Mass, Scalar Curvature, &

Kähler Geometry, III

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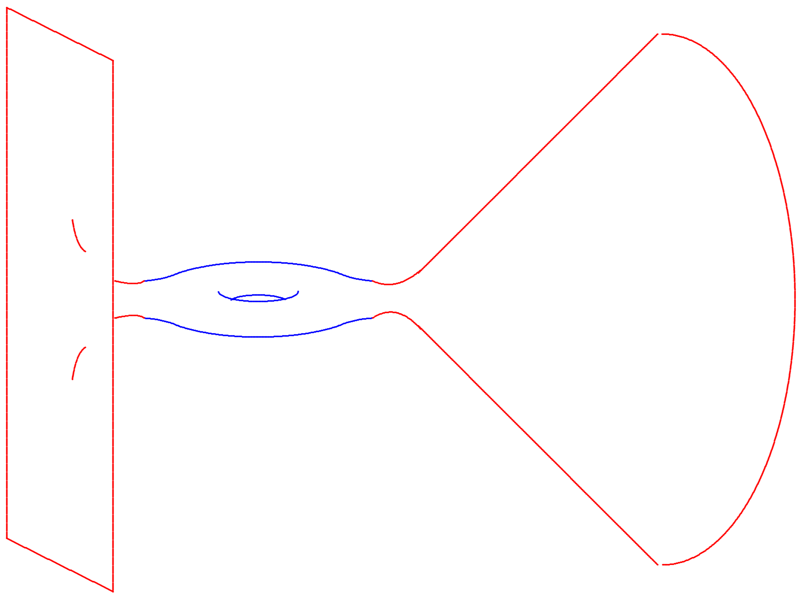
Seminario de Geometría
ICMAT, November 8, 2018

Definition. *Complete, non-compact* n*-manifold* (M^n, g) *is* asymptotically locally Euclidean



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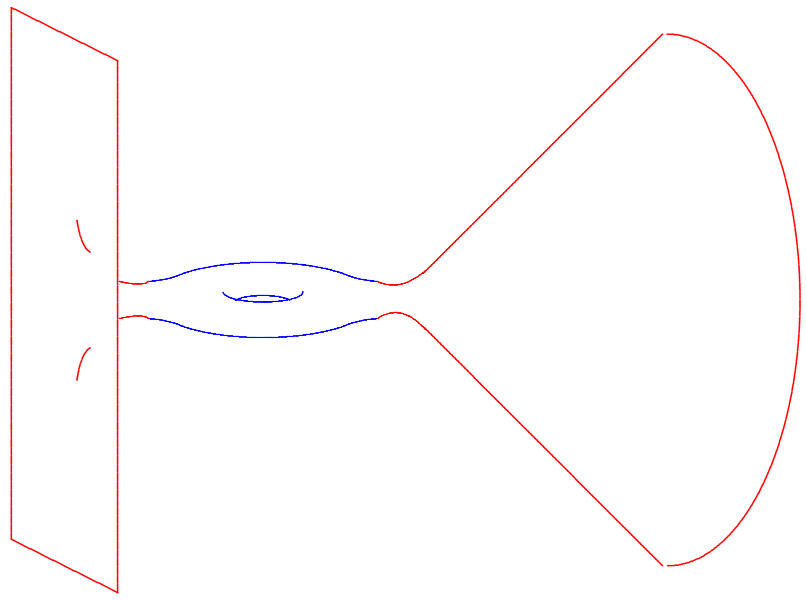

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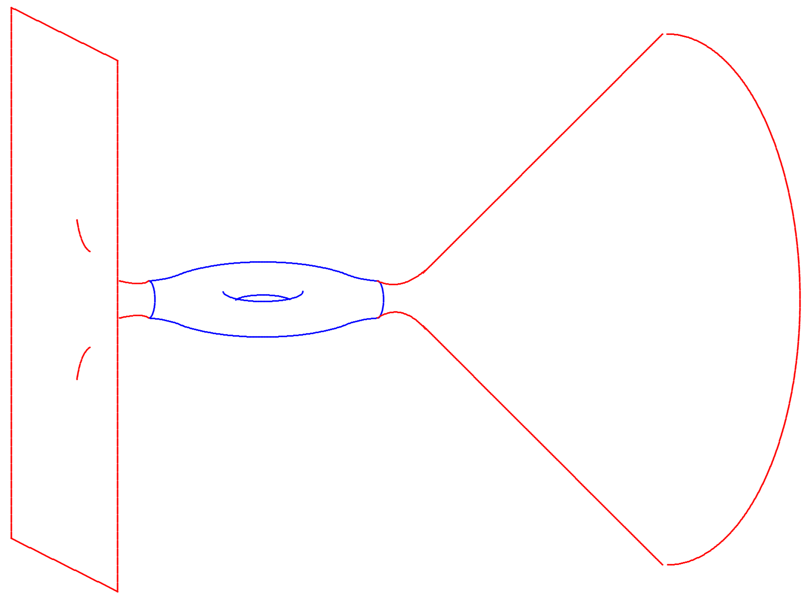

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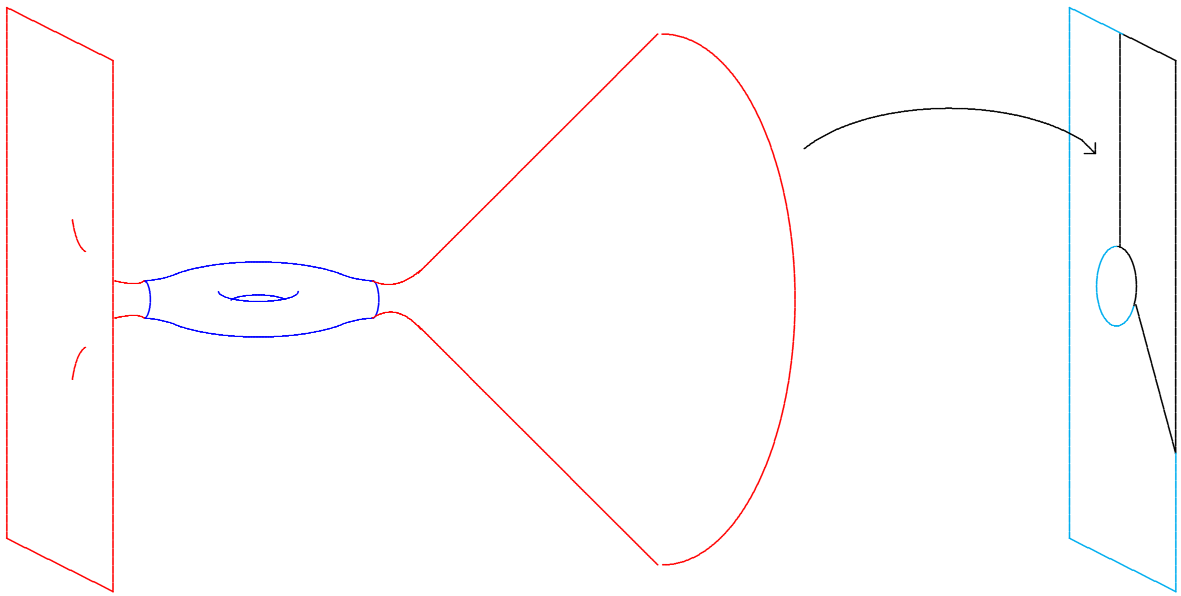

