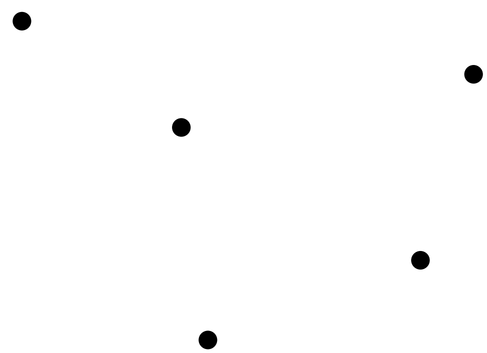

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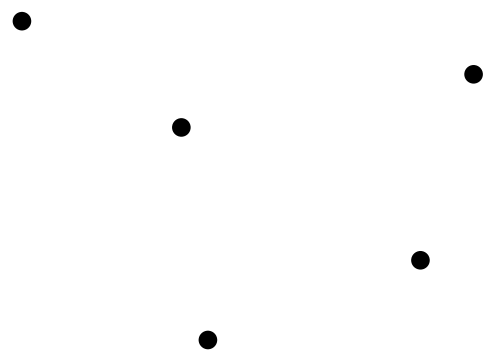

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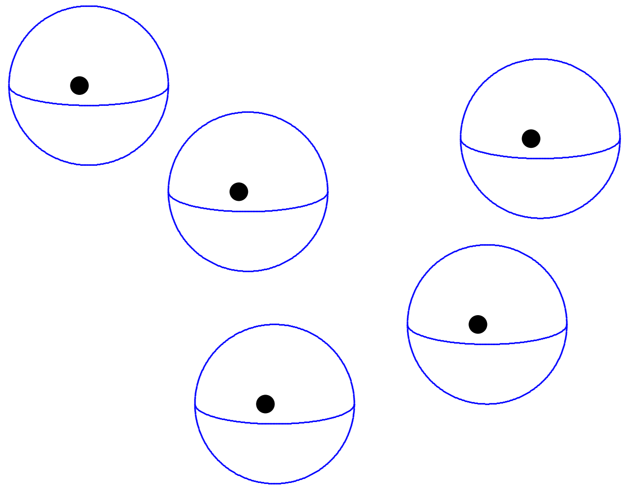

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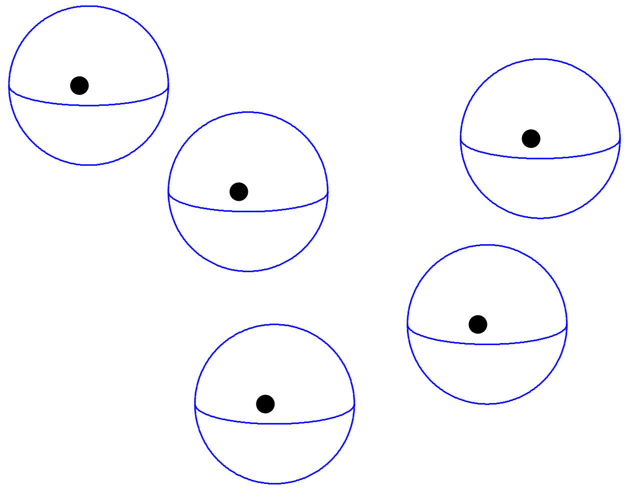

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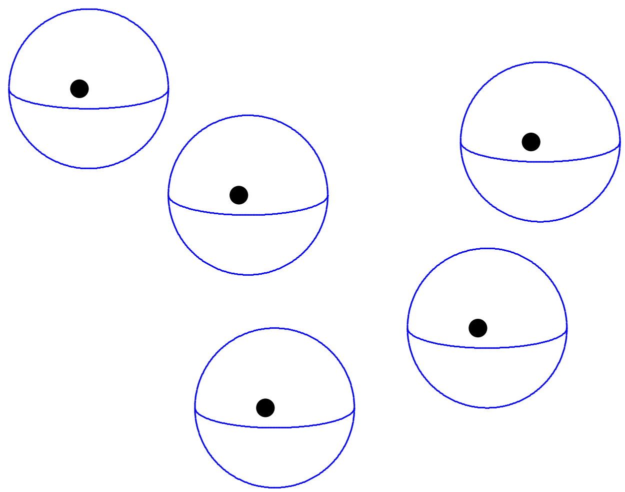

