Four-Manifolds, 

*Einstein Metrics, &*

*Differential Topology*

Claude LeBrun 
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IMPA, 6/11/13
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Now choosing \(T_pM \cong \mathbb{R}^n\) via some orthonormal basis gives us special coordinates on \(M\).
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“...the greatest blunder of my life!”

— A. Einstein, to G. Gamow
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$$\Delta x^j = 0 \implies r_{jk} = \frac{1}{2} \Delta g_{jk} + \text{lots.}$$
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**Proposition.** If $n \geq 3$, a Riemannian $n$-manifold $(M^n, g)$ is Einstein iff the trace-free part of its Ricci tensor vanishes:

$$ \hat{r} := r - \frac{s}{n} g = 0. $$
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Proof. Bianchi identity $\implies \nabla \cdot \hat{r} = \left(\frac{1}{2} - \frac{1}{n}\right)ds$. 

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- When $n \geq 6$, wide open. Maybe???
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\[ \text{Diagram of connected sum} \]
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Similar results for most simply connected spin 5-manifolds. (Boyer, Galicki, Kollar, et al.)
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**Theorem** (L). *There is only one Einstein metric on compact complex-hyperbolic 4-manifold $\mathcal{CH}^2/\Gamma$, up to scale and diffeos.*
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Enough rigidity apparently still holds in dimension four to call this a geometrization.
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By contrast, high-dimensional Einstein metrics too common; have little to do with geometrization.
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where \(\Lambda^\pm\) are \((\pm1)\)-eigenspaces of

\[
\star : \Lambda^2 \to \Lambda^2, \\
\star^2 = 1.
\]

\(\Lambda^+\) self-dual 2-forms.

\(\Lambda^-\) anti-self-dual 2-forms.
Riemann curvature of $g$

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\[
\begin{array}{|c|c|}
\hline
\Lambda^+ & W_+ + \frac{s}{12} \\
\hline
\Lambda^- & \dot{\rho} \\
\hline
\Lambda^{**} & \dot{\rho} \\
\hline
\Lambda^{-*} & W_- + \frac{s}{12} \\
\hline
\end{array}
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$$\begin{array}{cc}
\Lambda^+ & \Lambda^{*+} \\
W_+ + \frac{s}{12} & \hat{r} \\
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where

\[ s = \text{scalar curvature} \]

\[ \hat{r} = \text{trace-free Ricci curvature} \]

\[ W_+ = \text{self-dual Weyl curvature} \quad (\text{conformally invariant}) \]

\[ W_- = \text{anti-self-dual Weyl curvature} \]
Thus \((M^4, g)\) Einstein \iff \(\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2\) commutes with 
\[\star : \Lambda^2 \rightarrow \Lambda^2 : \]
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\mathcal{R} = \begin{pmatrix}
W_+ + \frac{s}{12} & \hat{r} \\
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Thus \((M^4, g)\) Einstein \iff \(R : \Lambda^2 \rightarrow \Lambda^2\) commutes with

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\[
R = \begin{pmatrix}
W_+ + \frac{s}{12} & \hat{r} \\
\hat{r} & W_- + \frac{s}{12}
\end{pmatrix}.
\]
Thus \((M^4, g)\) Einstein \iff 
\[
\mathcal{R} : \Lambda^2 \to \Lambda^2
\]
commutes with
\[
\star : \Lambda^2 \to \Lambda^2
\]

\[
\mathcal{R} = \begin{pmatrix}
W_+ + \frac{s}{12} & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
& W_- + \frac{s}{12} \\
& 0
\end{pmatrix}
\]
Corollary. A Riemannian 4-manifold \((M, g)\) is Einstein \iff sectional curvatures are equal for any pair of perpendicular 2-planes.
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Corollary. A Riemannian 4-manifold $(M, g)$ is Einstein $\iff$ sectional curvatures are equal for any pair of perpendicular 2-planes.
$(M, g)$ compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + \right) d\mu$$
\((M, g)\) compact oriented Riemannian.