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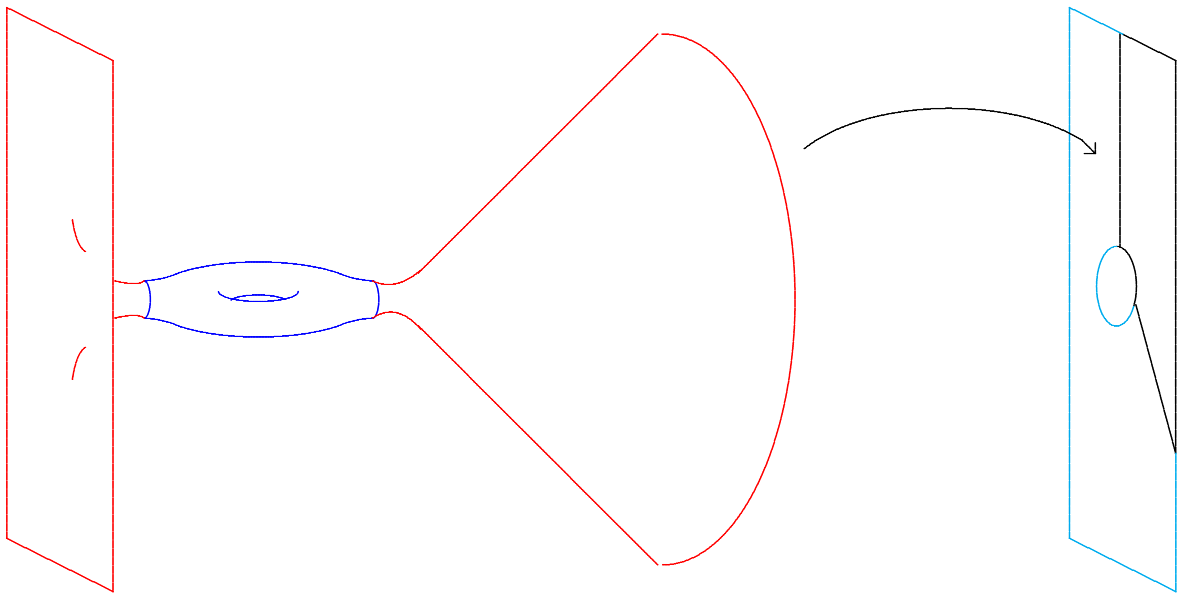

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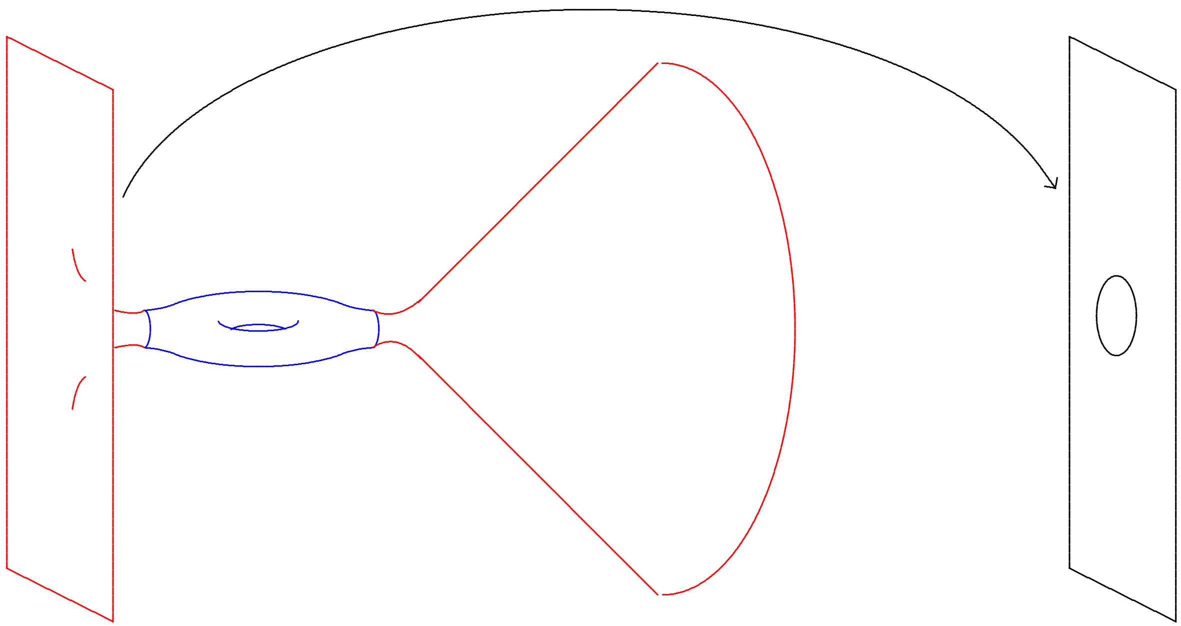

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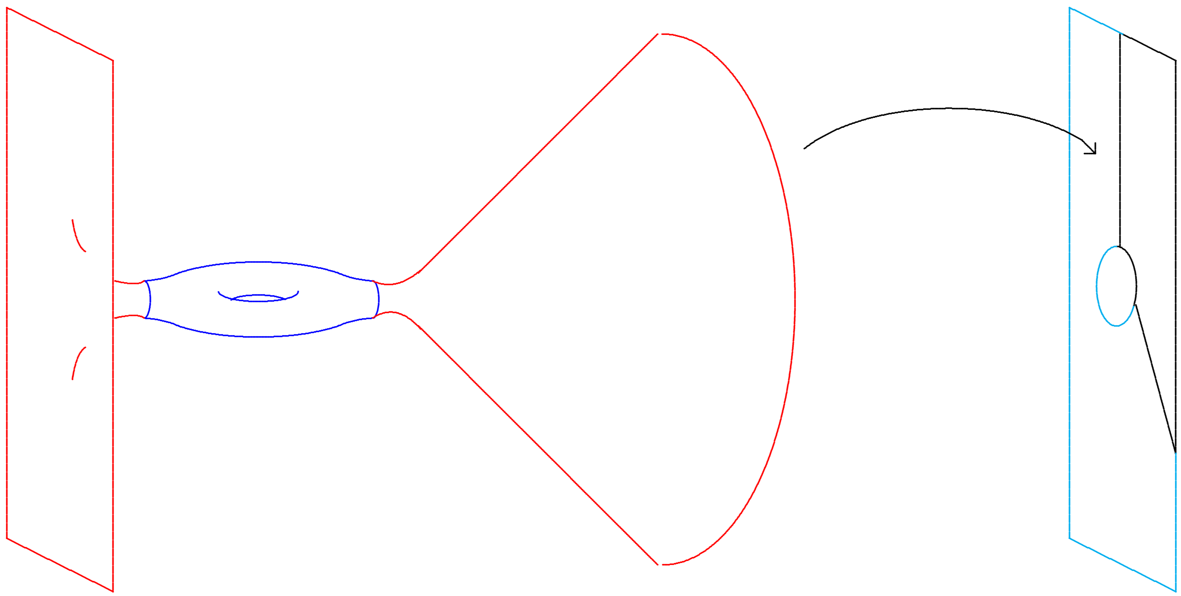