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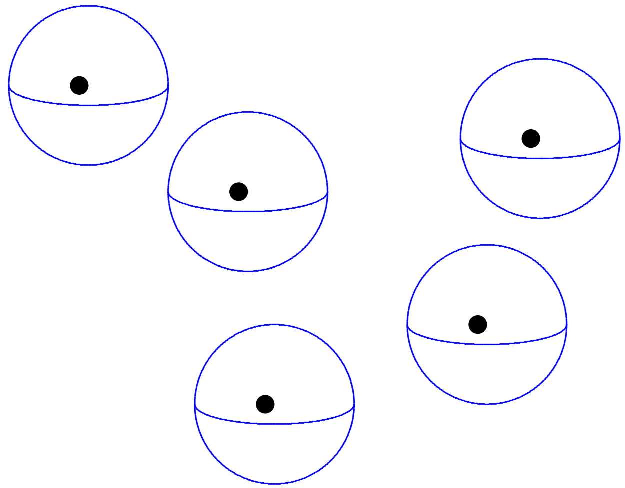

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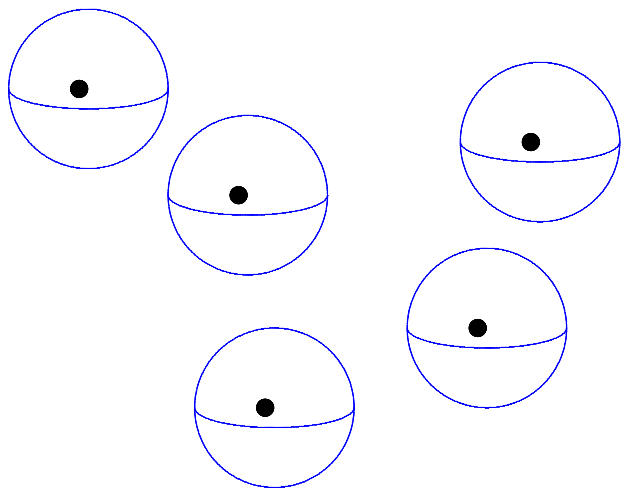

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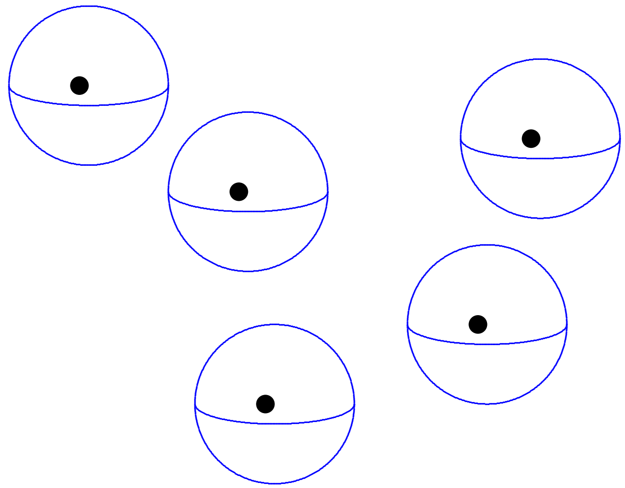

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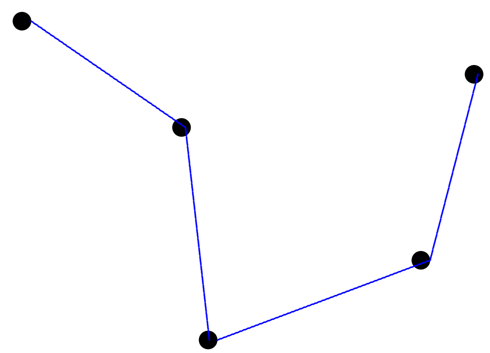