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$$\chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu$$
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4-dimensional Gauss-Bonnet formula

\[
\chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\hat{\rho}|^2}{2} \right) d\mu
\]

for Euler-characteristic \(\chi(M) = \sum_j (-1)^j b_j(M)\).
4-dimensional Hirzebruch signature formula

\[ \tau(M) = \frac{1}{12\pi^2} \int_M \left( |W_+|^2 \right) d\mu \]
4-dimensional Hirzebruch signature formula

$$\tau(M) = \frac{1}{12\pi^2} \int_M \left( |W_+|^2 - |W_-|^2 \right) d\mu$$
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\[ \tau(M) = \frac{1}{12\pi^2} \int_M \left( |W_+|^2 - |W_-|^2 \right) d\mu \]

for signature \( \tau(M) = b_+(M) - b_-(M) \).
4-dimensional Hirzebruch signature formula

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for signature \( \tau(M) = b_+(M) - b_-(M) \).

Here \( b_{\pm}(M) = \text{max dim subspaces} \subset H^2(M, \mathbb{R}) \) on which intersection pairing

\[ H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) \to \mathbb{R} \]

\[ ([\varphi], [\psi]) \mapsto \int_M \varphi \wedge \psi \]

is positive (resp. negative) definite.
Theorem (Freedman/Donaldson). Two smooth compact simply connected oriented 4-manifolds are orientedly homeomorphic if and only if
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Warning: “Exotic differentiable structures!”
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Warning: “Exotic differentiable structures!”

No diffeomorphism classification currently known!

Typically, one homeotype $\leftrightarrow \infty$ many diffeotypes.
**Theorem** (Freedman/Donaldson). *Two smooth compact simply connected oriented 4-manifolds are orientedly homeomorphic if and only if*

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Corollary. *Any smooth compact simply connected non-spin 4-manifold* $M$ *is homeomorphic to*
**Corollary.** Any smooth compact simply connected non-spin 4-manifold $M$ is homeomorphic to a connect sum

$$j\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2} = \underbrace{\mathbb{CP}^2 \# \cdots \# \mathbb{CP}^2}_j \# \underbrace{\overline{\mathbb{CP}^2} \# \cdots \# \overline{\mathbb{CP}^2}}_k$$
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where \( j = b_+(M) \) and \( k = b_-(M) \).
Corollary. Any smooth compact simply connected non-spin 4-manifold $M$ is homeomorphic to a connect sum

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where $j = b_+(M)$ and $k = b_-(M)$.

Convention:

$\overline{\mathbb{CP}^2} = \text{reverse oriented } \mathbb{CP}^2$. 
Corollary. Any smooth compact simply connected non-spin 4-manifold $M$ is homeomorphic to a connect sum $j\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$.
**Corollary.** Any smooth compact simply connected non-spin 4-manifold \( M \) is homeomorphic to a connect sum \( j\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2 \).

**Conjecture (11/8 Conjecture).** Any smooth compact simply connected spin 4-manifold \( M \) is (un-orientedly) homeomorphic to either \( S^4 \) or a connected sum \( jK3 \# k(S^2 \times S^2) \).
Corollary. Any smooth compact simply connected non-spin 4-manifold $M$ is homeomorphic to a connect sum $j\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$.

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Equivalent to asserting that such manifolds satisfy

$$b_2 \geq \frac{11}{8} |\tau|.$$
Corollary. Any smooth compact simply connected non-spin 4-manifold $M$ is homeomorphic to a connect sum $j\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$.

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Equivalent to asserting that such manifolds satisfy

$$b_2 \geq \frac{11}{8}|\tau|.$$ 

Certainly true of all examples in this lecture!
Question. Which smooth compact 4-manifolds $M^4$ admit Einstein metrics?
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Complex geometry provides rich source of examples.
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On suitable 4-manifolds, Seiberg-Witten theory allows one to mimic Kähler geometry when treating non-Kähler metrics.