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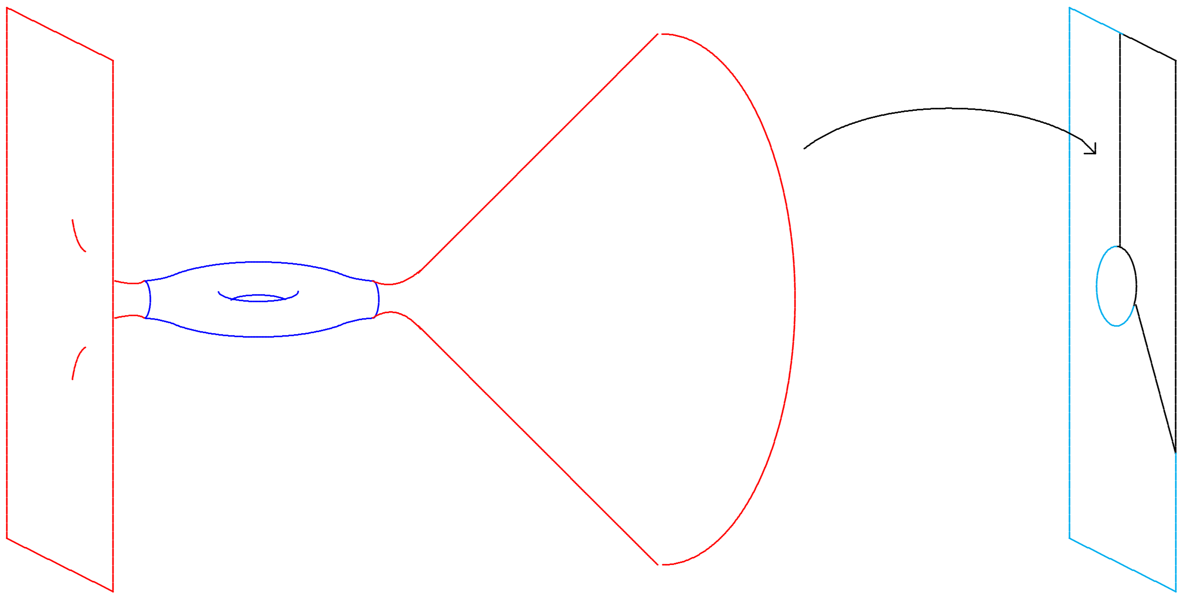

$$g_{jk} = \delta_{jk} + O(|x|^{1-\frac{n}{2}-\varepsilon})$$

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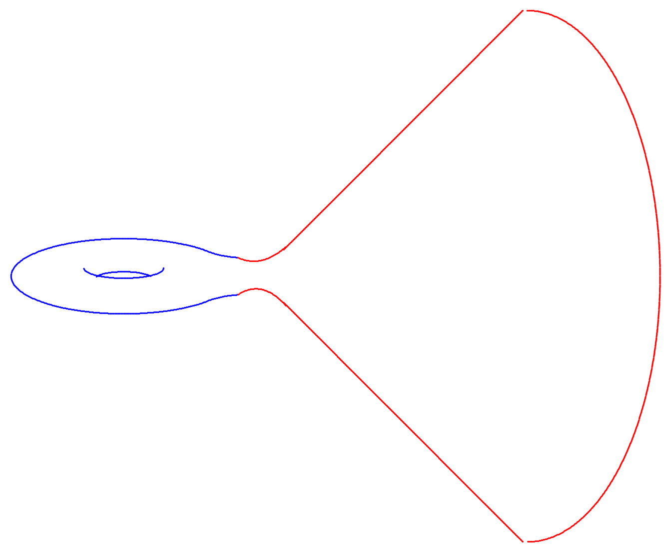
Lemma. *Any ALE Kähler manifold has only one end.*



$$n = 2m \geq 4$$

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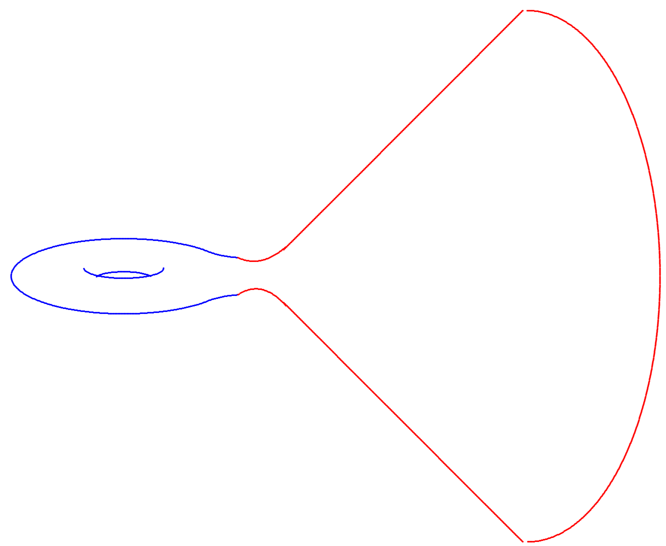

Proof later today!

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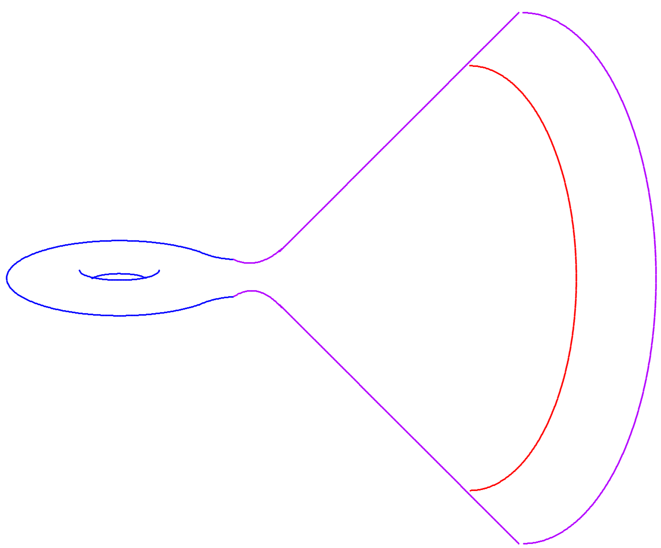

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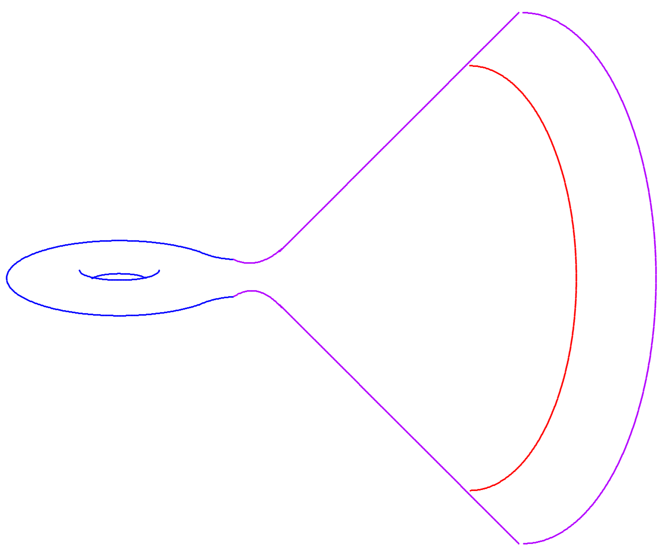