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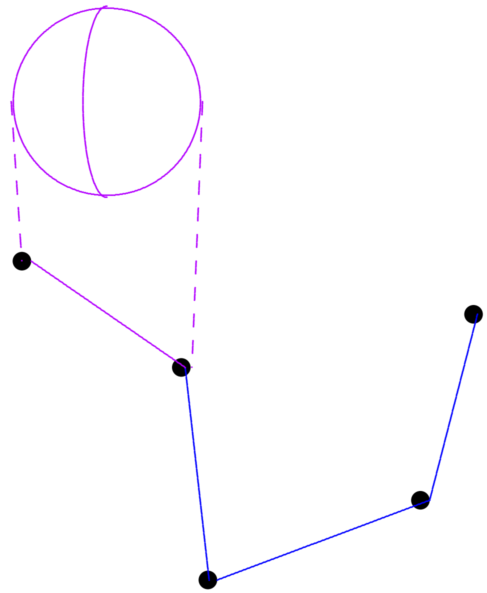

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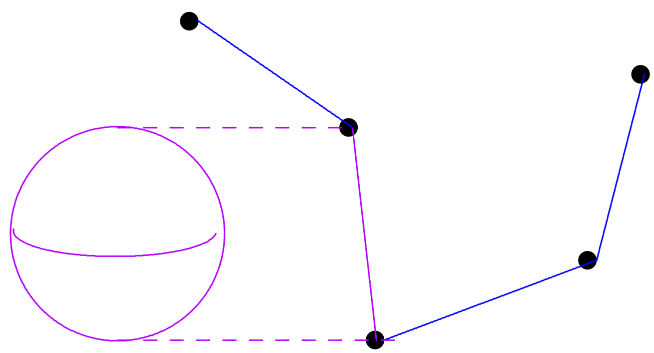

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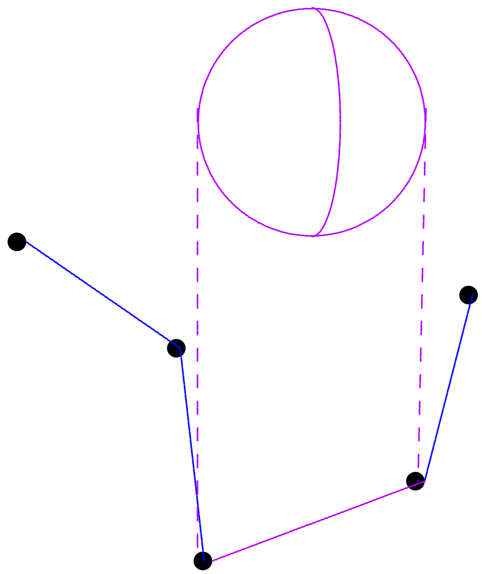

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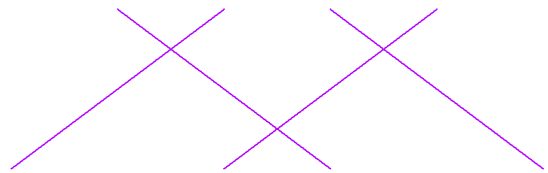

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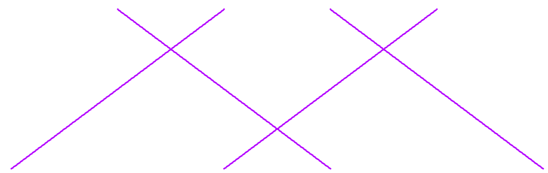

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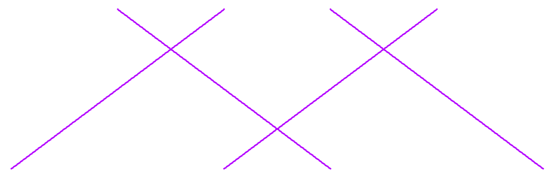

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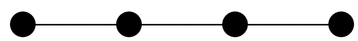

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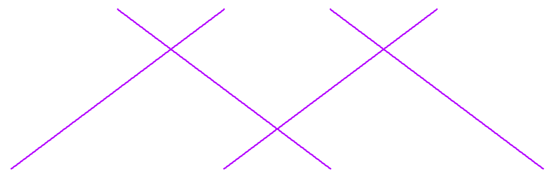

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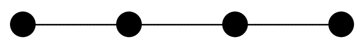

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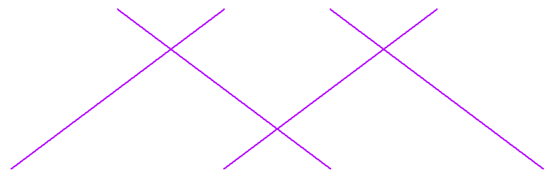

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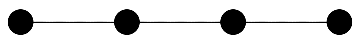

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Plumb together k copies of T^*S^2
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$M \to \mathbb{R}^3$ hyper-Kähler moment map of S^1 action.