**Today’s Main Question.** If $(M^4, J)$ is a compact complex surface, when does $M^4$ admit an Einstein metric $g$ (unrelated to $J$)?
Even Narrower Question. When does a compact complex surface \((M^4, J)\) admit an Einstein metric \(g\) which is Hermitian,
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\[ g(\cdot, \cdot) = g(J\cdot, J\cdot)? \]

Kähler if the 2-form

\[ \omega = g(J\cdot, \cdot) \]

is closed:

\[ d\omega = 0. \]

But we do not assume this!
Even Narrower Question. When does a compact complex surface \((M^4, J)\) admit an Einstein metric \(g\) which is Hermitian, in the sense that

\[ g(\cdot, \cdot) = g(J\cdot, J\cdot)? \]
Theorem. A compact complex surface \((M^4, J)\) admits an Einstein metric \(g\) which is Hermitian with respect to \(J\) \(\iff\) \(c_1(M^4, J)\) “has a sign.”
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More precisely, \(\exists\) such \(g\) with Einstein constant \(\lambda\) \(\iff\) there is a Kähler form \(\omega\) such that

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c_1(M^4, J) = \lambda[\omega].
\]
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Moreover, this metric is unique, up to isometry, if \(\lambda \neq 0\).
**Theorem.** A compact complex surface $(M^4, J)$ admits an Einstein metric $g$ which is Hermitian with respect to $J \iff c_1(M^4, J)$ “has a sign.”

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Aubin, Yau, Siu, Tian . . . Kähler case.
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Chen-L-Weber (’08), L (’12, ’13): non-Kähler case.
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Aubin, Yau, Siu, Tian . . . Kähler case.

Chen-L-Weber (’08), L (’12, ’13): non-Kähler case.

Only two metrics arise in non-Kähler case!
Corollary. The non-spin 4-manifolds
\[ \mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}, \quad 0 \leq k \leq 8, \]
all admit \( \lambda > 0 \) Einstein metrics.
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Blowing up:
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If \( N \) is a complex surface, may replace \( p \in N \) with \( \mathbb{CP}_1 \)
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Blowing up:

If \( N \) is a complex surface, may replace \( p \in N \) with \( \mathbb{CP}^1 \) to obtain blow-up
\[ M \approx N \# \overline{\mathbb{CP}}^2 \]
**Corollary.** The non-spin 4-manifolds

\[ \mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}, \quad 0 \leq k \leq 8, \]

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So does the spin 4-manifold

\[ S^2 \times S^2. \]

---

**Blowing up:**

If \( N \) is a complex surface, may replace \( p \in N \) with \( \mathbb{CP}^1 \) to obtain blow-up

\[ M \approx N \# \overline{\mathbb{CP}^2} \]

in which new \( \mathbb{CP}^1 \) has self-intersection \(-1\).
Theorem. Suppose that $M$ is a smooth compact oriented 4-manifold which admits a complex structure $J$. 
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Theorem. Suppose that $M$ is a smooth compact oriented 4-manifold which admits a complex structure $J$. Then $M$ also admits an (unrelated) Einstein metric $g$ with $\lambda > 0$

\[ \iff M \cong \begin{cases} \mathbb{C}P_2 \# k \overline{\mathbb{C}P_2}, & 0 \leq k \leq 8, \\
\text{or} \\
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Theorem. Suppose that $M$ is a smooth compact oriented 4-manifold which admits a complex structure $J$. Then $M$ also admits an (unrelated) Einstein metric $g$ with $\lambda > 0$

\[ \iff M \cong \begin{cases} \mathbb{CP}^2 \# k \overline{\mathbb{CP}}^2, & 0 \leq k \leq 8, \\ or \\ S^2 \times S^2 \end{cases} \]

$\implies$: Hitchin-Thorpe inequality, easy Seiberg-Witten.
Theorem. Suppose that $M$ is a smooth compact oriented 4-manifold which admits an integrable complex structure $J$. Then $M$ also admits an Einstein metric $g$ with $\lambda \geq 0$ if and only if

\[
M \overset{\text{diff}}{\approx} \begin{cases} 
\mathbb{CP}_2 \# k \overline{\mathbb{CP}_2}, \\
S^2 \times S^2, \\
K^3, \\
K^3 / \mathbb{Z}_2, \\
T^4, \\
T^4 / \mathbb{Z}_2, \\
T^4 / \mathbb{Z}_3, \\
T^4 / \mathbb{Z}_4, \\
T^4 / (\mathbb{Z}_2 \oplus \mathbb{Z}_2), \\
T^4 / (\mathbb{Z}_3 \oplus \mathbb{Z}_3), \\
T^4 / (\mathbb{Z}_2 \oplus \mathbb{Z}_4) & \text{if} \ k \leq 8.
\end{cases}
\]
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**Theorem.** Suppose that $M$ is a smooth compact oriented 4-manifold which admits an integrable complex structure $J$. Then $M$ also admits an Einstein metric $g$ with $\lambda \geq 0$ if and only if