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- $\clubsuit : H^2(M) \xrightarrow{\cong} H^2_c(M)$ *inverse of natural map.*

Scalar-flat Kähler surface:

$$m(M, g) = -\frac{1}{3\pi}\langle \clubsuit(c_1), [\omega]\rangle$$

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$$d(\theta\wedge\omega) = \rho\wedge\omega = \frac{s}{4}\omega^2 = 0.$$

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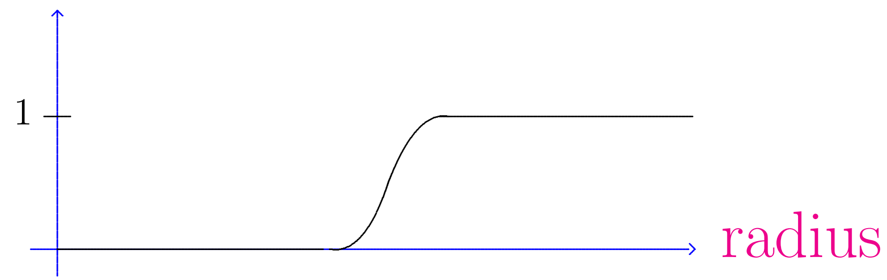

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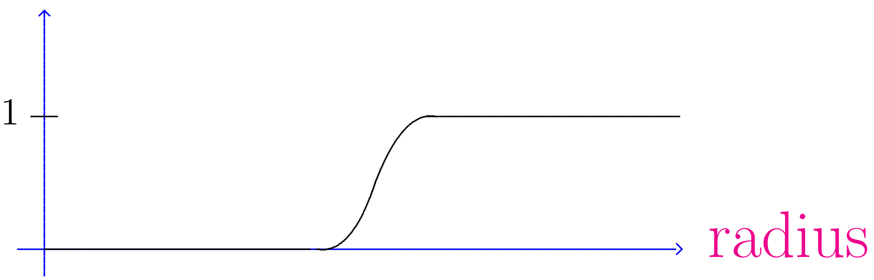

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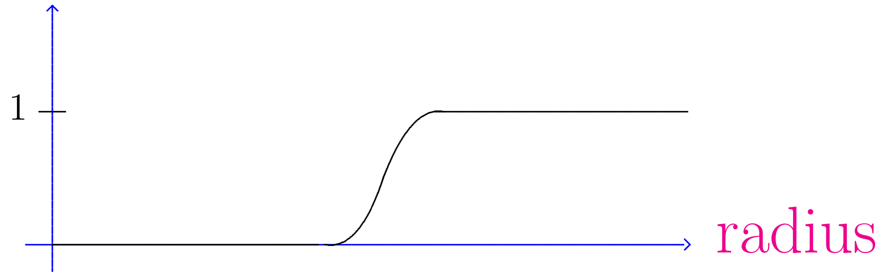

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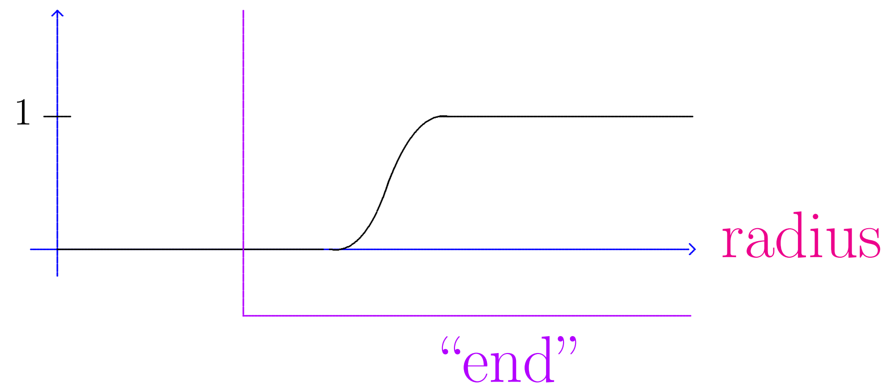

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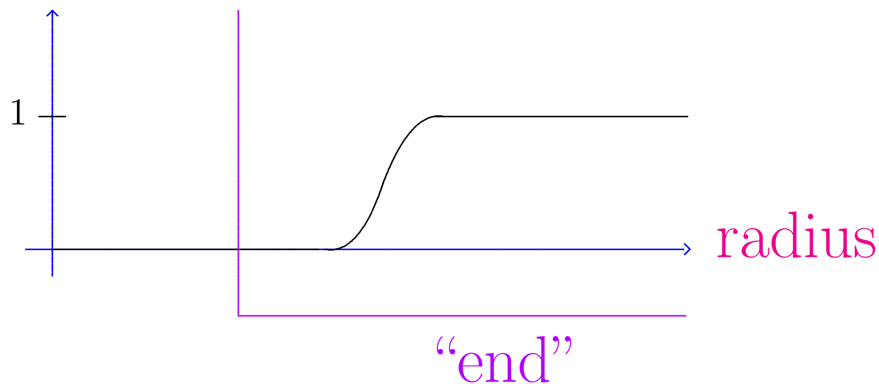

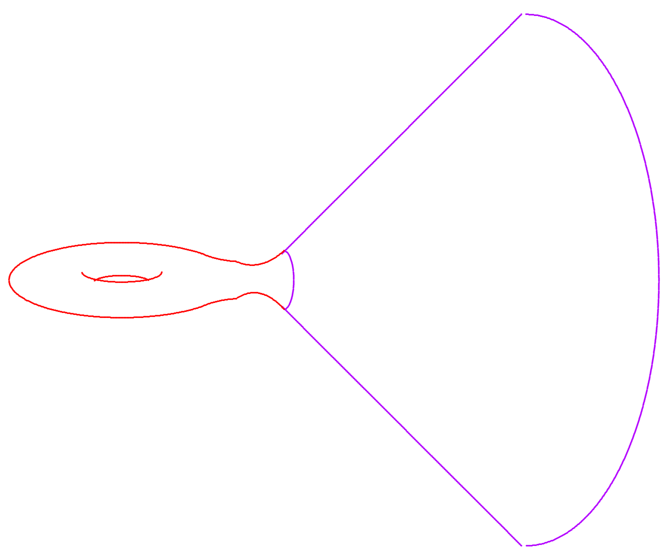

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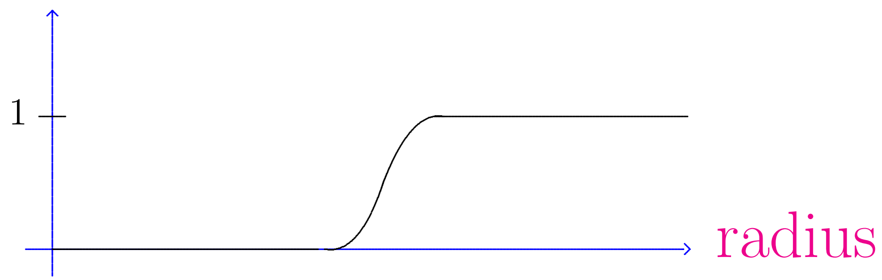