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Gibbons and Hawking were unaware of all this!

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cf. Bishop-Gromov inequality!

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This last property distinguishes the ALF spaces from other classes of gravitational instantons:

ALG, ALH, ALG*, ALH*, ...

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Taub-NUT becomes Hermitian metric on $\mathbb{C}^2$.

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$$V = 1 + \frac{1}{2\varrho}$$

Can also write as

$$g = \frac{r+1}{4r}dr^2 + r(1+r)[{\sigma_1}^2 + {\sigma_2}^2] + \frac{r}{r+1}{\sigma_3}^2$$

for left-invariant coframe $\{\sigma_j\}$ on $S^3 = \mathbf{SU}(2)$.

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This J determines opposite orientation from the hyper-Kähler complex structures.

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Non-Kähler, but conformally Kähler!

Hawking also explored non-hyper-Kähler examples. . .

Example.

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Bach-flat Kähler metrics are extremal!

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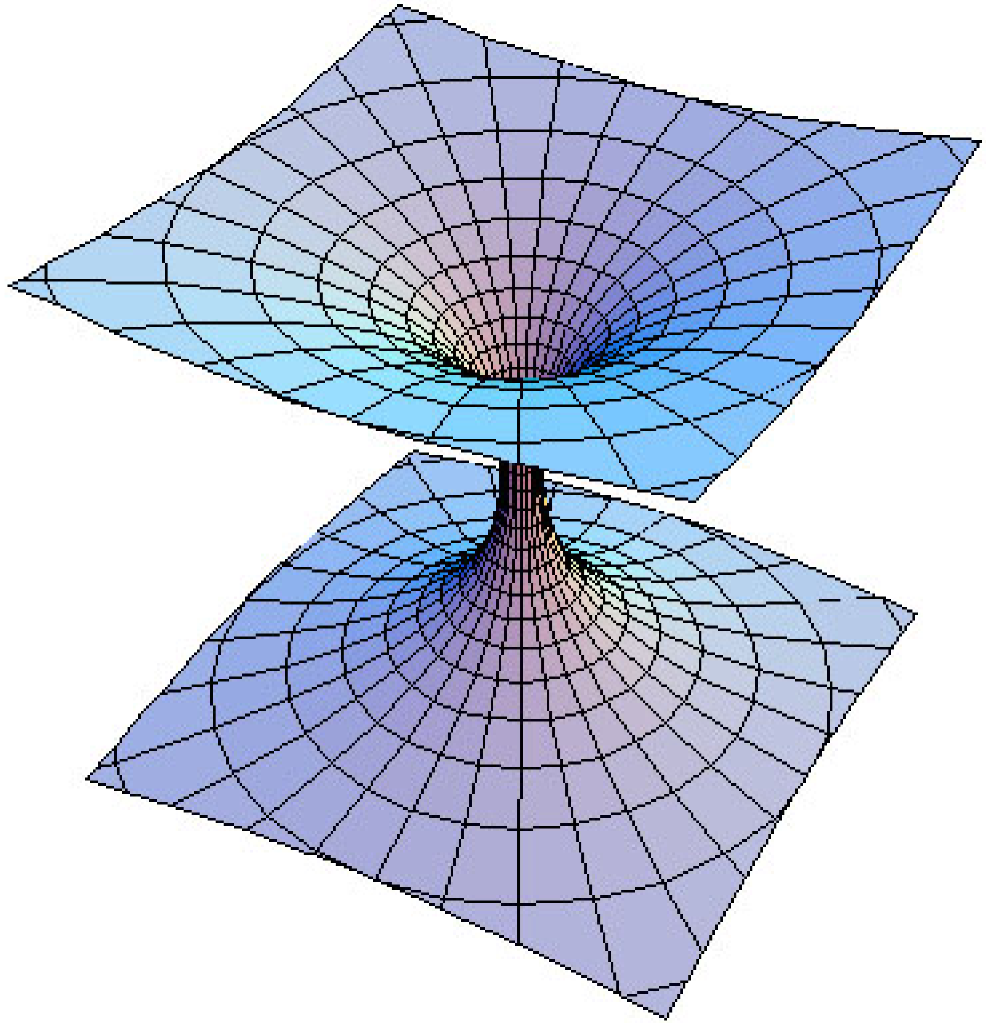
Makes h into extremal Kähler metric on $\mathbb{C} \times \mathbb{CP}_1$.