$$M \simeq \begin{cases} \mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2, & 0 \leq k \leq 8, \\ S^2 \times S^2, & \end{cases}$$
Theorem. Suppose that $M$ is a smooth compact oriented 4-manifold which admits an integrable complex structure $J$. Then $M$ also admits an Einstein metric $g$ with $\lambda \geq 0$ if and only if

$$M \overset{\text{diff}}{\approx} \begin{cases} \mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}, & 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ T^4, \\ T^4 / \mathbb{Z}_2, \\ T^4 / \mathbb{Z}_3, \\ T^4 / \mathbb{Z}_4, \\ T^4 / (\mathbb{Z}_2 \oplus \mathbb{Z}_2), \\ T^4 / (\mathbb{Z}_3 \oplus \mathbb{Z}_3), \\ T^4 / (\mathbb{Z}_2 \oplus \mathbb{Z}_4) \end{cases}$$
Theorem. Suppose that $M$ is a smooth compact oriented 4-manifold which admits an integrable complex structure $J$. Then $M$ also admits an Einstein metric $g$ with $\lambda \geq 0$ if and only if

$$M \cong \begin{cases} 
\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}, & 0 \leq k \leq 8, \\
S^2 \times S^2, \\
K3, \\
K3/\mathbb{Z}_2, \\
T^4, \\
T^4/\mathbb{Z}_2, \\
T^4/\mathbb{Z}_3, \\
T^4/\mathbb{Z}_4, \\
T^4/\left(\mathbb{Z}_2 \oplus \mathbb{Z}_2\right), \\
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\end{cases}$$
Theorem. Suppose that $M$ is a smooth compact oriented 4-manifold which admits an integrable complex structure $J$. Then $M$ also admits an Einstein metric $g$ with $\lambda \geq 0$ if and only if $M$ is diffeomorphic to

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S^2 \times S^2, \\
K3, \\
K3/\mathbb{Z}_2, \\
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Theorem. Suppose that $M$ is a smooth compact oriented 4-manifold which admits an integrable complex structure $J$. Then $M$ also admits an Einstein metric $g$ with $\lambda \geq 0$ if and only if

$$M \approx \begin{cases} \mathbb{CP}_2 \# k\overline{\mathbb{CP}_2}, & 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \end{cases}$$
Theorem. Suppose that $M$ is a smooth compact oriented 4-manifold which admits an integrable complex structure $J$. Then $M$ also admits an Einstein metric $g$ with $\lambda \geq 0$ if and only if

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Del Pezzo surfaces,
K3 surface, Enriques surface,
Abelian surface, Hyper-elliptic surfaces.
Theorem. Suppose that $M$ is a smooth compact oriented 4-manifold which admits an integrable complex structure $J$. Then $M$ also admits an Einstein metric $g$ with $\lambda \geq 0$ if and only if

$$M^{\text{diff}} \approx \begin{cases} 
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S^2 \times S^2, \\
S^2, \\
K3, \\
K3/\mathbb{Z}_2, \\
T^4, \\
T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\
T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). 
\end{cases}$$

Del Pezzo surfaces,
K3 surface, Enriques surface,
Abelian surface, Hyper-elliptic surfaces.