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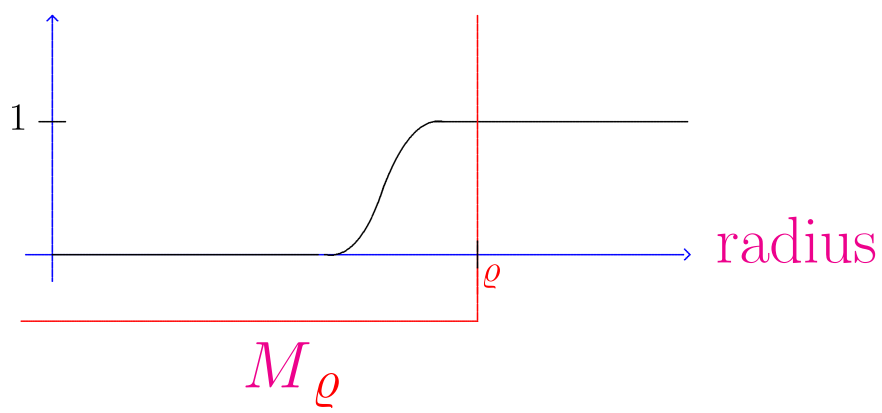

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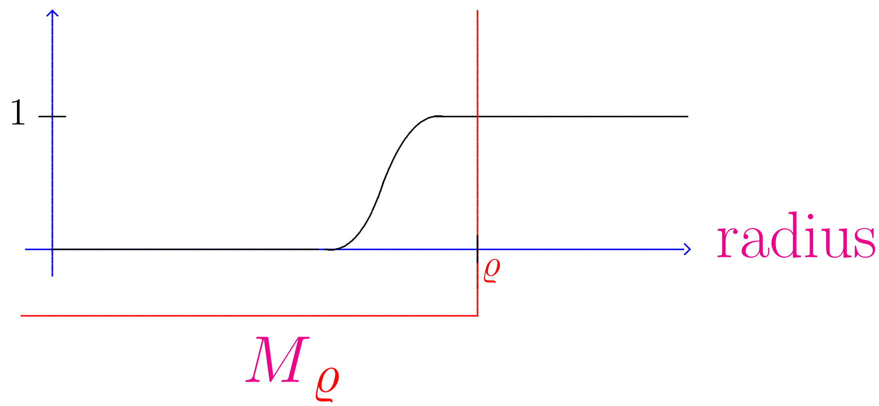

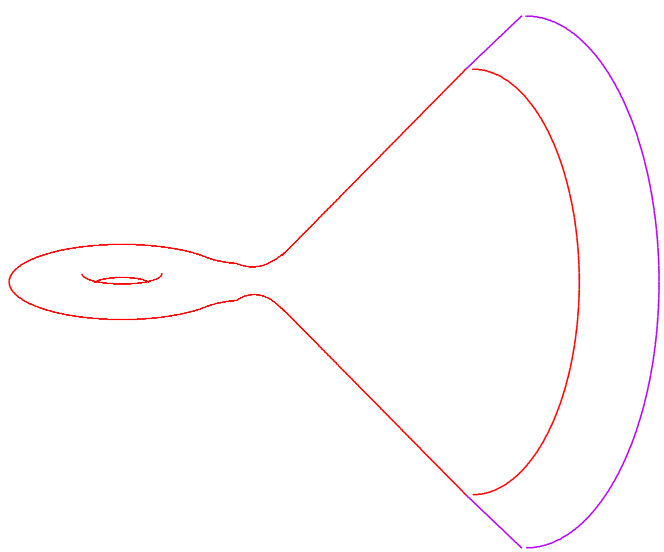

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where M_ϱ defined by radius $\leq \varrho$.

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scalar-flat $\Longrightarrow \rho \wedge \omega = 0.$

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because scalar-flat $\Longrightarrow \ \rho \wedge \omega = 0.$

Let $f : M \to \mathbb{R}$ be smooth cut-off function:

$\equiv 0$ away from end,

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as claimed.

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But were our assumptions justified?

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- $m = 2$;
- $s \equiv 0$; and
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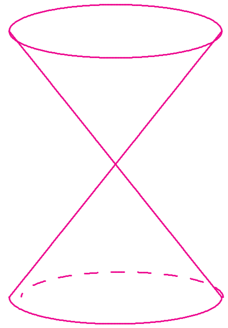

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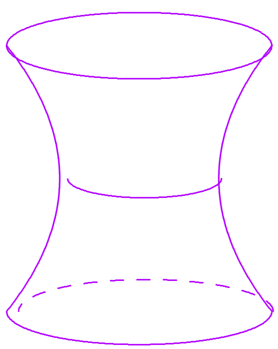

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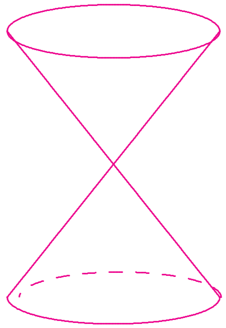

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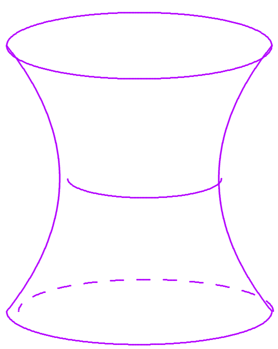

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In this case, capped-off end $\cong_{\text{bih}} \mathbb{CP}_m - B^{2m}$.

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When $m \geq 3$, can construct Kähler metric on $\widetilde{M}$.

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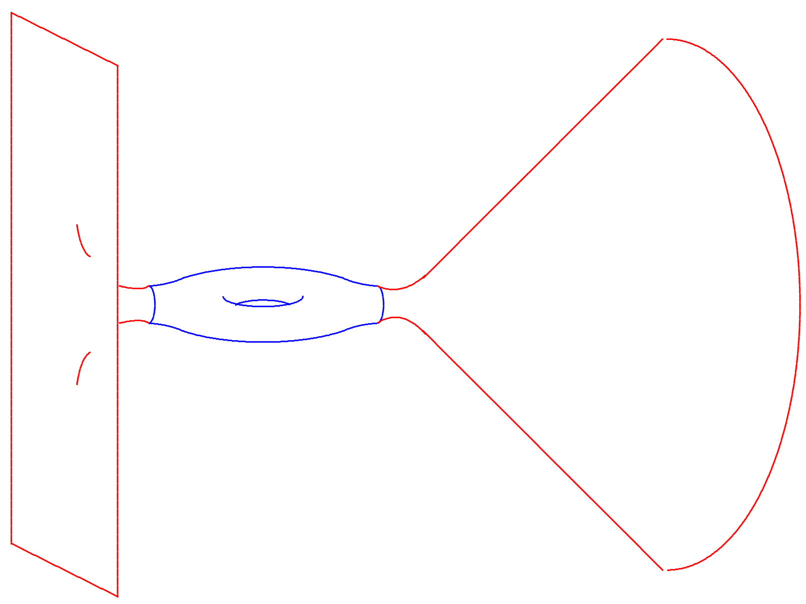