$\mathbb{R} \times S^2 \subset \mathbb{R}^2 \times S^2$

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This might lend some credence to the aphorism. . .

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Y. Chen & E. Teo, 2011

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The proof is actually constructive!

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By the Riemannian Goldberg-Sachs Theorem, the Hermitian assumption is equivalent to assuming that the Einstein metric g is conformally Kähler.

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It is false in all higher dimensions!

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Derdziński: the conformally related Kähler metric is also automatically extremal in the sense of Calabi!

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Ricci-flat case — not merely Einstein!

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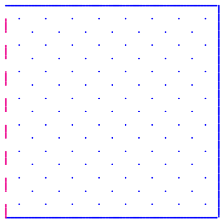

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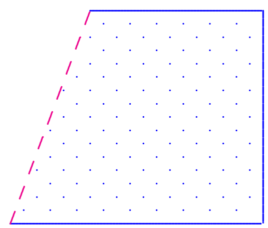

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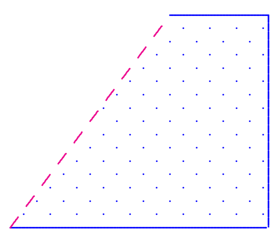

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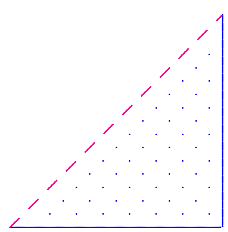

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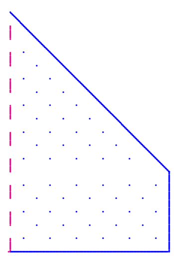

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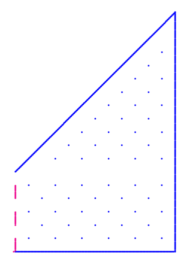

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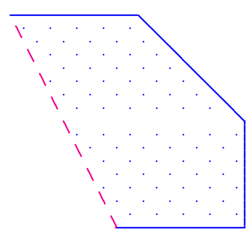

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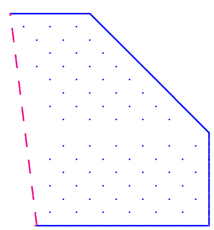

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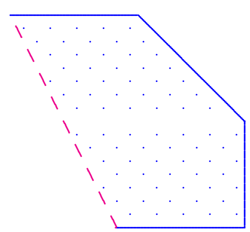

Chen-Teo

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Only depends on the conformal class

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Notational warning:
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$$2\sqrt{6}|W^+|_h + |s_h| \geq |\omega \wedge \star d\omega|^2$$

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Proposition.

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Proof.

$$3\, d[\omega \wedge \star d\omega] \geq \star\left(\frac{1}{2}|\nabla\omega|^2 + 3\,|d\omega|^2\right).$$

Proposition. *Let* (M, g) *be an oriented, simply-connected Riemannian* 4*-manifold that satisfies* $\delta W^+ = 0$ *and* $\det(W^+) > 0$ *everywhere. Also suppose that* M *is a nested union* $M = \cup_j U_j$ *of compact domains*

$$U_1 \subset U_2 \subset \cdots \subset U_j \subset \cdots$$

with smooth boundary such that

$$\lim_{j \to \infty} \int_{\partial U_j} \omega \wedge \star d\omega = 0.$$

Then $h = \alpha_g{}^{2/3} g$ *is a Kähler metric.*

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It's a pleasure to be here!