Similarly when $M$ symplectic instead of complex.
Hitchin-Thorpe Inequality:

\[(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W + |^2 - \frac{|\hat{\mathfrak{r}}|^2}{2} \right) d\mu_g \]
Hitchin-Thorpe Inequality:

\[
(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W| + \frac{1}{2} \right) d\mu_g
\]

Einstein \Rightarrow \quad = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W| + \frac{1}{2} \right) d\mu_g
Hitchin-Thorpe Inequality:

\[(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 - \frac{|\tilde{r}|^2}{2} \right) d\mu_g \]

Einstein \implies \quad = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 \right) d\mu_g

**Theorem** (Hitchin-Thorpe Inequality). *If smooth compact oriented* \(M^4\) *admits Einstein* \(g\), *then*

\[(2\chi + 3\tau)(M) \geq 0,\]

*with equality only if* \((M, g)\) *finitely covered by flat* \(T^4\) *or Calabi-Yau* \(K3\).
Seiberg-Witten theory:

generalized Kähler geometry of non-Kähler 4-manifolds.
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Can’t hope to generalize $\bar{\partial}$ operator to this setting.
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But $\bar{\partial} + \bar{\partial}^*$ does generalize:
Seiberg-Witten theory:

generalized Kähler geometry of non-Kähler 4-manifolds.

Can’t hope to generalize $\bar{\partial}$ operator to this setting.

But $\bar{\partial} + \bar{\partial}^*$ does generalize:

spin$^c$ Dirac operator, preferred connection on $L$. 
Let $J$ be any almost complex structure on $M$.

Let $L = \Lambda^{0,2}$ be its anti-canonical line bundle.
Let $J$ be any almost complex structure on $M$.

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Every unitary connection $A$ on $L$ induces

spin$^c$ Dirac operator

\[D_A : \Gamma(\mathbb{V}_+) \rightarrow \Gamma(\mathbb{V}_-) \]

generalizing $\bar{\partial} + \bar{\partial}^*$. 
Seiberg-Witten equations:

\[ D_A \Phi = 0 \]
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Non-linear, but elliptic once ‘gauge-fixing’
\[ d^*(A - A_0) = 0 \]
imposed to eliminate automorphisms of \( L \rightarrow M \).
Weitzenböck formula:

\[ 0 = 2\Delta |\Phi|^2 + 4|\nabla_A \Phi|^2 + s|\Phi|^2 + |\Phi|^4 \]
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If, in addition, \( c_1^2 > 0, \)

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If complex surface $M$ admits any Einstein metric, either

- on $\lambda \geq 0$ list; or else
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Minimality is harder!
A complex surface $X$ is called **minimal** if it is not the blow-up of another complex surface.
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Any complex surface $M$ can be obtained from a minimal surface $X$ by blowing up a finite number of times:

$$M \approx X \# k\mathbb{CP}^2$$

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One says that $X$ is minimal model of $M$.

A complex surface $M$ is of general type $\iff$ its minimal model $X$ satisfies

$$ c_1^2(X) > 0 $$

$$ c_1 \cdot [\omega] < 0 $$

for some Kähler class $[\omega]$. 
Theorem (Curvature Estimates). For any Riemannian metric $g$ on a compact complex surface $M$ of general type, the following curvature bounds are satisfied:
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\int_M s^2 d\mu_g \geq 32\pi^2 c_1^2(X)
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where $X$ is the minimal model of $M$.

Moreover, equality holds in either case iff $M = X$, and $g$ is Kähler-Einstein with $\lambda < 0$. 
Theorem (L ’01). Let $X$ be a minimal surface of general type, and let

$$M = X \# k\overline{\mathbb{CP}^2}.$$ 

Then $M$ cannot admit an Einstein metric if

$$k \geq c_1^2(X)/3.$$
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(Better than Hitchin-Thorpe by a factor of 3.)

So being “very” non-minimal is an obstruction.
Example.
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\[
\begin{array}{c}
N \\
\ \ \ B' \\
\end{array}
\rightarrow
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**Example.** Let $N$ be double branched cover $\mathbb{CP}^2$, ramified at a smooth octic:

![Diagram of the branched cover]

$c_1 < 0 \implies N$ carries an Einstein metric.
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In example:

$$c_1^2(X) = 3 \quad k = 1$$
$X$ is triple cover $\mathbb{CP}_2$ ramified at sextic

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Moral: Existence depends on diffeotype!
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But $M$ and $N$ are both simply connected & non-spin, and both have $c_2 = 2, h^2, 0 = 3$, so $\chi = 46, \tau = -30$. Hence Freedman $\Rightarrow$ $M$ homeomorphic to $N$.

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