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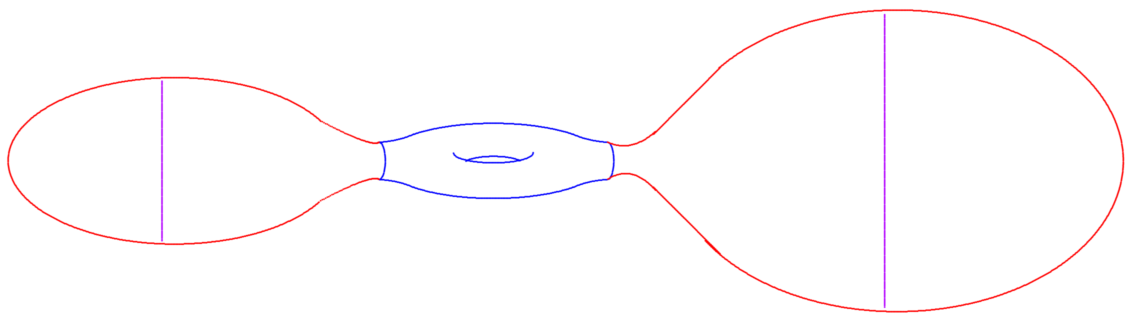

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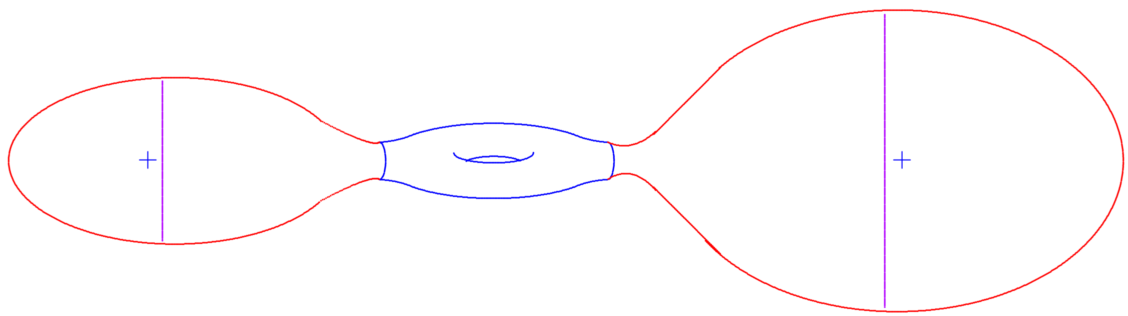

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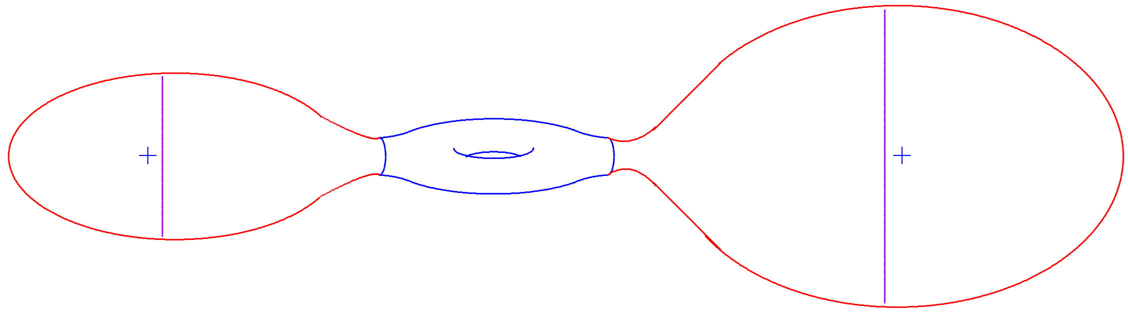

Intersection form

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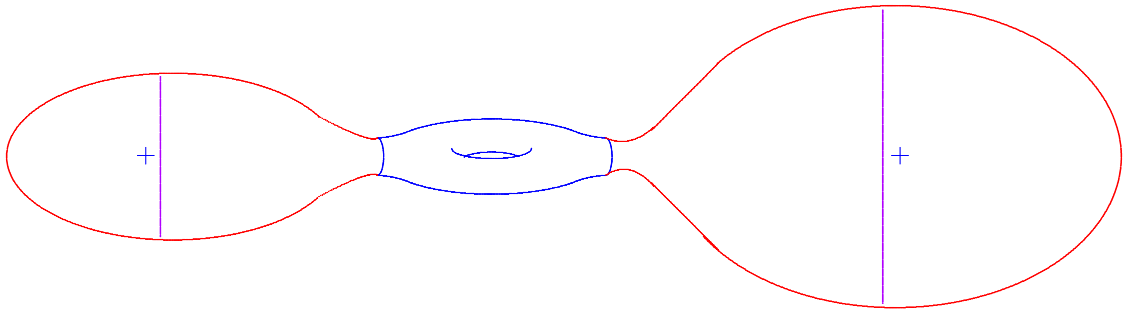

Intersection form

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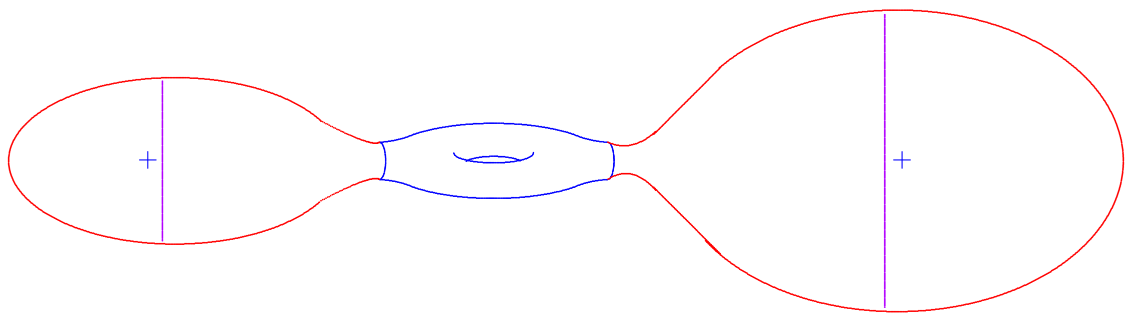

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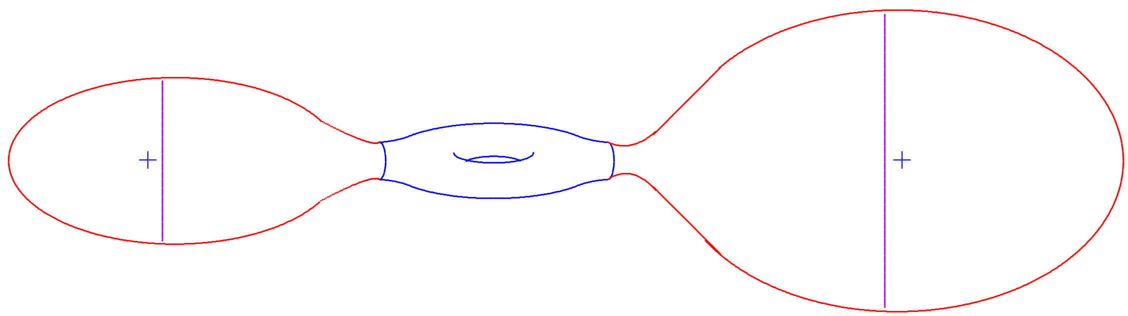

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thus has one positive direction for each end.

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Hodge theorem on intersection form

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Hodge theorem on intersection form

Form has only one positive direction in $H^{1,1}(\widehat{M}, \mathbb{R})$:

$$(+ - \cdots -)$$

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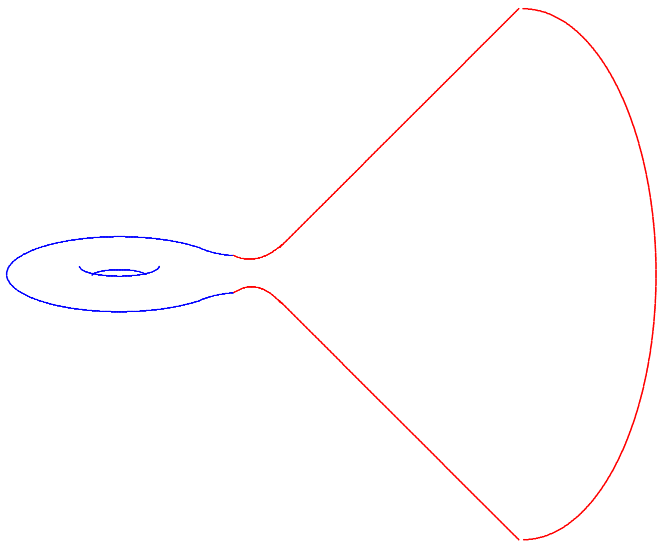
Lemma. *Any ALE Kähler manifold has only one end.*



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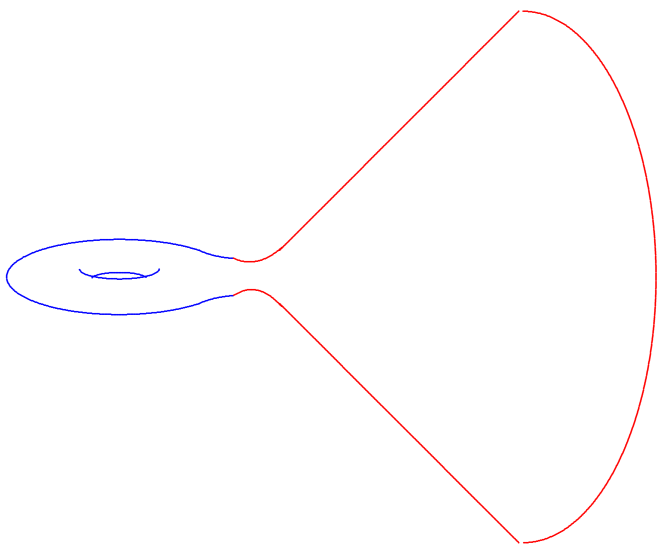

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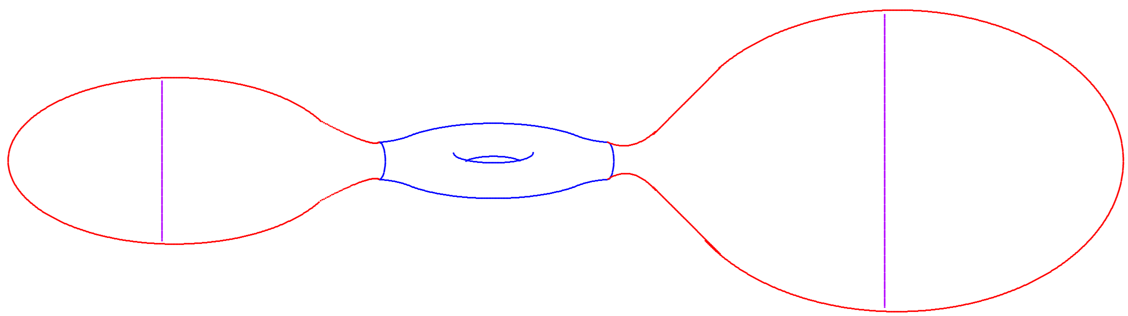

Compactify M as symplectic 4-manifold $\widehat{M}$.

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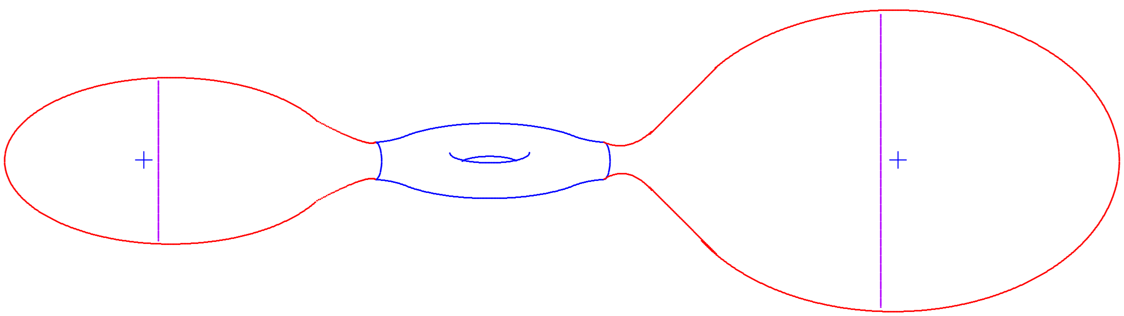

Each end-cap contains immersed symplectic 2-sphere of positive normal bundle.

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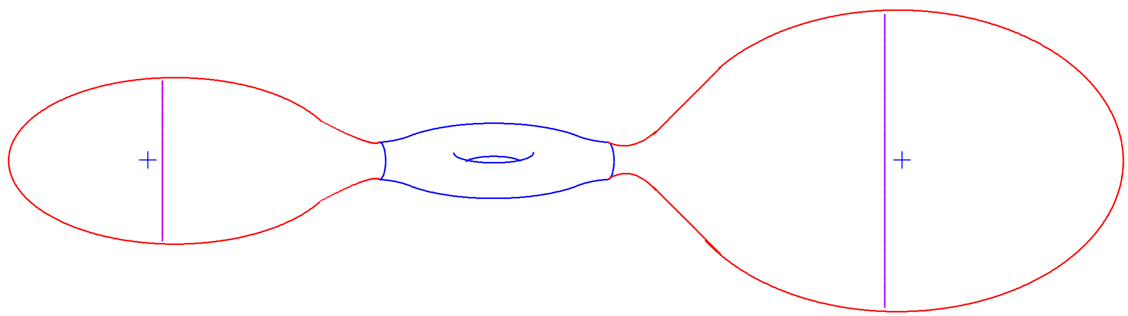

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McDuff $\Longrightarrow \widehat{M} \approx$ rational complex surface.

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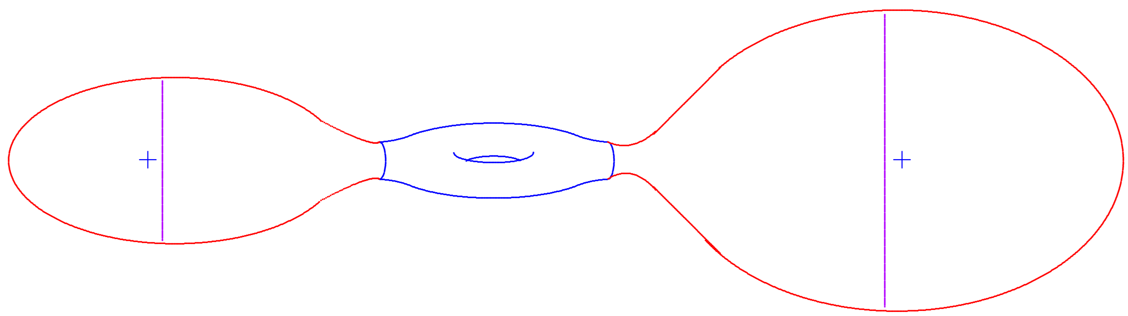

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McDuff $\Longrightarrow$ intersection form $(+- \cdots -)$.

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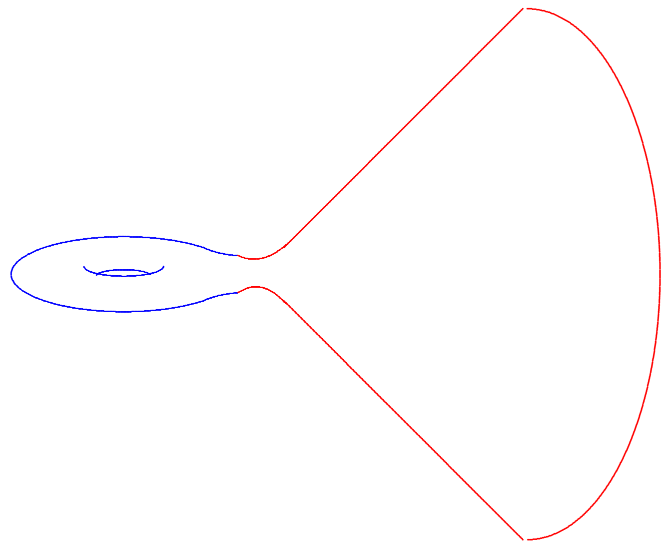

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$$\int_{S_\varrho/\Gamma} \left[g_{kj,k} - g_{kk,j} \right] \mathbf{n}^j d\mathfrak{a}_E - \frac{2}{(m-1)!} \int_{S_\varrho/\Gamma} \theta \wedge \omega^{m-1} = O(\varrho^{-2\varepsilon})$$

Proposition. *Let g be a C^2 Kähler metric on $(\mathbb{R}^{2m} - \mathbf{D}^{2m})/\Gamma$, $m \geq 2$, where $\Gamma \subset \mathbf{SO}(2m)$ is some finite group that acts without fixed-points on S^{2m-1}. In the given coordinate system, assume that g satisfies the weak fall-off hypothesis*

$$g_{jk} = \delta_{jk} + O(\varrho^{-\tau}), \qquad g_{jk,\ell} = O(\varrho^{-\tau-1})$$

where $\varrho = |x|$ and where $\tau = m - 1 + \varepsilon$ for some $\varepsilon > 0$. Then there is a continuously differentiable 1-form θ on $(\mathbb{R}^{2m} - \mathbf{D}^{2m})/\Gamma$ such that

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and such that $d\theta = \rho$, where ρ is the Ricci form of g with respect to a given compatible integrable almost-complex structure J.

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The mass formula then follows, much as before.

Theorem C. *Any ALE Kähler manifold* (M, g, J) *of complex dimension* m *has mass given by*

$$\mathfrak{m}(M,g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1}\rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m}\int_M s_g d\mu_g$$

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This has some interesting consequences...

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Proof actually shows something stronger!

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In (M, J), this gives desired Poincaré dual of c_1.

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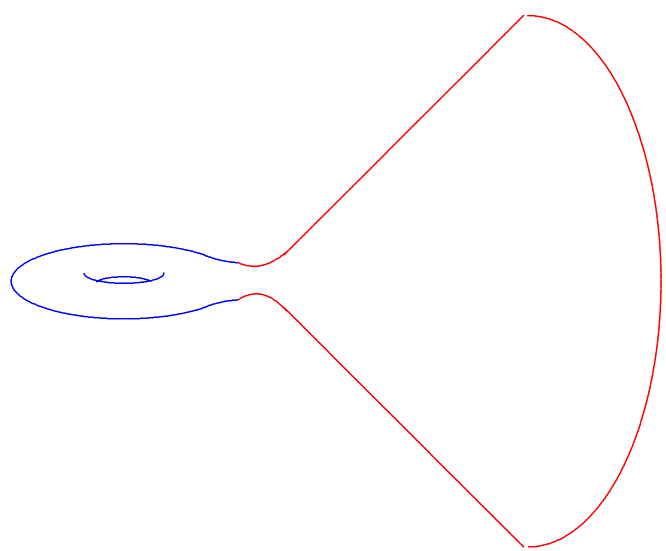

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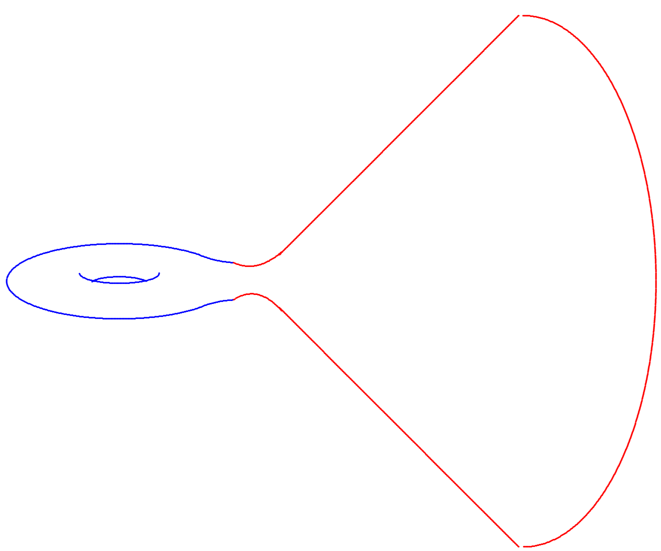

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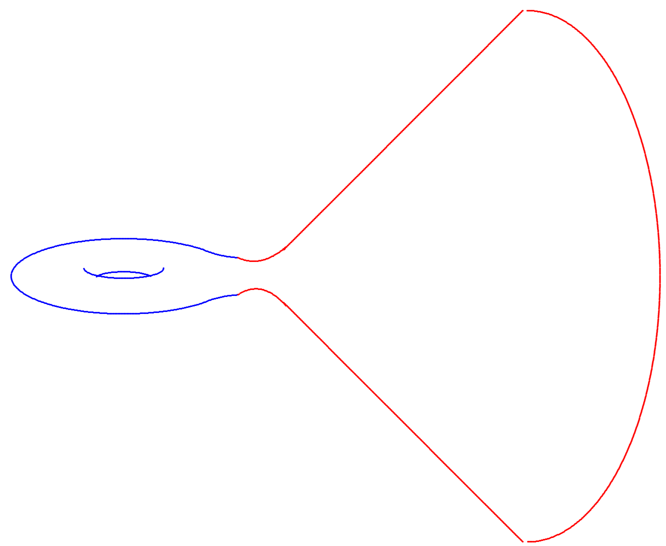
$$\mathfrak{m}(M, g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1}\rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m}\int_M s_g \, d\mu_g$$



End, Part III

¡Muchas Gracias por la Invitación!